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Research Article

Asymptotic Behavior of Bifurcation Curve for Sine-Gordon-Type Differential Equation

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We consider the nonlinear eigenvalue problems for the equation $-u''(t) + \sin u(t) = \lambda u(t)$, $u(t) > 0$, $t \in I =: (0, 1)$, $u(0) = u(1) = 0$, where $\lambda > 0$ is a parameter. It is known that for a given $\xi > 0$, there exists a unique solution pair $(u_\xi, \lambda(\xi)) \in C^2(\bar{I}) \times \mathbb{R}_+$ with $\|u_\xi\|_\infty = \xi$. We establish the precise asymptotic formulas for bifurcation curve $\lambda(\xi)$ as $\xi \to \infty$ and $\xi \to 0$ to see how the oscillation property of $\sin u$ has effect on the behavior of $\lambda(\xi)$. We also establish the precise asymptotic formula for bifurcation curve $\lambda(\alpha)$ $(\alpha = \|u_\lambda\|_2)$ to show the difference between $\lambda(\xi)$ and $\lambda(\alpha)$.

1. Introduction

We consider the following nonlinear eigenvalue problem:

$$-u''(t) + \sin u(t) = \lambda u(t), \quad t \in I =: (0, 1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(0) = u(1) = 0, \tag{1.3}$$

where $\lambda > 0$ is a parameter. This problem comes from sine-Gordon equation and has been investigated from a view point of bifurcation theory in L^∞-framework. Indeed, by using implicit function theorem, it has been shown in [1] that for $\xi > 0$, there exists a continuous function $\lambda = \lambda(\xi)$ such that $(u_\xi, \lambda(\xi)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (1.1)–(1.3) with $\|u_\xi\|_\infty = \xi$. Moreover, the solution set of of (1.1)–(1.3) is given by $\Gamma := \{(u_\xi, \lambda(\xi)) \in C^2(\bar{I}) \times \mathbb{R}_+;\ \xi > 0\}$.

Furthermore, it is well known that $u_\xi(t) \sim \xi \sin \pi t$ for $\xi \gg 1$ and $0 < \xi \ll 1$. Therefore, we have

$$\lambda(\xi) \longrightarrow \pi^2 \quad (\xi \longrightarrow \infty), \tag{1.4}$$

$$\lambda(\xi) \longrightarrow \pi^2 + 1 \quad (\xi \longrightarrow 0). \tag{1.5}$$

Equations (1.1)–(1.3) are the special case of the following semilinear equation:

$$-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I, \tag{1.6}$$

$$u(t) > 0, \quad t \in I, \tag{1.7}$$

$$u(0) = u(1) = 0. \tag{1.8}$$

The structures of the global behavior of the bifurcation curves of (1.6)–(1.8) have been studied by many authors in L^∞-framework. We refer to [2–6] and the references therein. In particular, if $f(u)/u$ is strictly increasing as $u \to \infty$, then we know from [3] that $\lambda(\xi)$ is also strictly increasing for $\xi > 0$ and the asymptotic behavior of $\lambda(\xi)$ as $\xi \to \infty$ is mainly determined by $f(\xi)/\xi$. For example, if $f(u) = u^p$ $(p > 1)$ in (1.6), then as $\xi \to \infty$ (cf. [7]),

$$\lambda(\xi) = \xi^{p-1} + O\left(e^{-\delta\sqrt{\xi}}\right), \tag{1.9}$$

where $\delta > 0$ is a constant. However, since $(\sin u)/u$ is not strictly increasing but oscillating as a function of $u \geq 0$, it is interesting to study whether the oscillation property of $\sin u$ has effect on the asymptotic shape of $\lambda(\xi)$ for $\xi > 0$ or not.

Motivated by this, we first establish the precise asymptotic formula for $\lambda(\xi)$ as $\xi \to \infty$.

Theorem 1.1. *As* $\xi \to \infty$,

$$\begin{aligned}\lambda(\xi) = {} & \pi^2 + 2\sqrt{\frac{2}{\pi}}\xi^{-3/2}\cos\left(\xi - \frac{3}{4}\pi\right) \\ & + 2\sqrt{\frac{2}{\pi}}\xi^{-5/2}\left\{-\frac{3}{8}\sin\left(\xi - \frac{3}{4}\pi\right) - \frac{1}{\sqrt{2}\pi^2}\cos\left(2\xi - \frac{1}{4}\pi\right)\right. \\ & \left. + \frac{1}{\pi^2}\cos\xi\cos\left(\xi - \frac{1}{4}\pi\right)\right\} + o\left(\xi^{-5/2}\right).\end{aligned} \tag{1.10}$$

The local behavior of $\lambda(\xi)$ as $\xi \to 0$ can be obtained formally by the method in [8]. However, it seems rather hard task to obtain the higher terms of the asymptotic expansion of $\lambda(\xi)$, since it is necessary to solve the equations derived from the asymptotic expansion of $\lambda(\xi)$ step by step.

Here, we introduce a simpler way on how to obtain the asymptotic expansion formula for $\lambda(\xi)$ as $\xi \to 0$.

