

## Research Article

# On the $q$ -Euler Numbers and Polynomials with Weight 0

**T. Kim and J. Choi**

*Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*

Correspondence should be addressed to T. Kim, tkkim@kw.ac.kr

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The purpose of this paper is to investigate some properties of  $q$ -Euler numbers and polynomials with weight 0. From those  $q$ -Euler numbers with weight 0, we derive some identities on the  $q$ -Euler numbers and polynomials with weight 0.

## 1. Introduction

Let  $p$  be a fixed odd prime number. Throughout this paper  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ , and  $\mathbb{C}_p$  will denote the ring of  $p$ -adic rational integers, the field of  $p$ -adic rational numbers, and the completion of algebraic closure of  $\mathbb{Q}_p$ . The  $p$ -adic absolute value is defined by  $|x|_p = 1/p^r$  where  $x = p^r s/t$  for  $s, t \in \mathbb{Z}$  with  $(p, t) = (p, s) = 1$  and  $r \in \mathbb{Q}$ . In this paper, we assume that  $\alpha \in \mathbb{Q}$  and  $q \in \mathbb{C}_p$  with  $|1 - q|_p < 1$ . As well-known definition, the Euler polynomials are defined by

$$\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.1)$$

with the usual convention about replacing  $E^n(x)$  by  $E_n(x)$  (see [1–15]).

In this special case,  $x = 0$ ,  $E_n(0) = E_n$  are called the  $n$ th Euler numbers (see [1]). Recently, the  $q$ -Euler numbers with weight  $\alpha$  are defined by

$$\tilde{E}_{0,q}^{(\alpha)} = 1, \quad q \left( q^\alpha \tilde{E}_q^{(\alpha)} + 1 \right)^n + \tilde{E}_{n,q}^{(\alpha)} = 0 \quad \text{if } n > 0, \quad (1.2)$$

with the usual convention about replacing  $(\tilde{E}_q^{(\alpha)})^n$  by  $\tilde{E}_{n,q}^{(\alpha)}$  (see [3, 12]). The  $q$ -number of  $x$  is defined by  $[x]_q = (1 - q^x)/(1 - q)$  (see [1–15]). Note that  $\lim_{q \rightarrow 1} [x]_q = x$ . Let us define

the notation of  $q$ -Euler numbers with weight 0 as  $\tilde{E}_{n,q}^{(0)} = \tilde{E}_{n,q}$ . The purpose of this paper is to investigate some interesting identities on the  $q$ -Euler numbers with weight 0.

## 2. On the Extended $q$ -Euler Numbers of Higher-Order with Weight 0

Let  $C(\mathbb{Z}_p)$  be the space of continuous functions on  $\mathbb{Z}_p$ . For  $f \in C(\mathbb{Z}_p)$ , the fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  is defined by Kim as follows:

$$\begin{aligned} I_q(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) \\ &= \lim_{N \rightarrow \infty} \frac{[2]_q}{1 + q^{p^N}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \end{aligned} \quad (2.1)$$

(see [1–12]). By (2.1), we get

$$q^n I_q(f_n) + (-1)^{n-1} I_q(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l) q^l, \quad (2.2)$$

where  $f_n(x) = f(x+n)$  and  $n \in \mathbb{N}$  (see [4, 5]).

By (1.2), (2.1), and (2.2), we see that

$$\int_{\mathbb{Z}_p} [x]_{q^n} d\mu_{-q}(x) = \tilde{E}_{n,q}^{(\alpha)} = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+q^{al+1}}. \quad (2.3)$$

In the special case,  $n = 1$ , we get

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q}{qe^t + 1} = \frac{1+q^{-1}}{e^t + q^{-1}} = \sum_{n=0}^{\infty} H_n(-q^{-1}) \frac{t^n}{n!}, \quad (2.4)$$

where  $H_n(-q^{-1})$  are the  $n$ th Frobenius-Euler numbers. From (2.4), we note that the  $q$ -Euler numbers with weight 0 are given by

$$\tilde{E}_{n,q} = \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = H_n(-q^{-1}), \quad \text{for } n \in \mathbb{Z}_+. \quad (2.5)$$

Therefore, by (2.5), we obtain the following theorem.

**Theorem 2.1.** For  $n \in \mathbb{Z}_+$ , one has

$$\tilde{E}_{n,q} = H_n(-q^{-1}), \quad (2.6)$$

where  $H_n(-q^{-1})$  are called the  $n$ th Frobenius-Euler numbers.

