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Research Article

A Weighted Variant of Riemann-Liouville Fractional Integrals on $\mathbb{R}^n$

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We introduce certain type of weighted variant of Riemann-Liouville fractional integral on $\mathbb{R}^n$ and obtain its sharp bounds on the central Morrey and λ-central BMO spaces. Moreover, we establish a sufficient and necessary condition of the weight functions so that commutators of weighted Hardy operators (with symbols in λ-central BMO space) are bounded on the central Morrey spaces. These results are further used to prove sharp estimates of some inequalities due to Weyl and Cesàro.

1. Introduction

Let $0 < \alpha < 1$. The well-known *Riemann-Liouville fractional integral* I_α is defined by

$$I_\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad x > 0, \tag{1.1}$$

for all locally integrable functions f on $(0,\infty)$. The study of Riemann-Liouville fractional integral has a very long history and number of papers involved its generalizations, variants, and applications. For the earlier development of this kind of integrals and many important applications in fractional calculus, we refer the interested reader to the book [1]. Among numerous material dealing with applications of fractional calculus to (ordinary or partial) differential equations, we choose to refer to [2] and references therein.

As the classical n-dimensional generalization of I_α, the well-known Riesz potential (the solution of Laplace equation) $\mathcal{I}_\alpha$ with $0 < \alpha < n$ is defined by setting, for all locally integrable functions f on $\mathbb{R}^n$,

$$\mathcal{I}_\alpha f(x) := C_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(t)}{|x-t|^{n-\alpha}} dt, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $C_{n,\alpha} := \pi^{n/2} 2^\alpha (\Gamma(\alpha/2))/(\Gamma((n-\alpha)/2))$. The importance of Riesz potentials lies in the fact that they are indeed smoothing operators and have been extensively used in many different areas such as potential analysis, harmonic analysis, and partial differential equations. Here we refer to the paper [3], which is devoted to the sharp constant in the Hardy-Littlewood-Sobolev inequality related to $\mathcal{I}_\alpha$.

This paper focused on another generalization, the weighted variants of Riemann-Liouville fractional integrals on $\mathbb{R}^n$. We investigate the boundedness of these weighted variants on the type of central Morrey and central Campanato spaces and also give the sharp estimates. This development begins with an equivalent definition of I_α as

$$x^\alpha I_\alpha f(x) = \int_0^1 f(tx) \frac{1}{\Gamma(\alpha)(1-t)^{1-\alpha}} dt, \quad x > 0. \tag{1.3}$$

More generally, we use a positive function (weight function) $\omega(t)$ to replace $1/(\Gamma(\alpha)(1-t)^{1-\alpha})$ in (1.3) and generalize the parameter x from the positive axle to the Euclidean space $\mathbb{R}^n$ therein. We then derive a weighted generalization of $|x|^\alpha I_\alpha$ on $\mathbb{R}^n$, which is called the weighted Hardy operator (originally named weighted Hardy-Littlewood avarage) H_ω.

More precise, let ω be a positive function on $[0,1]$. The *weighted Hardy operator* H_ω is defined by setting, for all complex-valued measurable functions f on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$H_\omega f(x) := \int_0^1 f(tx)\omega(t) dt. \tag{1.4}$$

Under certain conditions on ω, Carton-Lebrun and Fosset [4] proved that H_ω maps $L^p(\mathbb{R}^n)$, $1 < p < \infty$, into itself; moreover, the operator H_ω commutes with the Hilbert transform when $n = 1$, and with certain Calderón-Zygmund singular integrals including the Riesz transform when $n \geq 2$. Obviously, for $n = 1$ and $0 < \alpha < 1$, if we take $\omega(t) := 1/(\Gamma(\alpha)(1-t)^{1-\alpha})$, then as mentioned above, for all $x > 0$,

$$H_\omega f(x) = x^{-\alpha} I_\alpha f(x). \tag{1.5}$$

A further extension of [4] was due to Xiao [5] as follows.

Theorem A. *Let $1 < p < \infty$. Then, H_ω is bounded on $L^p(\mathbb{R}^n)$ if and only if*

$$\mathbb{A} := \int_0^1 t^{-n/p} \omega(t) dt < \infty. \tag{1.6}$$

Moreover,

$$\|H_\omega f\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \mathbb{A}. \tag{1.7}$$

Remark 1.1. Notice that the condition (1.6) implies that ω is integrable on $[0,1]$ since $\int_0^1 \omega(t)dt \le \int_0^1 t^{-n/p}\omega(t)dt$. We naturally assume ω is integrable on $[0,1]$ throughout this paper.

Obviously, Theorem A implies the celebrated result of Hardy et al. [6, Theorem 329], namely, for all $0<\alpha<1$ and $1<p<\infty$,

$$\|I_\alpha\|_{L^p(dx)\to L^p(x^{-p\alpha}dx)} = \frac{\Gamma(1-1/p)}{\Gamma(1+\alpha-1/p)}. \tag{1.8}$$

The constant $\mathbb{A}$ in (1.6) also seems to be of interest as it equals to $p/(p-1)$ if $\omega\equiv 1$ and $n=1$. In this case, H_ω is precisely reduced to the *classical Hardy operator* H defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x>0, \tag{1.9}$$

which is the most fundamental integral averaging operator in analysis. Also, a celebrated operator norm estimate due to Hardy et al. [6], that is,

$$\|H\|_{L^p(\mathbb{R}^+)\to L^p(\mathbb{R}^+)} = \frac{p}{p-1} \tag{1.10}$$

with $1<p<\infty$, can be deduced from Theorem A immediately.

Recall that $\mathrm{BMO}(\mathbb{R}^n)$ is defined to be the space of all $b\in L_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|b\|_{\mathrm{BMO}} := \sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B |b(x)-b_B|\,dx<\infty, \tag{1.11}$$

where $b_B = (1/|B|)\int_B b$ and the supremum is taken over all balls B in $\mathbb{R}^n$ with sides parallel to the axes. It is well known that $L^\infty(\mathbb{R}^n)\subsetneq \mathrm{BMO}(\mathbb{R}^n)$, since $\mathrm{BMO}(\mathbb{R}^n)$ contains unbounded functions such as $\log|x|$. Another interesting result of Xiao in [5] is that the weighted Hardy operator H_ω is bounded on $\mathrm{BMO}(\mathbb{R}^n)$, if and only if

$$\int_0^1 \omega(t)dt<\infty. \tag{1.12}$$

Moreover,

$$\|H_\omega\|_{\mathrm{BMO}(\mathbb{R}^n)\to\mathrm{BMO}(\mathbb{R}^n)} = \int_0^1 \varphi(t)dt. \tag{1.13}$$

In recent years, several authors have extended and considered the action of weighted Hardy operators on various spaces. We mention here, the work of Rim and Lee [7], Kuang [8], Krulić et al. [9], Tang and Zhai [10], Tang and Zhou [11].