Theorem 1.2. *Let an arbitrary integer $N > 0$ be fixed. Then as $\xi \to 0$,*

$$\lambda = \pi^2 + 1 - \frac{1}{8}\xi^2 + \frac{1}{192}\left(1 + \frac{1}{8\pi^2}\right)\xi^4 + \sum_{n=3}^{N} a_n \xi^{2n} + o\left(\xi^{2N}\right), \tag{1.11}$$

where $\{a_n\}$ $(n = 3, 4, \ldots)$ are the constants determined inductively.

Next, since (1.1)–(1.3) is regarded as an eigenvalue problem, we focus our attention on studying the structure of the solution set in L^2-framework. Suppose that $f(u) = u^p (p > 1)$ in (1.6). Then we know from [9] that, for a given $\alpha > 0$, there exists a unique solution pair $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ of (1.6)–(1.8) satisfying $\|u_\alpha\|_2 = \alpha$. Furthermore, $\lambda(\alpha)$ is an increasing function of $\alpha > 0$ and as $\alpha \to \infty$,

$$\lambda(\alpha) = \alpha^{p-1} + C_0 \alpha^{(p-1)/2} + O(1). \tag{1.12}$$

We see from (1.9) and (1.12) the difference between the asymptotic formulas for $\lambda(\xi)$ and $\lambda(\alpha)$ when $f(u) = u^p$ in (1.6). We refer to [4, 7, 9] for the works in this direction.

Motivated by this, it seems interesting to compare the asymptotic behavior of $\lambda(\alpha)$ and $\lambda(\xi)$ of (1.1)–(1.3) when $\xi \gg 1$ and $\alpha \gg 1$.

Now we consider (1.1)–(1.3) in L^2-framework. Let $\alpha > 0$ be a given constant. Assume that there exists a solution pair $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfying $\|u_\alpha\|_2 = \alpha$. Then, it is natural to expect that for $t \in \bar{I}$, as $\alpha \to \infty$,

$$\frac{u_\alpha(t)}{\alpha} \longrightarrow \sqrt{2} \sin \pi t. \tag{1.13}$$

Therefore, we expect that $\|u_\alpha\|_\infty \sim \sqrt{2}\, \|u_\alpha\|_2$ for $\alpha \gg 1$. To obtain the existence, we apply the variational method to our situation, namely, we consider the constrained minimization problem associated with (1.1)–(1.3). Let

$$M_\alpha := \left\{ v \in H_0^1(I) : \|v\|_2 = \alpha \right\}, \tag{1.14}$$

where $\|v\|_2$ is the usual L^2-norm of v, $\alpha > 0$ is a parameter, and $H_0^1(I)$ is the usual real Sobolev space. Then consider the following minimizing problem, which depends on $\alpha > 0$:

$$\text{Minimize } K(v) := \frac{1}{2}\|v'\|_2^2 + \int_I (1 - \cos v(t))\,dt \quad \text{under the constraint } v \in M_\alpha. \tag{1.15}$$

Let

$$\beta(\alpha) := \min_{v \in M_\alpha} K(v). \tag{1.16}$$

Then by Lagrange multiplier theorem, for a given $\alpha > 0$, there exists a pair $(u_\alpha, \lambda(\alpha)) \in M_\alpha \times \mathbb{R}_+$ which satisfies (1.1)–(1.3) with $K(u_\alpha) = \beta(\alpha)$. Here, $\lambda(\alpha)$, which is called the *variational* *eigenvalue*, is the Lagrange multiplier. By this variational framework, we parameterize the solution (u, λ) of (1.1)–(1.3) by α, that is, $(u, \lambda) = (u_\alpha, \lambda(\alpha)) \in M_\alpha \times \mathbb{R}_+$. Then we know from the arguments in [10, 11] that $\lambda(\alpha)$ is continuous function for $0 < \alpha \ll 1$ and $\alpha \gg 1$. Our next aim is to study precisely the asymptotic behavior of $\lambda(\alpha)$ as $\alpha \to \infty$.

Theorem 1.3. *As* $\alpha \to \infty$

$$\begin{aligned}\lambda(\alpha) = \pi^2 &+ 2^{3/4}\pi^{-1/2}\alpha^{-3/2}\cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right)\\ &- \pi^{-3}\alpha^{-2}\sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right)\cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right)\\ &+ 2^{1/4}\pi^{-1/2}\alpha^{-5/2}\left\{-\frac{3}{8}\sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) - \frac{1}{\sqrt{2}\pi^2}\cos\left(2\sqrt{2}\alpha - \frac{1}{4}\pi\right)\right.\\ &\qquad\left. + \frac{1}{\pi^2}\cos\left(\sqrt{2}\alpha\right)\cos\left(\sqrt{2}\alpha - \frac{1}{4}\pi\right) - \frac{1}{4}\pi^{-5}\cos^3\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right)\right\}\\ &+ o\left(\alpha^{-5/2}\right).\end{aligned} \tag{1.17}$$

By Theorems 1.1 and 1.3, we clearly understand the difference between $\lambda(\xi)$ and $\lambda(\alpha)$.

The remainder of this paper is organized as follows. In Section 2, we prove Theorem 1.1. We prove Theorem 1.2 in Section 3. Section 4 is devoted to the proof of Theorem 1.3.

