Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 571951, 9 pages
doi:10.1155/2012/571951

Research Article

Strict Monotonicity and Unique Continuation for the Third-Order Spectrum of Biharmonic Operator

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Received 28 September 2012; Accepted 5 November 2012

Academic Editor: Julio Rossi

We will study the spectrum for the biharmonic operator involving the laplacian and the gradient of the laplacian with weight, which we call third-order spectrum. We will show that the strict monotonicity of the eigenvalues of the operator $\Delta^2 u + 2\beta \cdot \nabla(\Delta u) + |\beta|^2 \Delta u$, where $\beta \in \mathbb{R}^N$, holds if some unique continuation property is satisfied by the corresponding eigenfunctions.

1. Introduction

We are concerned here with the eigenvalue problem:

$$\begin{aligned} &\text{Find } (\beta, \alpha, u) \in \mathbb{R}^N \times \mathbb{R} \times H \\ &\text{such that } \Delta^2 u + 2\beta \cdot \nabla(\Delta u) + |\beta|^2 \Delta u = \alpha m u \quad \text{in } \Omega, \\ &u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, $H = H^2(\Omega) \cap H_0^1(\Omega)$, Δ^2 denotes the biharmonic operator defined by $\Delta^2 u = \Delta(\Delta u)$, and $m \in M = \{m \in L^\infty(\Omega) / \operatorname{meas}\{x \in \Omega / m(x) > 0\} \neq 0\}$.

Based on the works of Anane et al. [1, 2], we will determine the spectrum of (1.1), which we call third-order spectrum for the biharmonic operator. This spectrum is defined to be the set of couples $(\beta, \alpha) \in \mathbb{R}^N \times \mathbb{R}$ such that the problem

$$\Delta^2 u + 2\beta \cdot \nabla(\Delta u) + |\beta|^2 \Delta u = \alpha m u \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

has a nontrivial solution $u \in H$. This spectrum, which is denoted by $\sigma_3(\Delta^2, m)$, is an infinite sequence of eigensurfaces $\Gamma_1^{\pm}, \Gamma_2^{\pm}, \ldots$, see Section 3. When $\beta = 0$, the zero-order spectrum is defined to be the set of eigenvalues $\alpha \in \mathbb{R}$ such that the problem

$$\Delta^2 u = \alpha m u \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

has a nontrivial solution $u \in H$. In this case the spectrum is denoted by $\sigma_0(\Delta^2, m)$. The eigenvalue problem (1.3), which is studied by Courant and Hilbert [3], admits an infinite sequence of real eigenvalues $(\alpha_n(m))_n$ satisfying

$$\frac{1}{\alpha_n(m)} = \sup_{F_n \in \mathcal{F}_n(H)} \min_{u \in F_n} \left(\int_\Omega m|u|^2 dx \right) \quad \forall n \geq 1, \tag{1.4}$$

where $\mathcal{F}_n(H)$ denotes the class of n-dimensional subspaces F_n of H.

Definition 1.1. We say that solutions of problem (1.1) satisfy the unique continuation property (U.C.P), if the unique solution $u \in L^2_{\text{Loc}}(\Omega)$ which vanishes on a set of positive measure in Ω is $u \equiv 0$.

In the literature there exist several works on unique continuation. We refer to the works of Jerison and Kenig [4] and Garofalo and Lin [5], among others. The unique continuation property as defined above differs from the usual notions of unique continuation, see [6] for more details.

Definition 1.2. We say that $\Gamma_k(\beta, \cdot)$ is strict monotone with respect to the weight if $\Gamma_k(\beta, m) > \Gamma_k(\beta, \widehat{m})$, for all $m \underset{\neq}{<} \widehat{m}$.

Here we use the notation $\underset{\neq}{<}$ to mean inequality almost everywhere together with strict inequality on a set of positive measure.