The main purpose of this paper is to make precise the mapping properties of weighted Hardy operators on the central Morrey and λ-central BMO spaces. The study of the central Morrey and λ-central BMO spaces are traced to the work of Wiener [12, 13] on describing the behavior of a function at the infinity. The conditions he considered are related to appropriate weighted L^q $(1 < q < \infty)$ spaces. Beurling [14] extended this idea and defined a pair of dual Banach spaces A^q and $B^{q'}$, where $1/q + 1/q' = 1$. To be precise, A^q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L^q spaces. The space $B^{q'}$ is expressed as the intersection of the corresponding weighted $L^{q'}$ spaces. Later, Feichtinger [15] observed that the space $B^{q'}$ can be equivalently described by the set of all locally q'-integrable functions f satisfying that

$$\|f\|_{B^{q'}} = \sup_{k\geq 0}\left(2^{-kn/q'}\|f\chi_k\|_{q'}\right) < \infty, \tag{1.14}$$

where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$, χ_k is the characteristic function of the annulus $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$, $k = 1, 2, 3, \ldots$, and $\|\cdot\|_{q'}$ is the norm in $L^{q'}$. By duality, the space A^q, called Beurling algebra now, can be equivalently described by the set of all locally q-integrable functions f satisfying that

$$\|f\|_{A^q} = \sum_{k=0}^{\infty} 2^{kn/q'}\|f\chi_k\|_q < \infty. \tag{1.15}$$

Based on these, Chen and Lau [16] and García-Cuerva [17] introduced an atomic space HA^q associated with the Beurling algebra A^q and identified its dual as the space CMO^q, which is defined to be the space of all locally q-integrable functions f satisfying that

$$\sup_{R\geq 1}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}\left|f(x) - f_{B(0,R)}\right|^q dx\right)^{1/q} < \infty. \tag{1.16}$$

By replacing $k \in \mathbb{N} \cup \{0\}$ with $k \in \mathbb{Z}$ in (1.3) and (1.6), we obtain the spaces $\dot{A}^q$ and $\dot{B}^{q'}$, which are the homogeneous version of the spaces A^q and $B^{q'}$, and the dual space of $\dot{A}^q$ is just $\dot{B}^{q'}$. Related to these homogeneous spaces, in [18, 19], Lu and Yang introduced the homogeneous counterparts of HA^q and CMO^q, denoted by $\mathrm{H\dot{A}}^q$ and $\mathrm{C\dot{M}O}^q$, respectively. These spaces were originally denoted by HK^q and CBMO_q in [18, 19]. Recall that the space $\mathrm{C\dot{M}O}^q$ is defined to be the space of all locally q-integrable functions f satisfying that

$$\sup_{R>0}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}\left|f(x) - f_{B(0,R)}\right|^q dx\right)^{1/q} < \infty. \tag{1.17}$$

It was also proved by Lu and Yang that the dual space of $\mathrm{H\dot{A}}^q$ is just $\mathrm{C\dot{M}O}^q$.

In 2000, Alvarez et al. [20] introduced the following λ-central bounded mean oscillation spaces and the central Morrey spaces, respectively.

Definition 1.2. Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The central Morrey space $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is defined to be the space of all locally q-integrable functions f satisfying that

$$\|f\|_{\dot{B}^{q,\lambda}} = \sup_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x)|^q dx\right)^{1/q} < \infty. \tag{1.18}$$

Definition 1.3. Let $\lambda < 1/n$ and $1 < q < \infty$. A function $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ is said to belong to the λ-central bounded mean oscillation space $\text{C}\dot{\text{M}}\text{O}^{q,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\text{C}\dot{\text{M}}\text{O}^{q,\lambda}} = \sup_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x) - f_{B(0,R)}|^q dx\right)^{1/q} < \infty. \tag{1.19}$$

We remark that if two functions which differ by a constant are regarded as a function in the space $\text{C}\dot{\text{M}}\text{O}^{q,\lambda}$, then $\text{C}\dot{\text{M}}\text{O}^{q,\lambda}$ becomes a Banach space. Apparently, (1.19) is equivalent to the following condition:

$$\sup_{R>0}\inf_{c\in\mathbb{C}}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x) - c|^q dx\right)^{1/q} < \infty. \tag{1.20}$$

Remark 1.4. $\dot{B}^{q,\lambda}$ is a Banach space which is continuously included in $\text{C}\dot{\text{M}}\text{O}^{q,\lambda}$. One can easily check $\dot{B}^{q,\lambda}(\mathbb{R}^n) = \{0\}$ if $\lambda < -1/q$, $\dot{B}^{q,0}(\mathbb{R}^n) = \dot{B}^q(\mathbb{R}^n)$, $\dot{B}^{q,-1/q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, and $\dot{B}^{q,\lambda}(\mathbb{R}^n) \supsetneq L^q(\mathbb{R}^n)$ if $\lambda > -1/q$. Similar to the classical Morrey space, we only consider the case $-1/q < \lambda \le 0$ in this paper.

Remark 1.5. The space $\text{C}\dot{\text{M}}\text{O}^{q,\lambda}$ when $\lambda = 0$ is just the space $\text{C}\dot{\text{M}}\text{O}^{q}$. It is easy to see that $\text{BMO} \subset \text{C}\dot{\text{M}}\text{O}^{q}$ for all $1 < q < \infty$. When $\lambda \in (0, 1/n)$, then the space $\text{C}\dot{\text{M}}\text{O}^{q,\lambda}$ is just the central version of the Lipschitz space $\text{Lip}_\lambda(\mathbb{R}^n)$.

Remark 1.6. If $1 < q_1 < q_2 < \infty$, then by Hölder's inequality, we know that $\dot{B}^{q_2,\lambda} \subset \dot{B}^{q_1,\lambda}$ for $\lambda \in \mathbb{R}$, and $\text{C}\dot{\text{M}}\text{O}^{q_2,\lambda} \subset \text{C}\dot{\text{M}}\text{O}^{q_1,\lambda}$ for $\lambda < 1/n$.

For more recent generalization about central Morrey and Campanato space, we refer to [21]. We also remark that in recent years, there exists an increasing interest in the study of Morrey-type spaces and the related theory of operators; see, for example, [22].