2. Proof of Theorem 1.1

In what follows, C denotes various positive constants independent of $\xi \gg 1$. We write $\lambda = \lambda(\xi)$ for simplicity. We know from [1] that if $(u_\xi, \lambda(\xi)) \in C^2(\overline{I}) \times \mathbb{R}_+$ satisfies (1.1)–(1.3), then

$$u_\xi(t) = u_\xi(1-t), \quad 0 \le t \le 1, \tag{2.1}$$

$$u_\xi\left(\frac{1}{2}\right) = \max_{0\le t\le 1} u_\xi(t) = \xi, \tag{2.2}$$

$$u'_\xi(t) > 0, \quad 0 \le t < \frac{1}{2}. \tag{2.3}$$

By (1.1), for $t \in \overline{I}$,

$$\left[u''_\xi(t) + \lambda u_\xi(t) - \sin u_\xi(t)\right]u'_\xi(t) = 0. \tag{2.4}$$

This implies that for $t \in \overline{I}$,

$$\frac{d}{dt}\left[\frac{1}{2}u'_\xi(t)^2 + \frac{1}{2}\lambda u_\xi(t)^2 + \cos u_\xi(t)\right] = 0. \tag{2.5}$$

By this, (2.2) and putting $t = 1/2$, we obtain

$$\frac{1}{2}u_\xi'(t)^2 + \frac{1}{2}\lambda u_\xi(t)^2 + \cos u_\xi(t) \equiv \text{ constant } = \frac{1}{2}\lambda\xi^2 + \cos\xi. \tag{2.6}$$

By this and (2.3), for $0 \le t \le 1/2$,

$$u_\xi'(t) = \sqrt{\lambda\left(\xi^2 - u_\xi(t)^2\right) + 2(\cos\xi - \cos u_\xi(t))}. \tag{2.7}$$

Then by putting $s = u_\xi(t)/\xi$, we obtain

$$\begin{aligned}
\frac{1}{2} &= \int_0^{1/2} dt = \int_0^{1/2} \frac{u_\xi'(t)}{\sqrt{\lambda\left(\xi^2 - u_\xi(t)^2\right) + 2(\cos\xi - \cos u_\xi(t))}}dt \\
&= \frac{1}{\sqrt{\lambda}}\int_0^1 \frac{1}{\sqrt{1 - s^2 + 2(\cos\xi - \cos\xi s)/(\lambda\xi^2)}}ds \\
&= \frac{1}{\sqrt{\lambda}}\left\{\int_0^1 \frac{1}{\sqrt{1-s^2}}ds + \left(\int_0^1 \frac{1}{\sqrt{1-s^2+B}}ds - \int_0^1 \frac{1}{\sqrt{1-s^2}}ds\right)\right\} \\
&= \frac{1}{\sqrt{\lambda}}\left(\frac{\pi}{2} + V\right),
\end{aligned} \tag{2.8}$$

where

$$V := -\int_0^1 \frac{B}{\sqrt{1-s^2+B}\sqrt{1-s^2}\left(\sqrt{1-s^2+B} + \sqrt{1-s^2}\right)}ds, \tag{2.9}$$

$$B := \frac{2}{\lambda\xi^2}(\cos\xi - \cos\xi s). \tag{2.10}$$

We put

$$V_1 = -\frac{1}{\lambda\xi^2}\int_0^1 \frac{\cos\xi - \cos\xi s}{(1-s^2)^{3/2}}ds, \tag{2.11}$$

$$V_2 = V - V_1. \tag{2.12}$$

Lemma 2.1. *For* $\xi \gg 1$

$$\begin{aligned}
V_1 = \sqrt{\frac{\pi}{2}}\frac{1}{\lambda\xi^{3/2}}\Bigg[&\left(1 + \frac{15}{128\xi^2}(1+o(1))\right)\cos\left(\xi - \frac{3}{4}\pi\right) \\
&- \frac{3}{8\xi}(1+o(1))\ \sin\left(\xi - \frac{3}{4}\pi\right)\Bigg].
\end{aligned} \tag{2.13}$$

Proof. By putting $s = \sin\theta$ in (2.11), integration by parts and l'Hopital's rule,

$$\begin{aligned}
-V_1 &= \frac{1}{\lambda\xi^2}\int_0^{\pi/2}\frac{1}{\cos^2\theta}(\cos\,\xi-\cos(\xi\sin\theta))d\theta \\
&= \frac{1}{\lambda\xi^2}\int_0^{\pi/2}(\tan\theta)'(\cos\xi-\cos(\xi\sin\theta))d\theta \\
&= \frac{1}{\lambda\xi^2}\lim_{t\to\pi/2}[\tan t(\cos\xi-\cos(\xi\sin t))]_0^t \\
&\quad -\frac{1}{\lambda\xi}\int_0^{\pi/2}\tan\theta\cos\theta\sin(\xi\sin\theta)d\theta \\
&= -\frac{1}{\lambda\xi}\int_0^{\pi/2}\sin\theta\sin(\xi\sin\theta)d\theta.
\end{aligned} \tag{2.14}$$

By [12, page 962],

$$\int_0^{\pi/2}\sin\theta\sin(\xi\sin\theta)d\theta = \frac{\pi}{2}J_1(\xi), \tag{2.15}$$

where $J_1(\xi)$ is Bessel function of the first kind. For $\xi \gg 1$, by [12, page 972], we have

$$\begin{aligned}
J_1(\xi) = \sqrt{\frac{2}{\pi\xi}}\Bigg[&\left(1+\frac{15}{128\xi^2}(1+o(1))\right)\cos\left(\xi-\frac{3}{4}\pi\right) \\
&-\frac{3}{8\xi}(1+o(1))\sin\left(\xi-\frac{3}{4}\pi\right)\Bigg].
\end{aligned} \tag{2.16}$$