Since the pioneer works of Carleman [7] in 1939 on the unique continuation, this notion has been the interest of many researchers in partial differential equations, see for instance [4, 5, 8]. In 1992, de Figueiredo and Gossez [6] proved that strict monotonicity holds if and only if some unique continuation property is satisfied by the corresponding eigenfunction of a uniformly elliptic operator of the second order. In 1993, Gossez and Loulit [8] have proved the unique continuation property in the linear case of the laplacian operator. The unique continuation property of the biharmonic operator was proved recently by Cuccu and Porru [9]. Our purpose in the fourth section is to show that strict monotonicity of eigensurfaces for problem (1.1) holds if some unique continuation property is satisfied by the corresponding eigenfunctions.

2. Preliminaries

Let H be a finite dimensional separable Hilbert space. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm of the space H, respectively. Let $T : H \to H$ be a compact operator.

Lemma 2.1. *All nonzero eigenvalues of the operator T are obtained by the following characterizations:*

$$\mu_n = \sup_{F_n \in \mathcal{F}_n(H)} \min\{(T(u),u) \text{ such that } \|\mathrm{u}\| = 1;\ \mathrm{u} \in \mathrm{F_n}\},$$
$$\mu_{-n} = \inf_{F_n \in \mathcal{F}_n(H)} \mathrm{Max}\{(T(u),u) \text{ such that } \|\mathrm{u}\| = 1;\ \mathrm{u} \in \mathrm{F_n}\}, \tag{2.1}$$

where $\mathcal{F}_n(H)$ denotes the class of n-dimensional subspaces F_n of H.

Moreover, zero is the only accumulation point of the set of all eigenvalues of T. Here, the eigenvalues are repeated with its order of multiplicity, and the eigenfunctions are mutually orthogonal [10].

3. Third-Order Spectrum of the Biharmonic Operator

We define the third-order eigenvalue problem of the biharmonic operator as follows:

$$\begin{gathered}\text{Find } (\beta,\alpha,u) \in \mathbb{R}^N \times \mathbb{R} \times H \setminus \{0\} \\ \text{such that } \Delta^2 u + 2\beta\cdot\nabla(\Delta u) + |\beta|^2 \Delta u = \alpha m u \text{ in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{3.1}$$

If (β,α,u) is a solution of (3.1) then (β,α) is called third-order eigenvalue and u is said to be the associated eigenfunction.

Lemma 3.1. *Problem* (3.1) *is equivalent to the following problem:*

$$\begin{gathered}\text{Find } (\alpha,u) \in \mathbb{R} \times H \setminus \{0\} \\ \text{such that } \Delta^{2,\beta} u = \alpha m e^{\beta\cdot x} u \quad \textit{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{3.2}$$

where $\Delta^{2,\beta} u = \Delta(e^{\beta\cdot x}\Delta u)$.

Proof. For any $\beta \in \mathbb{R}^N$, we have

$$\begin{aligned}\Delta\left(e^{\beta\cdot x}\Delta u\right) &= \nabla\left(\nabla\left(e^{\beta\cdot x}\Delta u\right)\right)\\ &= \nabla\left(\beta e^{\beta\cdot x}\Delta u + e^{\beta\cdot x}\nabla(\Delta u)\right)\\ &= e^{\beta\cdot x}\left[\Delta^2 u + 2(\beta\cdot\nabla(\Delta u)) + |\beta|^2\Delta u\right].\end{aligned} \tag{3.3}$$

Hence, problem (3.1) is equivalent to problem (3.2) □

Remark 3.2. Let $u \in H$; we denote by $\partial u/\partial \nu$ the normal derivative defined by $\partial u/\partial\nu = (\nabla u|_{\partial\Omega})\cdot\vec{\nu}$ where $\nabla u|_{\partial\Omega} \in (L^2(\partial\Omega))^N$ and $\partial u/\partial\nu \in L^2(\partial\Omega)$.