In this paper, we give sufficient and necessary conditions on the weight ω which ensure that the corresponding weighted Hardy operator H_ω is bounded on $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and $\text{C}\dot{\text{M}}\text{O}^{q,\lambda}(\mathbb{R}^n)$. Meanwhile, we can work out the corresponding operator norms. Moreover, we establish a sufficient and necessary condition of the weight functions so that commutators of weighted Hardy operators (with symbols in central Campanato-type space) are bounded on the central Morrey-type spaces. These results are further used to prove sharp estimates of some inequalities due to Weyl and Cesàro.

2. Sharp Estimates of H_ω

Let us state our main results.

Theorem 2.1. *Let* $1 < q < \infty$ *and* $-1/q < \lambda \le 0$. *Then* H_ω *is a bounded operator on* $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ *if and only if*

$$\mathbb{B} := \int_0^1 t^{n\lambda}\omega(t)dt < \infty. \tag{2.1}$$

Moreover, when (2.1) *holds, the operator norm of* H_ω *on* $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ *is given by*

$$\|H_\omega\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)\to\dot{B}^{q,\lambda}(\mathbb{R}^n)} = \mathbb{B}. \tag{2.2}$$

Proof. Suppose (2.1) holds. For any $R > 0$, using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|(H_\omega f)(x)\right|^q dx\right)^{1/q} \\
&\quad\le \int_0^1\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|f(tx)\right|^q dx\right)^{1/q}\omega(t)dt \\
&\quad= \int_0^1\left(\frac{1}{|B(0,tR)|^{1+\lambda q}}\int_{B(0,tR)}\left|f(x)\right|^q dx\right)^{1/q} t^{n\lambda}\omega(t)dt \\
&\quad\le \|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)}\int_0^1 t^{n\lambda}\omega(t)dt.
\end{aligned} \tag{2.3}$$

It implies that

$$\|H_\omega\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)\to\dot{B}^{q,\lambda}(\mathbb{R}^n)} \le \int_0^1 t^{n\lambda}\omega(t)dt. \tag{2.4}$$

Thus H_ω maps $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ into itself.

The proof of the converse comes from a standard calculation. If H_ω is a bounded operator on $\dot{B}^{q,\lambda}(\mathbb{R}^n)$, take

$$f_0(x) = |x|^{n\lambda}, \quad x \in \mathbb{R}^n. \tag{2.5}$$

Then

$$\|f_0\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} = \Omega_n^{-\lambda}\frac{1}{(nq\lambda+n)^{1/q}}, \tag{2.6}$$

where $\Omega_n = \pi^{n/2}/(\Gamma(1+n/2))$ is the volume of the unit ball in $\mathbb{R}^n$.

We have

$$H_\omega f_0 = f_0 \int_0^1 t^{n\lambda} \omega(t)\,dt, \tag{2.7}$$

$$\|H_\omega\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n) \to \dot{B}^{q,\lambda}(\mathbb{R}^n)} \geq \int_0^1 t^{n\lambda} \omega(t)\,dt. \tag{2.8}$$

(2.8) together with (2.4) yields the desired result. □

Corollary 2.2. (i) *For* $0 < \alpha < 1$, $1 < q < \infty$, *and* $-1/q < \lambda \leq 0$,

$$\|I_\alpha\|_{\dot{B}^{q,\lambda}(dx) \to \dot{B}^{q,\lambda}(x^{-q\alpha}dx)} = \frac{\Gamma(1+\lambda)}{\Gamma(1+\alpha+\lambda)}. \tag{2.9}$$

(ii) *For* $1 < q < \infty$ *and* $-1/q < \lambda \leq 0$,

$$\|H\|_{\dot{B}^{q,\lambda} \to \dot{B}^{q,\lambda}} = \frac{1}{1+\lambda}. \tag{2.10}$$

Next, we state the corresponding conclusion for the space $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$.

Theorem 2.3. *Let* $1 < q < \infty$ *and* $0 \leq \lambda < 1/n$. *Then* H_ω *is a bounded operator on* $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$ *if and only if* (2.1) *holds. Moreover, when* (2.1) *holds, the operator norm of* H_ω *on* $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$ *is given by*

$$\|H_\omega\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n) \to C\dot{M}O^{q,\lambda}(\mathbb{R}^n)} = \mathbb{B}. \tag{2.11}$$

Proof. Suppose (2.1) holds. If $f \in C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$, then for any $R > 0$ and ball $B(0,R)$, using Fubini's theorem, we see that

$$(H_\omega f)_{B(0,R)} = \int_0^1 \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} f(tx)\,dx \right) \omega(t)\,dt = \int_0^1 f_{B(0,tR)} \omega(t)\,dt. \tag{2.12}$$

Using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|(H_\omega f)(x)-(H_\omega f)_{B(0,R)}\right|^q dx\right)^{1/q}\\
&\quad=\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|\int_0^1\left(f(tx)-f_{B(0,tR)}\right)dt\right|^q dx\right)^{1/q}\\
&\quad\le\int_0^1\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|f(tx)-f_{B(0,tR)}\right|^q dx\right)^{1/q}\omega(t)dt\\
&\quad=\int_0^1\left(\frac{1}{|B(0,tR)|^{1+\lambda q}}\int_{B(0,tR)}\left|f(x)-f_{B(0,tR)}\right|^q dx\right)^{1/q}t^{n\lambda}\omega(t)dt\\
&\quad\le\|f\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)}\int_0^1 t^{n\lambda}\omega(t)dt,
\end{aligned}\tag{2.13}$$

which implies H_ω is bounded on $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$ and

$$\|H_\omega\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)\to C\dot{M}O^{q,\lambda}(\mathbb{R}^n)}\le\mathbb{B}.\tag{2.14}$$

Conversely, if H_ω is a bounded operator on $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$, take

$$f_0(x)=\begin{cases}|x|^{n\lambda}, & x\in\mathbb{R}_r^n,\\ -|x|^{n\lambda}, & x\in\mathbb{R}_l^n,\end{cases}\tag{2.15}$$

where $\mathbb{R}_r^n$ and $\mathbb{R}_l^n$ denote the right and the left halves of $\mathbb{R}^n$, separated by the hyperplane $x_1=0$, and x_1 is the first coordinate of $x\in\mathbb{R}^n$.