By this, (2.14) and (2.15), we obtain (2.13). Thus, the proof is complete. □

Remark 2.2. Taking (1.4) into account, (2.13) is written as

$$\begin{aligned}
V_1 = 2^{-1/2}\pi^{-3/2}\xi^{-3/2}(1+o(1))\Bigg[&\left(1+\frac{15}{128\xi^2}(1+o(1))\right)\cos\left(\xi-\frac{3}{4}\pi\right) \\
&-\frac{3}{8\xi}(1+o(1))\sin\left(\xi-\frac{3}{4}\pi\right)\Bigg].
\end{aligned} \tag{2.17}$$

After we obtain (2.31) later, then (2.13) will be improved in the form (2.32).

Lemma 2.3. *For $\xi \gg 1$,*

$$\begin{aligned}
V_2 = &-2^{-1/2}\pi^{-7/2}(1+o(1))\xi^{-5/2}\left\{\frac{1}{\sqrt{2}}\cos\left(2\xi-\frac{1}{4}\pi\right)-\cos\xi\cos\left(\xi-\frac{1}{4}\pi\right)\right\} \\
&+o\left(\xi^{-5/2}\right).
\end{aligned} \tag{2.18}$$

Proof. For $\xi \gg 1$ and $0 \le s \le 1$, by mean value theorem,

$$|B| \le C\xi^{-1}(1-s) \le C\xi^{-1}\left(1-s^2\right). \tag{2.19}$$

By this and Lebesgue's convergence theorem, we have

$$\begin{aligned}
V_2 &= -\frac{2}{\lambda\xi^2}\int_0^1 \frac{\cos\xi - \cos\xi s}{\sqrt{1-s^2}} \\
&\quad \times \left(\frac{1}{\sqrt{1-s^2+B}\left(\sqrt{1-s^2+B}+\sqrt{1-s^2}\right)} - \frac{1}{\sqrt{1-s^2}\left(2\sqrt{1-s^2}\right)}\right) ds \\
&= -(1+o(1))\frac{2}{\lambda\xi^2}\int_0^1 \frac{\cos\xi - \cos\xi s}{\sqrt{1-s^2}} \\
&\quad \times \frac{2\left(1-s^2\right) - \left(1-s^2+B+\sqrt{1-s^2}\sqrt{1-s^2+B}\right)}{\sqrt{1-s^2+B}\left(\sqrt{1-s^2+B}+\sqrt{1-s^2}\right)\sqrt{1-s^2}2\sqrt{1-s^2}} ds \\
&= -(1+o(1))\frac{2}{\lambda\xi^2}\int_0^1 \frac{\cos\xi - \cos\xi s}{\sqrt{1-s^2}} \cdot \frac{1-s^2-B-\sqrt{1-s^2}\sqrt{1-s^2+B}}{4(1-s^2)^2} ds \\
&= -(1+o(1))\frac{1}{2\lambda\xi^2}\int_0^1 \frac{\cos\xi - \cos\xi s}{\sqrt{1-s^2}} \cdot \frac{\left(1-s^2-B\right)^2 - \left(1-s^2\right)\left(1-s^2+B\right)}{(1-s^2)^2\left[(1-s^2-B)+\sqrt{1-s^2}\sqrt{1-s^2+B}\right]} ds \\
&= \frac{3}{4}(1+o(1))\frac{1}{\lambda\xi^2}\int_0^1 \frac{\cos\xi - \cos\xi s}{\sqrt{1-s^2}} \cdot \frac{\left(1-s^2\right)B}{\left(1-s^2\right)^3} ds \\
&= \frac{3}{2}(1+o(1))\frac{1}{\lambda^2\xi^4}\int_0^1 \frac{(\cos\xi - \cos\xi s)^2}{\left(1-s^2\right)^{5/2}} ds \\
&= \frac{3}{2}(1+o(1))\frac{1}{\lambda^2\xi^4}\int_0^{\pi/2} \frac{(\cos\xi - \cos(\xi\sin\theta))^2}{\cos^4\theta} d\theta \\
&= \frac{3}{2}(1+o(1))\frac{1}{\lambda^2\xi^4} V_3,
\end{aligned} \tag{2.20}$$

where

$$V_3 := \int_0^{\pi/2} \frac{(\cos\xi - \cos(\xi\sin\theta))^2}{\cos^4\theta} d\theta. \tag{2.21}$$

We know

$$\int \frac{1}{\cos^4\theta} d\theta = \frac{1}{3}\sin\theta\left(\frac{1}{\cos^3\theta} + \frac{2}{\cos\theta}\right). \tag{2.22}$$