Definition 3.3. A weak solution of (3.2) is a function u in $H \setminus \{0\}$ witch satisfies, for $(\beta,\alpha) \in \mathbb{R}^N \times \mathbb{R}$ and for all $\varphi \in H$,

$$\int_\Omega e^{\beta\cdot x}\Delta u\Delta\varphi\, dx = \alpha\int_\Omega e^{\beta\cdot x} m u\,\varphi\, dx. \tag{3.4}$$

Definition 3.4. For $(\beta,\alpha) \in \mathbb{R}^N\times\mathbb{R}$, we say that $u \in H$ is a classical solution of problem (3.1) if $u \in C^4(\overline{\Omega})$.

Proposition 3.5. *If $u \in H$ is a weak solution of* (3.2) *and $u \in C^4(\overline{\Omega})$, then u is a classical solution of* (3.2).

Proof. Let $u \in C^4(\overline{\Omega})$ be a weak solution of (3.2), then we have

$$\int_\Omega e^{\beta\cdot x}\Delta u\Delta\varphi\, dx = \alpha\int_\Omega e^{\beta\cdot x} m u\varphi\, dx \quad \forall\varphi \in H. \tag{3.5}$$

Using the Green formula, we obtain

$$\begin{aligned}\int_\Omega \Delta\left(e^{\beta\cdot x}\Delta u\right)\varphi\, dx &= -\int_\Omega \nabla\left(e^{\beta\cdot x}\Delta u\right)\cdot\nabla\varphi\, dx + \int_{\partial\Omega}\varphi\frac{\partial}{\partial\nu}\left(e^{\beta\cdot x}\Delta u\right)dx,\\ \int_\Omega e^{\beta\cdot x}\Delta u\Delta\varphi\, dx &= -\int_\Omega \nabla\left(e^{\beta\cdot x}\Delta u\right)\cdot\nabla\varphi\, dx + \int_{\partial\Omega} e^{\beta\cdot x}\Delta u\frac{\partial\varphi}{\partial\nu}dx.\end{aligned} \tag{3.6}$$

Then we have

$$\int_\Omega \Delta\left(e^{\beta\cdot x}\Delta u\right)\varphi\, dx = \int_\Omega e^{\beta\cdot x}\Delta u\Delta\varphi\, dx + \int_{\partial\Omega}\varphi\frac{\partial}{\partial\nu}\left(e^{\beta\cdot x}\Delta u\right)dx - \int_{\partial\Omega} e^{\beta\cdot x}\Delta u\frac{\partial\varphi}{\partial\nu}dx. \tag{3.7}$$

Thus, the prove is complete. □

Theorem 3.6. *Let $\mathcal{S}_\beta = \{u \in H/\|u\|^2_{2,2,\beta} = \int_\Omega e^{\beta\cdot x}|\Delta u|^2 dx = 1\}$, then we have*

(a) $\sigma_3(\Delta^2, m) = \bigcup_{n=1}^{+\infty} G(\Gamma_n(m,\cdot))$, *where the function* $\Gamma_n(m,\cdot)\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ *is defined by*

$$\frac{1}{\Gamma_n(m,\beta)} = \sup_{F_n \in \mathcal{F}_n(H)} \min_{u \in F_n} \left\{ \int_\Omega e^{\beta\cdot x} m|u|^2 dx,\ u \in \mathcal{S}_\beta \cap F_n \right\} \quad \forall \beta \in \mathbb{R}^N, \tag{3.8}$$

where $\mathcal{F}_n(H)$ *denotes the class of* n*-dimensional subspaces* F_n *of* H *and* $G(\Gamma_n(m,\cdot)) \subset \mathbb{R}^N \times \mathbb{R}$ *is the graph of* $\Gamma_n(m,\cdot)$.

(b) $\int_\Omega e^{\beta\cdot x} m|u|^2 dx \le (\Gamma_1(m,\beta))^{-1} \|u\|_{2,2,\beta}^2$.

(c) *For all* $\beta \in \mathbb{R}^N$: $\lim_{n\to+\infty} \Gamma_n(m,\beta) = +\infty$.