Thus, by a standard calculation, we see that $(f_0)_{B(0,R)}=0$ and

$$\begin{gathered}
\|f_0\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)}=\Omega_n^{-\lambda}\frac{1}{(nq\lambda+n)^{1/q}},\\
H_\omega f_0=f_0\int_0^1 t^{n\lambda}\omega(t)dt.
\end{gathered}\tag{2.16}$$

From this formula we have

$$\|H_\omega\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)\to C\dot{M}O^{q,\lambda}(\mathbb{R}^n)}\ge\mathbb{B}.\tag{2.17}$$

The proof is complete. □

Corollary 2.4. (i) *For* $1 < q < \infty$ *and* $0 \le \lambda < 1$, *we have*

$$\|H\|_{C\dot{M}O^{q,\lambda} \to C\dot{M}O^{q,\lambda}} = \frac{1}{1+\lambda}. \tag{2.18}$$

(ii) *For* $1 < q < \infty$, *we have* $\|H\|_{C\dot{M}O^{q} \to C\dot{M}O^{q}} = 1$.

3. A Characterization of Weight Functions via Commutators

A well-known result of Coifman et al. [23] states that the commutator generated by Calderón-Zygmund singular integrals and BMO functions is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Recently, we introduced the commutators of weighted Hardy operators and BMO functions introduced in [24]. For any locally integrable function b on $\mathbb{R}^n$ and integrable function $\omega : [0,1] \to [0,\infty)$, the *commutator of the weighted Hardy operator* H_ω^b is defined by

$$H_\omega^b f := bH_\omega f - H_\omega(bf). \tag{3.1}$$

It is easy to see that when $b \in L^\infty(\mathbb{R}^n)$ and ω satisfies the condition (1.6), then the commutator H_ω^b is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. An interesting choice of b is that it belongs to the class of BMO($\mathbb{R}^n$). When symbols $b \in$ BMO($\mathbb{R}^n$), the condition (1.6) on weight functions ω can not ensure the boundedness of H_ω^b on $L^p(\mathbb{R}^n)$. Via controlling H_ω^b by the Hardy-Littlewood maximal operators instead of sharp maximal functions, we [24] established a sufficient and necessary (more stronger) condition on weight functions ω which ensures that H_ω^b is bounded on $L^p(\mathbb{R}^n)$, where $1 < p < \infty$. More recently, Fu and Lu [25] studied the boundedness of H_ω^b on the classical Morrey spaces. Tang et al. [26] and Tang and Zhou [11] obtained the corresponding result on some Herz-type and Triebel-Lizorkin-type spaces. We also refer to the work [27] for more general m-linear Hardy operators.

Similar to [24], we are devoted to the construction of a sufficient and necessary condition (which is stronger than $\mathbb{B} = \infty$ in Theorem 2.1) on the weight functions so that commutators of weighted Hardy operators (with symbols in λ-central BMO space) are bounded on the central Morrey spaces. For the boundedness of commutators with symbols in central BMO spaces, we refer the interested reader to [28, 29] and Mo [30].

Theorem 3.1. *Let* $1 < q_1 < q < \infty, 1/q_1 = 1/q + 1/q_2$, $-1/q < \lambda < 0$. *Assume further that* ω *is a positive integrable function on* $[0,1]$. *Then, the commutator* H_ω^b *is bounded from* $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ *to* $\dot{B}^{q_1,\lambda}(\mathbb{R}^n)$, *for any* $b \in C\dot{M}O^{q_2}(\mathbb{R}^n)$, *if and only if*

$$\mathbb{C} := \int_0^1 t^{n\lambda}\omega(t)\log\frac{2}{t}dt < \infty. \tag{3.2}$$

Remark 3.2. The condition (2.1), that is, $\mathbb{B} < \infty$, is weaker than $\mathbb{C} < \infty$. In fact, let

$$\mathbb{D} := \int_0^1 t^{n\lambda}\omega(t)\log\frac{1}{t}dt < \infty. \tag{3.3}$$

By $\mathbb{C} = \mathbb{B}\log 2 + \mathbb{D}$, we know that $\mathbb{C} < \infty$ implies $\mathbb{B} < \infty$. But the following example shows that $\mathbb{B} < \infty$ does not imply $\mathbb{C} < \infty$. For $0 < \beta < 1$, if we take

$$e^{s(-n\lambda-1)}\tilde{\omega}(s) = \begin{cases} s^{-1+\beta}, & 0 < s \leq 1, \\ s^{-1-\beta}, & 1 < s < \infty, \\ 0, & s = 0, \infty, \end{cases} \tag{3.4}$$

and $\omega(t) = \tilde{\omega}(\log(1/t))$, where $0 \leq t \leq 1$, then $\mathbb{B} < \infty$ and $\mathbb{C} = \infty$.

Proof. (i) Let $R \in (0,\infty)$. Denote $B(0,R)$ by B and $B(0,tR)$ by tB. Assume $\mathbb{C} < \infty$. We get

$$\begin{aligned}\left(\frac{1}{|B|}\int_B \left|H_\omega^b f(x)\right|^{q_1} dx\right)^{1/q_1} &\leq \left(\frac{1}{|B|}\int_B \left(\int_0^1 \left|(b(x)-b(tx))f(tx)\right|\omega(t)dt\right)^{q_1} dx\right)^{1/q_1} \\ &\leq \left(\frac{1}{|B|}\int_B \left(\int_0^1 \left|(b(x)-b_B)f(tx)\right|\omega(t)dt\right)^{q_1} dx\right)^{1/q_1} \\ &\quad + \left(\frac{1}{|B|}\int_B \left(\int_0^1 \left|(b_B-b_{tB})f(tx)\right|\omega(t)dt\right)^{q_1} dx\right)^{1/q_1} \\ &\quad + \left(\frac{1}{|B|}\int_B \left(\int_0^1 \left|(b(tx)-b_{tB})f(tx)\right|\omega(t)dt\right)^{q_1} dx\right)^{1/q_1} \\ &:= I_1 + I_2 + I_3.\end{aligned} \tag{3.5}$$

By the Minkowski inequality and the Hölder inequality (with $1/q_1 = 1/q + 1/q_2$), we have

$$\begin{aligned} I_1 &\leq \int_0^1 \left(\frac{1}{|B|}\int_B \left|(b(x)-b_B)f(tx)\right|^{q_1} dx\right)^{1/q_1} \omega(t)dt \\ &\leq \int_0^1 \left(\frac{1}{|B|}\int_B |b(x)-b_B|^{q_2} dx\right)^{1/q_2} \left(\frac{1}{|B|}\int_B \left|f(tx)\right|^q dx\right)^{1/q} \omega(t)dt \\ &\leq |B|^\lambda \|b\|_{C\dot{M}O^{q_2}} \int_0^1 \left(\frac{1}{|tB|^{1+q\lambda}}\int_{tB} \left|f(x)\right|^q dx\right)^{1/q} t^{n\lambda}\omega(t)dt \\ &\leq |B|^\lambda \|b\|_{C\dot{M}O^{q_2}} \left\|f\right\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda}\omega(t)dt. \end{aligned} \tag{3.6}$$