Taking (2.22) into account and integration by parts in V_3, we obtain that

$$\begin{aligned} V_3 &= \lim_{\theta\to\pi/2}\left[\frac{1}{3}\sin\theta\left(\frac{1}{\cos^3\theta}+\frac{2}{\cos\theta}\right)(\cos\xi-\cos(\xi\sin\theta))^2\right]_0^\theta \\ &\quad -\frac{2}{3}\xi\int_0^{\pi/2}\sin\theta\left(\frac{1}{\cos^2\theta}+2\right)(\cos\xi-\cos(\xi\sin\theta))\sin(\xi\sin\theta)d\theta \\ &:= \frac{1}{3}V_4-\frac{2}{3}\xi(V_5+V_6), \end{aligned} \tag{2.23}$$

where

$$V_4 := \lim_{\theta\to\pi/2}\sin\theta\left(\frac{1}{\cos^3\theta}+\frac{2}{\cos\theta}\right)(\cos\xi-\cos(\xi\sin\theta))^2 \tag{2.24}$$

$$V_5 := \int_0^{\pi/2}\frac{\sin\theta}{\cos^2\theta}(\cos\xi-\cos(\xi\sin\theta))\sin(\xi\sin\theta)d\theta, \tag{2.25}$$

$$V_6 := 2\int_0^{\pi/2}\sin\theta(\cos\xi-\cos(\xi\sin\theta))\sin(\xi\sin\theta)d\theta. \tag{2.26}$$

Then by l'Hopital's rule,

$$\begin{aligned} V_4 &= \lim_{\theta\to\pi/2}\sin\theta\left(\frac{1}{\cos^3\theta}+\frac{2}{\cos\theta}\right)(\cos\xi-\cos(\xi\sin\theta))^2 \\ &= \lim_{\theta\to\pi/2}\frac{(1+2\cos^2\theta)(\cos\theta-\cos(\xi\sin\theta))^2}{\cos^3\theta} \\ &= \lim_{\theta\to\pi/2}\frac{(\cos\xi-\cos(\xi\sin\theta))^2}{\cos^3\theta} \\ &= \lim_{\theta\to\pi/2}-\frac{2\xi(\cos\xi-\cos(\xi\sin\theta))\sin(\xi\sin\theta)}{3\cos\theta\sin\theta} \\ &= \lim_{\theta\to\pi/2}-\frac{2\xi\sin\xi}{3}\frac{(\cos\xi-\cos(\xi\sin\theta))}{\cos\theta\sin\theta} \\ &= \lim_{\theta\to\pi/2}-\frac{2\xi^2\sin\xi}{3}\frac{\sin(\xi\sin\theta)\cos\theta}{\cos(2\theta)} = 0. \end{aligned} \tag{2.27}$$

We next calculate V_5. We know from [12, pages 442 and 972] that for $z \gg 1$,

$$\begin{aligned} \int_0^{\pi/2}\cos(z\cos\theta)d\theta &= \frac{\pi}{2}J_0(z) \\ &= \sqrt{\frac{\pi}{2}}(1+o(1))z^{-1/2}\cos\left(z-\frac{1}{4}\pi\right), \end{aligned} \tag{2.28}$$

where $J_0(z)$ is Bessel function. Integration by parts in (2.25), applying the l'Hopital's rule, putting $\theta = \pi/2 - \eta$ and taking (2.28) into account, we obtain

$$\begin{aligned} V_5 &= \lim_{\theta\to\pi/2}\left[\frac{1}{\cos\theta}(\cos\xi - \cos(\xi\sin\theta))\sin(\xi\sin\theta)\right]_0^{\theta} \\ &\quad - \xi\int_0^{\pi/2}\Big(\sin^2(\xi\sin\theta) + \cos\xi\cos(\xi\sin\theta) - \cos^2(\xi\sin\theta)\Big)d\theta \\ &= \xi\int_0^{\pi/2}\cos(2\xi\sin\theta)d\theta - \xi\cos\xi\int_0^{\pi/2}\cos(\xi\sin\theta)d\theta \\ &= \xi\int_0^{\pi/2}\cos(2\xi\cos\eta)d\eta - \xi\cos\xi\int_0^{\pi/2}\cos(\xi\cos\eta)d\eta \\ &= \sqrt{\frac{\pi}{2}}\xi^{1/2}(1+o(1))\left(\frac{1}{\sqrt{2}}\cos\left(2\xi - \frac{1}{4}\pi\right) - \cos\xi\cos\left(\xi - \frac{1}{4}\pi\right)\right). \end{aligned} \tag{2.29}$$

Clearly,

$$V_6 = O(1). \tag{2.30}$$

By (1.4), (2.20), (2.23), (2.27), (2.29), and (2.30), we obtain (2.18). Thus the proof is complete. □

Proof of Theorem 1.1. By (2.8), Lemmas 2.1 and 2.3,

$$\lambda = \pi^2 + 4\pi V + 4V^2 = \pi^2 + 4\pi V_1 + O\left(\xi^{-5/2}\right) = \pi^2 + O\left(\xi^{-3/2}\right). \tag{2.31}$$

By this and Lemma 2.1,

$$\begin{aligned} V_1 &= \sqrt{\frac{\pi}{2}}\frac{1}{\xi^{3/2}}\left(\pi^2 + O(\xi^{-3/2})\right)^{-1} \\ &\quad\times\left(\cos\left(\xi - \frac{3}{4}\pi\right) - \frac{3}{8}(1+o(1))\xi^{-1}\sin\left(\xi - \frac{3}{4}\pi\right) + O\left(\xi^{-2}\right)\right) \\ &= 2^{-1/2}\pi^{-3/2}\xi^{-3/2}\left(\cos\left(\xi - \frac{3}{4}\pi\right) - \frac{3}{8}\xi^{-1}\sin\left(\xi - \frac{3}{4}\pi\right)\right) + o\left(\xi^{-5/2}\right). \end{aligned} \tag{2.32}$$