Proof. Let $(\beta,\alpha,u) \in H\setminus\{0\}$, then (β,α,u) is a solution of (3.1) if and only if (α,u) is a solution of problem (3.2). We prove that the map

$$\begin{gathered} l : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow \mathbb{R}, \\ (u,v) \longmapsto l(u,v) = \left\langle \Delta^{2,\beta} u, v \right\rangle \end{gathered} \tag{3.9}$$

defines a scalar product on $H = H^2(\Omega) \cap H_0^1(\Omega)$ equivalent to the usual scalar product $\int_\Omega \Delta u \Delta v\, dx$.

The map $l(\cdot,\cdot)$ is a continuous symmetric bilinear form. Since Δ^2 satisfies the condition of the uniform ellipticity, then we have

$$\begin{aligned} l(u,u) &= \left\langle \Delta^{2,\beta} u, u \right\rangle \\ &= \int_\Omega e^{\beta\cdot x} |\Delta u|^2 dx \ge c' \|u\|_{2,2}^2, \end{aligned} \tag{3.10}$$

where $c\prime = \min_{x\in\overline{\Omega}} e^{\beta\cdot x}$. Therefore, the bilinear form $l(\cdot,\cdot)$ is coercive. On the other hand, the operator

$$\begin{gathered} T^{2,\beta} : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow H^2(\Omega) \cap H_0^1(\Omega), \\ u \longmapsto T^{2,\beta}(u) = \left(\Delta^{2,\beta}\right)^{-1} \left(m e^{\beta\cdot x} u\right) \end{gathered} \tag{3.11}$$

is well defined, linear, symmetric, and compact on H. Then, problem (3.2) can be written as

$$T^{2,\beta}(u) = \frac{1}{\alpha} u, \quad u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3.12}$$

Note that $\alpha = 0$ is not an eigenvalue of (3.2). It follows that (α, β) is an eigenvalue of (1.1) if and only if $1/\alpha$ is eigenvalue of the operator $T^{2,\beta}$. By Lemma 2.1, the eigenvalues are given by the characterizations

$$\begin{aligned} \frac{1}{\alpha_n} &= \sup_{F_n \in \mathcal{F}_n(H)} \min\left\{ l\left(T^{2,\beta}(u), u\right) \text{ such that } \|u\| = 1;\ u \in F_n \right\}, \\ \frac{1}{\alpha_{-n}} &= \inf_{F_n \in \mathcal{F}_n(H)} \max\left\{ l\left(T^{2,\beta}(u), u\right) \text{ such that } \|u\| = 1;\ u \in F_n \right\}. \end{aligned} \tag{3.13}$$

In addition, we have

$$\begin{aligned} l\left(T^{2,\beta}u, u\right) &= \left\langle \Delta^{2,\beta}\left(\left(\Delta^{2,\beta}\right)^{-1} m e^{\beta \cdot x} u\right), u \right\rangle \\ &= \int_\Omega m e^{\beta \cdot x} |u|^2 dx, \end{aligned} \tag{3.14}$$

and $\|u\|^2 = (u, u) = \langle \Delta^{2,\beta} u, u \rangle = \|u\|^2_{2,2,\beta}$, then relation (3.8) is satisfied. Since $me^{\beta . x} \in M$, then we have $\Gamma_n(m, \beta) > 0$ for all $n \in \mathbb{N}^*$. As zero is the only accumulation point of the sequence $(1/\alpha_n)_n$, it follows that $\Gamma_n(m, \beta) \to +\infty$ when $n \to +\infty$. Therefore, the proof is completed. □

4. Strict Monotonicity and Unique Continuation

In this section, we will show that strict monotonicity of eigensurfaces for problem (3.1) holds if some unique continuation property is satisfied by the corresponding eigenfunctions.

Theorem 4.1. *Let m and $\widehat{m}$ be two weights with $m \underset{\neq}{<} \widehat{m}$ and $k \in \mathbb{N}$. If the eigenfunctions φ_k associated to $\Gamma_k(\beta, m)$ satisfy the (U.C.P) then $\Gamma_k(\beta, m) > \Gamma_k(\beta, \widehat{m})$.*

Theorem 4.2. *Let m be a weight and $k \in \mathbb{N}$. If the eigenfunctions φ_k associated to $\Gamma_k(\beta, m)$ do not satisfy the (U.C.P) then there exists a weight $\widehat{m}$ with $m \underset{\neq}{<} \widehat{m}$, such that, for some $i \in \mathbb{N}$ with $\Gamma_i(\beta, m) = \Gamma_k(\beta, m)$, one has $\Gamma_i(\beta, m) = \Gamma_i(\beta, \widehat{m})$.*

As a consequence of Theorems 4.1 and 4.2 we have the following result.