Similarly, we have

$$
\begin{aligned}
I_3 &\leq \int_0^1 \left(\frac{1}{|B|} \int_B \left| (b(tx) - b_{tB}) f(tx) \right|^{q_1} dx \right)^{1/q_1} \omega(t) dt \\
&\leq \int_0^1 \left(\frac{1}{|tB|} \int_{tB} |b(x) - b_{tB}|^{q_2} dx \right)^{1/q_2} \left(\frac{1}{|tB|} \int_{tB} \left| f(x) \right|^{q} dx \right)^{1/q} \omega(t) dt \\
&\leq |B|^{\lambda} \|b\|_{C\dot{M}O^{q_2}} \int_0^1 \left(\frac{1}{|tB|^{1+q\lambda}} \int_{tB} \left| f(x) \right|^{q} dx \right)^{1/q} t^{n\lambda} \omega(t) dt \\
&\leq C|B|^{\lambda} \|b\|_{C\dot{M}O^{q_2}} \left\| f \right\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda} \omega(t) dt.
\end{aligned}
\tag{3.7}
$$

Now we estimate I_2,

$$
\begin{aligned}
I_2 &\leq \int_0^1 \left(\frac{1}{|B|} \int_B \left| f(tx) \right|^{q_1} dx \right)^{1/q_1} |b_B - b_{tB}| \omega(t) dt \\
&\leq \left\| f \right\|_{\dot{B}^{q,\lambda}} \int_0^1 |tB|^{\lambda} |b_B - b_{tB}| \omega(t) dt \\
&= \left\| f \right\|_{\dot{B}^{q,\lambda}} \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} |tB|^{\lambda} |b_B - b_{tB}| \omega(t) dt \\
&\leq \left\| f \right\|_{\dot{B}^{q,\lambda}} \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} |tB|^{\lambda} \left\{ \left(\sum_{i=0}^{k} |b_{2^{-i}B} - b_{2^{-i-1}B}| \right) + |b_{2^{-k-1}B} - b_{tB}| \right\} \omega(t) dt.
\end{aligned}
\tag{3.8}
$$

We see that

$$
\begin{aligned}
\sum_{i=0}^{k} |b_{2^{-i}B} - b_{2^{-i-1}B}| &\leq C \sum_{i=0}^{k} \left(\frac{1}{|2^{-i}B|} \int_{2^{-i}B} \left| b(y) - b_{2^{-i}B} \right|^{q_2} dy \right)^{1/q_2} \\
&\leq C \|b\|_{C\dot{M}O^{q_2}} (k+1).
\end{aligned}
\tag{3.9}
$$

Therefore,

$$
I_2 \leq C|B|^{\lambda} \|b\|_{C\dot{M}O^{q_2}} \left\| f \right\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda} \omega(t) \log \frac{1}{t} dt. \tag{3.10}
$$

Combining the estimates of I_1, I_2, and I_3, we conclude that H_{ω}^{b} is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1}(\mathbb{R}^n)$.

Conversely, assume that for any $b \in C\dot{M}O^{q_2}$, H_{ω}^{b} is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_2,\lambda}(\mathbb{R}^n)$. We need to show that $\mathbb{C} < \infty$. Since $\mathbb{C} = \mathbb{B} \log 2 + \mathbb{D}$, we will prove that $\mathbb{B} < \infty$ and $\mathbb{D} < \infty$, respectively. To this end, let

$$
b_0(x) = \log|x| \tag{3.11}
$$

for all $x \in \mathbb{R}^n$. Then it follows from Remark 1.5 that $b_0 \in \mathrm{BMO} \subset C\dot{M}O^{q_2}$, and

$$\left\| H_\omega^{b_0} \right\|_{\dot{B}^{q,\lambda} \to \dot{B}^{q_1,\lambda}} < \infty. \tag{3.12}$$

Let $f_0(x) = |x|^{n\lambda}, x \in \mathbb{R}^n$. Then

$$\begin{gathered} \|f_0\|_{\dot{B}^{q,\lambda}} = \Omega_n^{-\lambda} \frac{1}{(nq\lambda + n)^{1/q}}, \\ H_\omega^{b_0} f_0(x) = |x|^{n\lambda} \int_0^1 t^{n\lambda} \omega(t) \log \frac{1}{t} dt. \end{gathered} \tag{3.13}$$

For $\lambda > -1/q > -1/q_1$, we obtain

$$\left\| H_\omega^{b_0} f_0 \right\|_{\dot{B}^{q_1,\lambda}} = \Omega_n^{-\lambda} \frac{1}{(nq_1\lambda + n)^{1/q_1}} \int_0^1 t^{n\lambda} \omega(t) \log \frac{1}{t} dt. \tag{3.14}$$

So,

$$\left\| H_\omega^{b_0} \right\|_{\dot{B}^{q_1,\lambda} \to \dot{B}^{q,\lambda}} \geq C_{n,\lambda,q,q_1} \int_0^1 t^{n\lambda} \omega(t) \log \frac{1}{t} dt. \tag{3.15}$$

Therefore, we have

$$\mathbb{D} < \infty. \tag{3.16}$$

On the other hand,

$$\begin{gathered} \int_0^{1/2} t^{n\lambda} \omega(t) dt \leq C \int_0^{1/2} t^{n\lambda} \omega(t) \log \frac{1}{t} dt < \infty, \\ \int_{1/2}^1 t^{n\lambda} \omega(t) dt < \infty, \end{gathered} \tag{3.17}$$

since $t^{n\lambda}$ and $\omega(t)$ are integrable functions on $[1/2, 1]$. Combining the above estimates, we get

$$\mathbb{B} < \infty. \tag{3.18}$$

Combining (3.18) and (3.16), we then obtain the desired result. □

Notice that comparing with Theorems 2.1 and 2.3, we need a priori assumption in Theorem 3.1 that ω is integrable on $[0, 1]$. However, by Remark 1.1, this assumption is reasonable in some sense.

When $b \in C\dot{M}O^{q_2,\lambda_2}(\mathbb{R}^n)$ with $\lambda_2 > 0$, namely, b is a central λ-Lipschitz function, we have the following conclusion. The proof is similar to that of Theorem 3.1. We give some details here.