By this, (2.31) and Lemmas 2.1 and 2.3,

$$\begin{aligned}
\lambda &= \pi^2 + 4\pi(V_1 + V_2) + O\left(V^2\right) \\
&= \pi^2 + 4\pi\left\{2^{-1/2}\pi^{-3/2}\xi^{-3/2}\left(\cos\left(\xi - \frac{3}{4}\pi\right) - \frac{3}{8}\xi^{-1}\sin\left(\xi - \frac{3}{4}\pi\right)\right)\right. \\
&\qquad \left. -2^{-1/2}\pi^{-7/2}\xi^{-5/2}\left(\frac{1}{\sqrt{2}}\cos\left(2\xi - \frac{1}{4}\pi\right) - \cos\xi\cos\left(\xi - \frac{1}{4}\pi\right)\right)\right\} \\
&\quad + o\left(\xi^{-5/2}\right).
\end{aligned} \tag{2.33}$$

By this, we obtain (1.10). Thus, the proof is complete. □

3. Proof of Theorem 1.2

We write $\lambda = \lambda(\xi)$ for simplicity. We prove (1.11) by showing the calculation to get a_2. The argument to obtain a_n $(n \geq 3)$ is the same as that to obtain a_2. The argument in this section is a variant used in [11, Section 2]. By (2.8) and (2.10), we have

$$\frac{1}{2} = \frac{1}{\sqrt{\lambda}}\int_0^1 \frac{1}{\sqrt{1 - s^2 + B}}ds. \tag{3.1}$$

Since $0 < \xi \ll 1$, by Taylor expansion, for $0 \leq s \leq 1$, we obtain

$$\cos\xi - \cos\xi s = \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\xi^{2k}\left(1 - s^{2k}\right). \tag{3.2}$$

By this and (3.1),

$$\sqrt{\lambda} = 2\int_0^1 \frac{1}{\sqrt{1 - s^2}}\left(1 + \frac{2}{\lambda\xi^2}\frac{1}{1 - s^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\xi^{2k}\left(1 - s^{2k}\right)\right)^{-1/2} ds. \tag{3.3}$$

By using this, direct calculation gives us Theorem 1.2. For completeness, we calculate (1.11) up to the third term.

Step 1. We have

$$1 + \frac{2}{\lambda\xi^2}\frac{1}{1 - s^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\xi^{2k}\left(1 - s^{2k}\right) = 1 - \frac{1}{\lambda} + \frac{1}{12\lambda}\frac{1 - s^4}{1 - s^2}\xi^2 + o\left(\xi^2\right). \tag{3.4}$$

By (3.3), (3.4), and Taylor expansion,

$$\begin{aligned}\sqrt{\lambda} &= 2\int_0^1 \frac{1}{\sqrt{1-s^2}}\left(1-\frac{1}{\lambda}+\frac{1}{12\lambda}\left(1+s^2\right)\xi^2+o\left(\xi^2\right)\right)^{-1/2} ds \\ &= \frac{2\sqrt{\lambda}}{\sqrt{\lambda-1}}\int_0^1 \frac{1}{\sqrt{1-s^2}}\left(1-\frac{1}{24(\lambda-1)}\left(1+s^2\right)\xi^2+o\left(\xi^2\right)\right) ds.\end{aligned} \tag{3.5}$$

By this, (1.5) and direct calculation, we obtain

$$\sqrt{\lambda-1} = \pi - \frac{1}{16\pi}\xi^2 + o\left(\xi^2\right). \tag{3.6}$$

This implies

$$\lambda = \pi^2 + 1 - \frac{1}{8}\xi^2 + o\left(\xi^2\right). \tag{3.7}$$

Step 2. Now we calculate the third term of $\lambda(\xi)$. First, we note that

$$\int_0^1 \frac{1+s^2+s^4}{\sqrt{1-s^2}} ds = \frac{15}{16}\pi, \qquad \int_0^1 \frac{\left(1+s^2\right)^2}{\sqrt{1-s^2}} ds = \frac{19}{16}\pi. \tag{3.8}$$

By this, (1.5), (3.3), (3.7), Taylor expansion, and the same calculation as that to obtain (3.5),

$$\begin{aligned}\sqrt{\lambda-1} &= 2\int_0^1 \frac{1}{\sqrt{1-s^2}}\left\{1-\frac{1}{2}\left(\frac{1}{12(\lambda-1)}\left(1+s^2\right)\xi^2\right.\right. \\ &\qquad \left.\left. -\frac{1}{360(\lambda-1)}\left(1+s^2+s^4\right)\xi^4\right)+\frac{3}{8}\frac{1}{144(\lambda-1)^2}\left(1+s^2\right)^2\xi^4+o\left(\xi^4\right)\right\} ds \\ &= \pi - \frac{1}{16\pi}\xi^2\left(1+\frac{1}{8\pi^2}\xi^2+o\left(\xi^2\right)\right)+\frac{1}{360\pi}\frac{15}{16}\xi^4+\frac{1}{192\pi^3}\frac{19}{16}\xi^4+o\left(\xi^4\right) \\ &= \pi - \frac{1}{16\pi}\xi^2+\frac{1}{384\pi}\left(1-\frac{5}{8\pi^2}\right)\xi^4+o\left(\xi^4\right).\end{aligned} \tag{3.9}$$

By this, we obtain (1.11) up to the third term. Thus, the proof is complete.