Let us define the generating function of the  $q$ -Euler numbers with weight 0 as follows:

$$\tilde{F}_q(t) = \sum_{n=0}^{\infty} \tilde{E}_{n,q} \frac{t^n}{n!}. \tag{2.7}$$

Then, by (2.3) and (2.7), we get

$$\tilde{F}_q(t) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{mt} = \frac{1+q}{qe^t+1}. \tag{2.8}$$

Now we define the  $q$ -Euler polynomials with weight 0 as follows:

$$\sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!} = \frac{1+q}{qe^t+1} e^{xt}. \tag{2.9}$$

Thus, (2.4) and (2.9), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{1+q}{qe^t+1} e^{xt} = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!}. \tag{2.10}$$

From (2.10), we have

$$\sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!} = \left( \frac{1+q^{-1}}{e^t+q^{-1}} \right) e^{xt} = \sum_{n=0}^{\infty} H_n(-q^{-1}, x) \frac{t^n}{n!}, \tag{2.11}$$

where  $H_n(-q^{-1}, x)$  are called the  $n$ th Frobenius-Euler polynomials (see [9]).

Therefore, by (2.11), we obtain the following theorem.

**Theorem 2.2.** For  $n \in \mathbb{Z}_+$ , one has

$$\tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(x) = H_n(-q^{-1}, x), \tag{2.12}$$

where  $H_n(-q^{-1}, x)$  are called the  $n$ th Frobenius-Euler polynomials.

From (2.2) and Theorem 2.2, we note that

$$q^n H_m(-q^{-1}, n) + H_m(-q^{-1}) = [2]_q \sum_{l=0}^{n-1} (-1)^l l^m q^l, \tag{2.13}$$

where  $n \in \mathbb{N}$  with  $n \equiv 1 \pmod{2}$ .

Therefore, by (2.13), we obtain the following corollary.

**Corollary 2.3.** For  $n \in \mathbb{N}$ , with  $n \equiv 1 \pmod{2}$  and  $m \in \mathbb{Z}_+$ , one has

$$q^n H_m(-q^{-1}, n) + H_m(-q^{-1}) = [2]_q \sum_{l=0}^{n-1} (-1)^l l^m q^l. \tag{2.14}$$

In particular,  $q = 1$ , we get  $E_m(n) + E_m = 2 \sum_{l=0}^{n-1} (-1)^l l^m$ , where  $E_m$  and  $E_m(n)$  are called the  $m$ th Euler numbers and polynomials which are defined by

$$\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}, \quad \frac{2}{e^t + 1} e^{xt} = \sum_{m=0}^{\infty} E_m(x) \frac{t^m}{m!}. \quad (2.15)$$

By (2.2), we easily see that

$$q \int_{\mathbb{Z}_p} f(x+1) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2]_q f(0). \quad (2.16)$$

Thus, by (2.16), we get

$$\begin{aligned} [2]_q &= q \int_{\mathbb{Z}_p} e^{(x+1)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) \\ &= \sum_{n=0}^{\infty} \left( q \int_{\mathbb{Z}_p} (x+1)^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( q H_n(-q^{-1}, 1) + H_n(-q^{-1}) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.17)$$

Therefore, by (2.16), we obtain the following theorem.

**Theorem 2.4.** For  $n \in \mathbb{Z}_+$ , one has

$$q H_n(-q^{-1}, 1) + H_n(-q^{-1}) = \begin{cases} 1 + q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \quad (2.18)$$

where  $H_n(-q^{-1}, x)$  are called the  $n$ th Frobenius-Euler polynomials and  $H_n(-q^{-1})$  are called the  $n$ th Frobenius-Euler numbers. In particular,  $q = 1$ , we have

$$E_n(1) + E_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \quad (2.19)$$

where  $E_n$  are called the  $n$ th Euler numbers.

From (2.5) and Theorem 2.2, we note that

$$\tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = \sum_{l=0}^n \binom{n}{l} \int_{\mathbb{Z}_p} y^l d\mu_{-q}(y) x^{n-l} = \sum_{l=0}^n \binom{n}{l} \tilde{E}_{n,q} x^{n-l} = (x + \tilde{E}_q)^n, \quad (2.20)$$

where the usual convention about replacing  $(\tilde{E}_q)^l$  by  $\tilde{E}_{l,q}$ . By Theorems 2.2 and 2.4, we get

$$q \tilde{E}_{n,q}(1) + \tilde{E}_{n,q} = \begin{cases} [2]_{q'}, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \quad (2.21)$$

From (2.20) and (2.21), we have

$$q\left(\tilde{E}_q + 1\right)^n + \tilde{E}_{n,q} = \begin{cases} [2]_{q'} & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \quad (2.22)$$

For  $n \in \mathbb{N}$ , by (2.20) and (2.22), we have

$$\begin{aligned} q^2 \tilde{E}_{n,q}(2) &= q^2 \left(\tilde{E}_q + 1 + 1\right)^n = q^2 \sum_{l=1}^n \binom{n}{l} \left(\tilde{E}_q + 1\right)^l + q(1 + q - \tilde{E}_{0,q}) = q + q^2 - q \sum_{l=0}^n \binom{n}{l} \tilde{E}_{l,q} \\ &= q + q^2 - q\left(\tilde{E}_q + 1\right)^n = q + q^2 + \tilde{E}_{n,q} - q[2]_q \delta_{0,n}. \end{aligned} \quad (2.23)$$

Therefore, by (2.23), we obtain the following theorem.