Theorem 3.3. *Let $1 < q_1 < q < \infty$, $1/q_1 = 1/q + 1/q_2$, $-1/q < \lambda < 0$, $-1/q_1 < \lambda_1 < 0$, $0 < \lambda_2 < 1/n$, and $\lambda_1 = \lambda + \lambda_2$. If (2.1) holds true, then for all $b \in C\dot{M}O^{q_2,\lambda_2}(\mathbb{R}^n)$, the corresponding commutator H_ω^b is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda_1}(\mathbb{R}^n)$.*

Proof. Let I_1, I_2, and I_3 be as in the proof of Theorem 3.1. Then, following the estimates of I_1 and I_3 in the proof of Theorem 3.1, we see that

$$\begin{aligned} I_1 &\le |B|^{\lambda_1} \|b\|_{C\dot{M}O^{q_2,\lambda_2}} \|f\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda} \omega(t)\,dt, \\ I_3 &\le |B|^{\lambda_1} \|b\|_{C\dot{M}O^{q_2,\lambda_2}} \|f\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda_1} \omega(t)\,dt \\ &\le |B|^{\lambda_1} \|b\|_{C\dot{M}O^{q_2,\lambda_2}} \|f\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda} \omega(t)\,dt. \end{aligned} \tag{3.19}$$

For I_2, we also have

$$I_2 \le \|f\|_{\dot{B}^{q,\lambda}} \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} |tB|^{\lambda} \left\{ \left(\sum_{i=0}^{k} |b_{2^{-i}B} - b_{2^{-i-1}B}| \right) + |b_{2^{-k-1}B} - b_{tB}| \right\} \omega(t)\,dt. \tag{3.20}$$

Since now $0 < \lambda_2 < 1/n$, we see that

$$\begin{aligned} \sum_{i=0}^{k} |b_{2^{-i}B} - b_{2^{-i-1}B}| &\le C \sum_{i=0}^{k} \left(\frac{1}{|2^{-i}B|} \int_{2^{-i}B} |b(y) - b_{2^{-i}B}|^{q_2} dy \right)^{1/q_2} \\ &\le C \|b\|_{C\dot{M}O^{q_2,\lambda_2}} |B|^{\lambda_2} \sum_{i=0}^{k} 2^{-in\lambda_2} \\ &\le C \|b\|_{C\dot{M}O^{q_2,\lambda_2}} |B|^{\lambda_2}. \end{aligned} \tag{3.21}$$

Therefore,

$$I_2 \le C |B|^{\lambda_1} \|b\|_{C\dot{M}O^{q_2,\lambda_2}} \|f\|_{\dot{B}^{q,\lambda}} \int_0^1 t^{n\lambda} \omega(t)\,dt. \tag{3.22}$$

Combining the estimates of I_1, I_2, and I_3, we conclude that H_ω^b is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda_1}(\mathbb{R}^n)$. □

Different from Theorem 3.1, it is still unknown whether the condition (2.1) in Theorem 3.3 is sharp. That is, whether the fact that H_ω^b is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda_1}(\mathbb{R}^n)$ for all $b \in C\dot{M}O^{q_2,\lambda_2}(\mathbb{R}^n)$ induces (2.1)?

More general, we may extend the previous results to the kth order commutator of the weighted Hardy operator. Given $k \geq 1$ and a vector $\vec{b} = (b_1, \ldots, b_k)$, we define the higher order commutator of the weighted Hardy operator as

$$H_\omega^{\vec{b}} f(x) = \int_0^1 \left(\prod_{j=1}^k (b_j(x) - b_j(tx)) \right) f(tx)\omega(t)dt, \quad x \in \mathbb{R}^n. \tag{3.23}$$

When $k = 0$, we understand that $H_\omega^{\vec{b}} = H_\omega$. Notice that if $k = 1$, then $H_\omega^{\vec{b}} = H_\omega^b$.

Using the method in the proof of Theorems 3.1 and 3.3, we can also get the following Theorem 3.4. For the sake of convenience, we give the sketch of the proof of Theorem 3.4(i) here.

Theorem 3.4. *Let $k \geq 2$, $1 < q_1 < q, q_2, \ldots, q_k < \infty$, $1/q_1 = 1/q + \sum_{i=2}^k 1/q_i$, $-1/q < \lambda < 0$, $-1/q_1 < \lambda_1 < 0$, $0 \leq \lambda_2, \ldots, \lambda_k < 1/n$, and $\lambda_1 = \lambda + \sum_{i=2}^k \lambda_i$.*

(i) *Assume further that ω is a positive integrable function on $[0, 1]$. The commutator $H_\omega^{\vec{b}}$ is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda}(\mathbb{R}^n)$, for any $\vec{b} = (b_2, \ldots, b_k) \in C\dot{M}O^{q_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k}(\mathbb{R}^n)$, if and only if*

$$\int_0^1 t^{n\lambda}\omega(t) \left(\log \frac{2}{t} \right)^{k-1} dt < \infty. \tag{3.24}$$

(ii) *Let $\lambda_2, \ldots, \lambda_k > 0$ and $\vec{b} = (b_2, \ldots, b_k) \in C\dot{M}O^{q_2,\lambda_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k,\lambda_k}(\mathbb{R}^n)$. If* (2.1) *holds true, then the corresponding commutator $H_\omega^{\vec{b}}$ is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1,\lambda_1}(\mathbb{R}^n)$.*

Proof. Let $R \in (0, \infty)$. Denote $B(0, R)$ by B and $B(0, tR)$ by tB. Assume $\mathbb{C} < \infty$. We get

$$\begin{aligned}
&\left(\frac{1}{|B|} \int_B \left| H_\omega^{\vec{b}} f(x) \right|^{q_1} dx \right)^{1/q_1} \\
&\quad \leq \left\{ \frac{1}{|B|} \int_B \left[\int_0^1 \left| \left(\prod_{j=2}^k (b_j(x) - b_j(tx)) \right) f(tx) \right| \omega(t)dt \right]^{q_1} dx \right\}^{1/q_1} \\
&\quad \leq C \sum_{I \subset \{2,\ldots,k\}} \sum_{J \subset \{2,\ldots,k\}, J \cap I = \emptyset} \left\{ \frac{1}{|B|} \int_B \left[\int_0^1 \left| \left(\prod_{i \in I} \prod_{j \in J} \prod_{m \in \{2,\ldots,k\} \setminus (I \cup J)} (b_i(x) - b_i(tx)) \right. \right. \right. \right. \\
&\qquad \left. \left. \left. \left. \times \left(b_j(x) - (b_j)_B \right) (b_m(tx) - (b_m)_{tB}) \right) f(tx) \right| \omega(t)dt \right]^{q_1} dx \right\}^{1/q_1}.
\end{aligned} \tag{3.25}$$