4. Proof of Theorem 1.3

In this section, we assume that $\alpha \gg 1$. We write $\lambda = \lambda(\alpha)$ for simplicity. We consider the solution pair $(\lambda(\alpha), u_\alpha) \in \mathbb{R}_+ \times M_\alpha$. We obtain from the same argument as that in [10, Theorem 1.2] that

$$\frac{u_\alpha(t)}{\alpha} \longrightarrow \sqrt{2}\sin \pi t \tag{4.1}$$

uniformly on $[0,1]$ as $\alpha \to \infty$. By this, we have

$$\|u_\alpha\|_\infty = \sqrt{2}\alpha(1 + o(1)). \tag{4.2}$$

Furthermore, by [13, Lemma 2.4], we see that $\beta(\alpha)$ is continuous for $\alpha > 0$. By multiplying u_α by (1.1) and integration by parts, we obtain

$$\lambda(\alpha)\alpha^2 = \|u'_\alpha\|_2^2 + \int_0^1 u_\alpha(t)\sin u_\alpha(t)dt. \tag{4.3}$$

By this and (1.16), for $\alpha \gg 1$,

$$\lambda(\alpha)\alpha^2 = 2\beta(\alpha) + \int_0^1 u_\alpha(t)\sin u_\alpha(t)dt - 2\int_0^1 (1 - \cos u_\alpha(t))dt. \tag{4.4}$$

This along with (4.1) implies that $\lambda(\alpha)$ is continuous for $\alpha \gg 1$.

Lemma 4.1. *For $\alpha \gg 1$,*

$$\|u_\alpha\|_\infty^2 = \left(1 - \frac{2}{\sqrt{\lambda}}\left(\frac{\pi}{4} + U\right)\right)^{-1}\alpha^2, \tag{4.5}$$

where

$$U = -\int_0^1 \frac{\sqrt{1-s^2}B}{\sqrt{1-s^2+B}\left(\sqrt{1-s^2+B} + \sqrt{1-s^2}\right)}ds. \tag{4.6}$$

Proof. By (2.7), (2.10), and putting $\theta = u_\alpha$ and $s = \theta/\|u_\alpha\|_\infty$,

$$\begin{aligned}
\|u_\alpha\|_\infty^2 - \alpha^2 &= 2\int_0^{1/2} \frac{\left(\|u_\alpha\|_\infty^2 - u_\alpha(t)^2\right)u_\alpha'(t)}{\sqrt{\lambda\left(\|u_\alpha\|_\infty^2 - u_\alpha(t)^2\right) + 2(\cos\|u_\alpha\|_\infty - \cos u_\alpha(t))}}dt \\
&= 2\int_0^{\|u_\alpha\|_\infty} \frac{\|u_\alpha\|_\infty^2 - \theta^2}{\sqrt{\lambda\left(\|u_\alpha\|_\infty^2 - \theta^2\right) + 2(\cos\|u_\alpha\|_\infty - \cos\theta)}}d\theta \\
&= 2\frac{\|u_\alpha\|_\infty^2}{\sqrt{\lambda}}\int_0^1 \frac{1-s^2}{\sqrt{1-s^2+B}}ds \\
&= 2\frac{\|u_\alpha\|_\infty^2}{\sqrt{\lambda}}\left[\int_0^1\sqrt{1-s^2}ds + \int_0^1\left(\frac{1-s^2}{\sqrt{1-s^2+B}} - \sqrt{1-s^2}\right)ds\right] \\
&= 2\frac{\|u_\alpha\|_\infty^2}{\sqrt{\lambda}}\left(\frac{\pi}{4} + U\right).
\end{aligned} \tag{4.7}$$

Now, the result follows easily from (4.7). Thus, the proof is complete. □

Lemma 4.2. *For* $\alpha \gg 1$,

$$\|u_\alpha\|_\infty = \sqrt{2}\alpha - 2^{-3/4}\pi^{-5/2}\alpha^{-1/2}\cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) + o\left(\alpha^{-1/2}\right). \tag{4.8}$$

Proof. By (2.10) and (4.6),

$$|U| \le \frac{C}{\lambda\|u_\alpha\|_\infty^2}\left|\int_0^1 \frac{\cos\|u_\alpha\|_\infty - \cos(\|u_\alpha\|_\infty s)}{\sqrt{1-s^2}}ds\right| \le C\left(\|u_\alpha\|_\infty^{-2}\right). \tag{4.9}$$

By this, (2.8), Lemma 2.1, and Taylor expansion,

$$\begin{aligned}
1 - \frac{2}{\sqrt{\lambda}}\left(\frac{\pi}{4} + U\right) &= 1 - 2(\pi + 2V)^{-1}\left(\frac{\pi}{4} + U\right) \\
&= \frac{1}{2} - \frac{2}{\pi}\left(U - \frac{V}{2}(1 + o(1))\right) = \frac{1}{2} + \frac{1}{\pi}V(1 + o(1)).
\end{aligned} \tag{4.10}$$