**Theorem 2.5.** For  $n \in \mathbb{N}$ , one has

$$q^2 \tilde{E}_{n,q}(2) = q + q^2 + \tilde{E}_{n,q}. \quad (2.24)$$

For  $n \in \mathbb{Z}_+$ , we have

$$\tilde{E}_{n,q^{-1}}(1-x) = \int_{\mathbb{Z}_p} (1-x+x_1)^n d\mu_{-q^{-1}}(x_1) = (-1)^n \int_{\mathbb{Z}_p} (x_1+x)^n d\mu_{-q}(x_1) = (-1)^n \tilde{E}_{n,q}(x). \quad (2.25)$$

Therefore, by (2.25), we obtain the following theorem.

**Theorem 2.6.** For  $n \in \mathbb{Z}_+$ , one has

$$\tilde{E}_{n,q^{-1}}(1-x) = (-1)^n \tilde{E}_{n,q}(x). \quad (2.26)$$

From (2.20), we have

$$\int_{\mathbb{Z}_p} (1-x)^n d\mu_{-q}(x) = (-1)^n \int_{\mathbb{Z}_p} (x-1)^n d\mu_{-q}(x) = (-1)^n \tilde{E}_{n,q}(-1). \quad (2.27)$$

By Theorem 2.6 and (2.27), we get

$$\int_{\mathbb{Z}_p} (1-x)^n d\mu_{-q}(x) = \tilde{E}_{n,q^{-1}}(2) = 1 + q + q^2 \tilde{E}_{n,q^{-1}} \quad \text{if } n > 0. \quad (2.28)$$

Therefore, by (2.28), we obtain the following theorem.

**Theorem 2.7.** For  $n \in \mathbb{N}$ , one has

$$\int_{\mathbb{Z}_p} (1-x)^n d\mu_{-q}(x) = 1 + q + q^2 \tilde{E}_{n,q^{-1}}. \quad (2.29)$$

Let  $C(\mathbb{Z}_p)$  be the space of continuous functions on  $\mathbb{Z}_p$ . For  $f \in C(\mathbb{Z}_p)$ ,  $p$ -adic analogue of Bernstein operator of order  $n$  for  $f$  is given by

$$\mathbb{B}_n(f | x) = \sum_{k=0}^n B_{k,n}(x) f\left(\frac{k}{n}\right) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad (2.30)$$

where  $n, k \in \mathbb{Z}_+$  (see [1, 6, 7]).

For  $n, k \in \mathbb{Z}_+$ ,  $p$ -adic Bernstein polynomial of degree  $n$  is defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in \mathbb{Z}_p \quad (2.31)$$

(see [1, 6, 7]).

Let us take the fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  for one Bernstein polynomials in (2.31) as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-q}(x) &= \binom{n}{k} \int_{\mathbb{Z}_p} x^k (1-x)^{n-k} d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \tilde{E}_{k+l,q}. \end{aligned} \quad (2.32)$$

By simple calculation, we easily get

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x) d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} (1-x)^{n-l} d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} (1+q+q^2 \tilde{E}_{n-l,q^{-1}}) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} q^2 \tilde{E}_{n-l,q^{-1}} + [2]_q \binom{n}{k} (-1)^k \delta_{0,k} \quad \text{if } n > k. \end{aligned} \quad (2.33)$$

Therefore, by (2.32) and (2.33), we obtain the following theorem.

**Theorem 2.8.** For  $n \in \mathbb{Z}_+$  with  $n > k > 0$ , one has

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \tilde{E}_{k+l,q} = \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} q^2 \tilde{E}_{n-l,q^{-1}}. \quad (2.34)$$

In particular,  $k = 0$ , we get

$$\sum_{l=0}^n \binom{n}{l} (-1)^l \tilde{E}_{l,q} = q^2 \tilde{E}_{n,q^{-1}} + [2]_q. \quad (2.35)$$

By Theorems 2.1 and 2.2, we get

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l H_{k+l}(-q^{-1}) = \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} q^2 H_{n-l}(-q), \quad (2.36)$$

where  $n, k \in \mathbb{Z}_+$  with  $n > k > 0$ .

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