Then, applying the Minkowski inequality and the Hölder inequality (with $1/q_1 = 1/q + \sum_{i=2}^{k} 1/q_i$), and repeating the arguments in the proof of Theorem 3.1, $H_\omega^{\vec{b}}$ is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1}(\mathbb{R}^n)$ for any $\vec{b} = (b_2, \ldots, b_k) \in C\dot{M}O^{q_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k}(\mathbb{R}^n)$, provided

$$\int_0^1 t^{n\lambda}\omega(t)\left(\log\frac{2}{t}\right)^{k-1} dt < \infty. \tag{3.26}$$

Conversely, assume that $H_\omega^{\vec{b}}$ is bounded from $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q_1}(\mathbb{R}^n)$ for any $\vec{b} = (b_2, \ldots, b_k) \in C\dot{M}O^{q_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k}(\mathbb{R}^n)$. We choose $\vec{b} = (b_2, \ldots, b_k)$ with $b_j(x) = \log|x|$ for all $x \in \mathbb{R}^n$ and $j \in \{2, \ldots, k\}$. Then $\vec{b} \in C\dot{M}O^{q_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k}(\mathbb{R}^n)$. Repeating the argument in the proof of Theorem 3.1 then yields the desired conclusion. □

We point out that, it is still unknown whether the condition (2.1) in Theorem 3.4(ii) is sharp.

4. Adjoint Operators and Related Results

In this section, we focus on the corresponding results for the adjoint operators of weighted Hardy operators.

Recall that the *weighted Cesàro operator* G_ω is defined by

$$G_\omega f(x) = \int_0^1 f\left(\frac{x}{t}\right) t^{-n}\omega(t)dt, \quad x \in \mathbb{R}^n. \tag{4.1}$$

If $0 < \alpha < 1$, $n = 1$, and $\omega(t) = 1/(\Gamma(\alpha)((1/t) - 1)^{1-\alpha})$, then $G_\omega f(\cdot)$ is reduced to $(\cdot)^{1-\alpha} J_\alpha f(\cdot)$, where J_α is a variant of Weyl integral operator and defined by

$$J_\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}}\frac{dt}{t} \tag{4.2}$$

for all $x \in (0, \infty)$. When $\omega \equiv 1$ and $n = 1$, G_ω is the classical *Cesàro operator*:

$$Gf(x) = \begin{cases} \displaystyle\int_x^\infty \frac{f(y)}{y} dy, & x > 0, \\ \displaystyle -\int_{-\infty}^x \frac{f(y)}{y} dy, & x < 0. \end{cases} \tag{4.3}$$

It was pointed out in [5] that the weighted Hardy operator H_ω and the weighted Cesàro operator G_ω are adjoint mutually, namely,

$$\int_{\mathbb{R}^n} g(x) H_\omega f(x) dx = \int_{\mathbb{R}^n} f(x) G_\omega g(x) dx \tag{4.4}$$

for all admissible pairs f and g.

Since $\dot{A}^q$ and $\dot{B}^{q'}$ are a pair of dual Banach spaces, it follows from Theorem 2.1 the following.

Theorem 4.1. *Let $1 < q < \infty$. Then G_ω is bounded on $\dot{A}^q(\mathbb{R}^n)$ if and only if*

$$\mathbb{E} := \int_0^1 \omega(t)dt < \infty. \tag{4.5}$$

Moreover, when (4.5) holds, the operator norm of G_ω on $\dot{A}^q(\mathbb{R}^n)$ is given by

$$\|G_\omega\|_{\dot{A}^q(\mathbb{R}^n)\to\dot{A}^q(\mathbb{R}^n)} = \mathbb{E}. \tag{4.6}$$

Corollary 4.2. (i) *For $0 < \alpha < 1$ and $1 < q < \infty$,*

$$\|J_\alpha\|_{\dot{A}^q(dx)\to\dot{A}^q(x^{q(1-\alpha)}dx)} = \frac{\Gamma(1)}{\Gamma(1+\alpha)}. \tag{4.7}$$

(ii) *For $1 < q < \infty$, we have*

$$\|G\|_{\dot{A}^q(\mathbb{R}^n)\to\dot{A}^q(\mathbb{R}^n)} = 1. \tag{4.8}$$

Since the dual space of $H\dot{A}^q(1 < q < \infty)$ is isomorphic to $\mathrm{C\dot{M}O}^{q'}$ (see [18, 19]), Theorem 2.3 implies the following result.

Theorem 4.3. *Let $1 < q < \infty$. Then G_ω is a bounded operator on $H\dot{A}^q(\mathbb{R}^n)$ if and only if (4.5) holds. Moreover, when (4.5) holds, the operator norm of G_ω on $H\dot{A}^q(\mathbb{R}^n)$ is given by*

$$\|G_\omega\|_{H\dot{A}^q(\mathbb{R}^n)\to H\dot{A}^q(\mathbb{R}^n)} = \mathbb{E}. \tag{4.9}$$

Corollary 4.4. *For $1 < q < \infty$, we have*

$$\|G\|_{H\dot{A}^q\to H\dot{A}^q} = 1. \tag{4.10}$$

Following the idea in Section 3, we define the higher order commutator of the weighted Cesàro operator as

$$G_\omega^{\vec{b}} f(x) = \int_0^1 \left(\prod_{j=1}^k \left(b_j\left(\frac{x}{t}\right) - b_j(x)\right)\right) f\left(\frac{x}{t}\right) t^{-n}\omega(t)dt, \quad x \in \mathbb{R}^n. \tag{4.11}$$

When $k = 0$, $G_\omega^{\vec{b}}$ is understood as G_ω. Notice that if $k = 1$, then $G_\omega^{\vec{b}} = G_\omega^b$. Similar to the proofs of Theorems 3.1 and 3.3, we have the following result.