By this, (4.5), (2.12), (2.13), (2.18), Taylor expansion, and (4.2),

$$\begin{aligned}
\|u_\alpha\|_\infty &= \left(\frac{1}{2}+\frac{1}{\pi}V(1+o(1))\right)^{-1/2}\alpha \\
&= \sqrt{2}\left(1-\frac{1}{\pi}V(1+o(1))\right)\alpha \\
&= \sqrt{2}\alpha - 2^{-3/4}\pi^{-5/2}\alpha^{-1/2}\cos\left(\sqrt{2}\alpha-\frac{3}{4}\pi\right)+o\left(\alpha^{-1/2}\right).
\end{aligned} \tag{4.11}$$

Thus, the proof is complete. □

Proof of Theorem 1.3. By Lemma 4.2, we put

$$\begin{aligned}
\|u_\alpha\|_\infty &= \sqrt{2}\alpha + A\alpha^{-1/2}+o\left(\alpha^{-1/2}\right), \\
A &= -2^{-3/4}\pi^{-5/2}\cos\left(\sqrt{2}\alpha-\frac{3}{4}\pi\right).
\end{aligned} \tag{4.12}$$

Then substitute (4.12) for (1.10) and use Taylor expansion to obtain

$$\begin{aligned}
\lambda &= \pi^2 + 2\sqrt{\frac{2}{\pi}}\left(\sqrt{2}\alpha + A\alpha^{-1/2}+o\left(\alpha^{-1/2}\right)\right)^{-3/2}\cos\left(\sqrt{2}\alpha + A\alpha^{-1/2}+o\left(\alpha^{-1/2}\right)-\frac{3}{4}\pi\right) \\
&\quad + 2\sqrt{\frac{2}{\pi}}\left(\sqrt{2}\alpha\right)^{-5/2}\left\{-\frac{3}{8}\sin\left(\sqrt{2}\alpha-\frac{3}{4}\pi\right)-\frac{1}{\sqrt{2}\pi^2}\cos\left(2\sqrt{2}\alpha-\frac{1}{4}\pi\right)\right. \\
&\qquad\qquad \left. +\frac{1}{\pi^2}\cos\left(\sqrt{2}\alpha\right)\cos\left(\sqrt{2}\alpha-\frac{1}{4}\pi\right)\right\}+o\left(\alpha^{-5/2}\right) \\
&= \pi^2 + 2^{3/4}\pi^{-1/2}\alpha^{-3/2}\left(1+\frac{1}{\sqrt{2}}A\alpha^{-3/2}+o\left(\alpha^{-3/2}\right)\right)^{-3/2} \\
&\quad \times\left\{\cos\left(\sqrt{2}\alpha-\frac{3}{4}\pi\right)\cos\left(A\alpha^{-1/2}(1+o(1))\right)\right. \\
&\qquad \left. -\sin\left(\sqrt{2}\alpha-\frac{3}{4}\pi\right)\sin\left(A\alpha^{-1/2}(1+o(1))\right)\right\} \\
&\quad + 2\sqrt{\frac{2}{\pi}}\left(\sqrt{2}\alpha\right)^{-5/2}\left\{-\frac{3}{8}\sin\left(\sqrt{2}\alpha-\frac{3}{4}\pi\right)-\frac{1}{\sqrt{2}\pi^2}\cos\left(2\sqrt{2}\alpha-\frac{1}{4}\pi\right)\right. \\
&\qquad\qquad \left. +\frac{1}{\pi^2}\cos\left(\sqrt{2}\alpha\right)\cos\left(\sqrt{2}\alpha-\frac{1}{4}\pi\right)\right\}+o\left(\alpha^{-5/2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \pi^2 + 2^{3/4}\pi^{-1/2}\alpha^{-3/2}\left(1 - \frac{3}{2\sqrt{2}}A\alpha^{-3/2} + o\left(\alpha^{-3/2}\right)\right) \\
&\quad \times \left\{\left(1 - \frac{1}{2}A^2\alpha^{-1}(1+o(1))\right)\cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right)\right. \\
&\qquad \left. - A\alpha^{-1/2}(1+o(1))\sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right)\right\} \\
&\quad + 2^{1/4}\pi^{-1/2}\alpha^{-5/2}\left\{-\frac{3}{8}\sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) - \frac{1}{\sqrt{2}\pi^2}\cos\left(2\sqrt{2}\alpha - \frac{1}{4}\pi\right)\right. \\
&\qquad \left. + \frac{1}{\pi^2}\cos\left(\sqrt{2}\alpha\right)\cos\left(\sqrt{2}\alpha - \frac{1}{4}\pi\right)\right\} + o\left(\alpha^{-5/2}\right) \\
&= \pi^2 + 2^{3/4}\pi^{-1/2}\alpha^{-3/2}\cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \\
&\quad - \pi^{-3}\alpha^{-2}\sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right)\cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \\
&\quad + 2^{1/4}\pi^{-1/2}\alpha^{-5/2}\left\{-\frac{3}{8}\sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) - \frac{1}{\sqrt{2}\pi^2}\cos\left(2\sqrt{2}\alpha - \frac{1}{4}\pi\right)\right. \\
&\qquad \left. + \frac{1}{\pi^2}\cos\left(\sqrt{2}\alpha\right)\cos\left(\sqrt{2}\alpha - \frac{1}{4}\pi\right) - \frac{1}{4}\pi^{-5}\cos^3\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right)\right\} \\
&\quad + o\left(\alpha^{-5/2}\right).
\end{aligned}
\tag{4.13}
$$

Thus, we obtain (1.17) and the proof is complete. □

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