Theorem 4.5. *Let $k \geq 2$, $1 < q_1 < q, q_2, \ldots, q_k < \infty$, $1/q_1 = 1/q + \sum_{i=2}^k 1/q_i$, $-1/q < \lambda < 0$, $-1/q_1 < \lambda_1 < 0$, $0 \leq \lambda_2, \ldots, \lambda_k < 1/n$, and $\lambda_1 = \lambda + \sum_{i=2}^k \lambda_i$.*

(i) *Assume further that* ω *is a positive integrable function on* $[0,1]$. *The commutator* $G_{\omega}^{\vec{b}}$ *is bounded from* $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ *to* $\dot{B}^{q_1,\lambda}(\mathbb{R}^n)$, *for any* $\vec{b} = (b_2, \ldots, b_k) \in C\dot{M}O^{q_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k}(\mathbb{R}^n)$, *if and only if*

$$\int_0^1 t^{-n(\lambda+1)}\omega(t)\left(\log\frac{2}{t}\right)^{k-1} dt < \infty. \tag{4.12}$$

(ii) *Let* $\lambda_2, \ldots, \lambda_k > 0$ *and* $\vec{b} = (b_2, \ldots, b_k) \in C\dot{M}O^{q_2,\lambda_2}(\mathbb{R}^n) \times \cdots \times C\dot{M}O^{q_k,\lambda_k}(\mathbb{R}^n)$. *Then the corresponding commutator* $G_{\omega}^{\vec{b}}$ *is bounded from* $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ *to* $\dot{B}^{q_1,\lambda_1}(\mathbb{R}^n)$, *provided that*

$$\int_0^1 t^{-n(\lambda+1)}\omega(t)dt < \infty. \tag{4.13}$$

We conclude this paper with some comments on the discrete version of the weighted Hardy and Cesàro operators.

Let $\mathbb{N}_0$ be the set of all nonnegative integers and $2^{-\mathbb{N}_0}$ denote the set $\{2^{-j} : j \in \mathbb{N}_0\}$. Let now φ be a nonnegative function defined on $2^{-\mathbb{N}_0}$ and f be a complex-valued measurable function on $\mathbb{R}^n$. The *discrete weighted Hardy operator* $\widetilde{H}_{\omega}$ is defined by

$$\left(\widetilde{H}_{\omega}f\right)(x) = \sum_{k=0}^{\infty} 2^{-k} f\left(2^{-k}x\right)\omega\left(2^{-k}\right), \quad x \in \mathbb{R}^n, \tag{4.14}$$

and the corresponding *discrete weighted Cesàro operator* is defined by setting, for all $x \in \mathbb{R}^n$,

$$\left(\widetilde{G}_{\omega}f\right)(x) = \sum_{k=0}^{\infty} f\left(2^{k}x\right)2^{k(n-1)}\omega\left(2^{-k}\right). \tag{4.15}$$

We remark that, by the same argument as above with slight modifications, all the results related to the operators H_{ω} and G_{ω} in Sections 1–4 are also true for their discrete versions $\widetilde{H}_{\omega}$ and $\widetilde{G}_{\omega}$.

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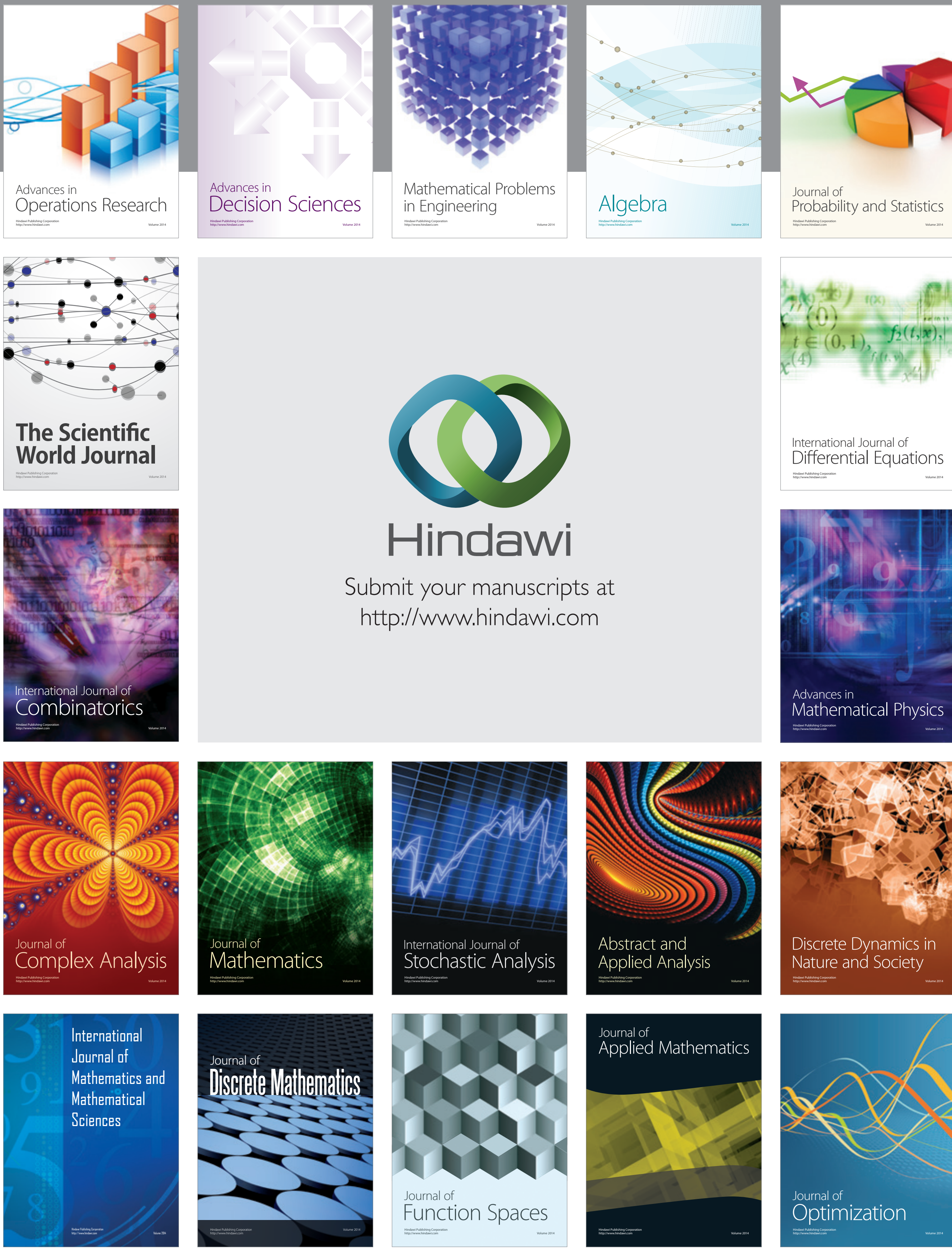

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