Corollary 4.3. *Let $m \in L^\infty(\Omega)$ and $k \in \mathbb{N}$. If $\Gamma_k(\beta, 1) \underset{\neq}{<} m \underset{\neq}{<} \Gamma_{k+1}(\beta, 1)$, then the only solution of the problem*

$$(P_m) \begin{cases} \Delta^2 u + 2\beta \cdot \nabla(\Delta u) + |\beta|^2 \Delta u = mu & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \Omega \end{cases} \tag{4.1}$$

is $u \equiv 0$.

Proof of Theorem 4.1. Let $k \in \mathbb{N}$; we define the space

$$F_k = \langle \varphi_1, \varphi_2, \dots, \varphi_k \rangle, \tag{4.2}$$

spanned by the eigenfunctions φ_i associated to $\Gamma_i(\beta, m)$ with

$$\int_\Omega e^{\beta \cdot x} |\varphi_i|^2 dx = 1, \quad \text{for } i = 1, \dots, k. \tag{4.3}$$

We have

$$\frac{1}{\Gamma_k(\beta, m)} = \min_{u \in F_k} \left\{ \int_\Omega m e^{\beta \cdot x} |u|^2 dx \text{ such that } \int_\Omega e^{\beta \cdot x} |\Delta u|^2 dx = 1 \right\} = \int_\Omega m e^{\beta \cdot x} \varphi_k^2 dx. \tag{4.4}$$

Let $u \in F_k$, with $\int_\Omega e^{\beta \cdot x} |\Delta u|^2 dx = 1$. We have either u achieves the infimum in (4.4) or not. In the case u is an eigenfunction associated to $\Gamma_k(\beta, m)$, then by the (U.C.P) and since $u^2(x) > 0$ a.e. $x \in \Omega$, we have

$$\frac{1}{\Gamma_k(\beta, m)} = \int_\Omega m e^{\beta \cdot x} u^2 dx < \int_\Omega \widehat{m} e^{\beta \cdot x} u^2 dx = \frac{1}{\Gamma_k(\beta, \widehat{m})}. \tag{4.5}$$

Thus, $\Gamma_k(\beta, m) > \Gamma_k(\beta, \widehat{m})$. In the other case, we have

$$\frac{1}{\Gamma_k(\beta, m)} < \int_\Omega m e^{\beta \cdot x} |u|^2 dx < \int_\Omega \widehat{m} e^{\beta \cdot x} |u|^2 dx = \frac{1}{\Gamma_k(\beta, \widehat{m})}. \tag{4.6}$$

Thus, in both cases we have

$$\frac{1}{\Gamma_k(\beta, m)} < \int_\Omega \widehat{m} e^{\beta \cdot x} u^2 dx. \tag{4.7}$$

It follows that

$$\frac{1}{\Gamma_k(\beta, m)} < \inf_{u \in F_k} \left\{ \int_\Omega \widehat{m} e^{\beta \cdot x} u^2 dx; \int_\Omega e^{\beta \cdot x} |\Delta u|^2 dx = 1 \right\}. \tag{4.8}$$

This yields the desired inequality $(1/\Gamma_k(\beta, m)) < (1/\Gamma_k(\beta, \widehat{m}))$. Hence, we have $\Gamma_k(\beta, m) > \Gamma_k(\beta, \widehat{m})$. □

Proof of Theorem 4.2. Denote by u an eigenfunction associated to $\Gamma_k(\beta, m)$ which vanishes on a set of positive measure. Take i such that $\Gamma_k(\beta, m) = \Gamma_i(\beta, m) < \Gamma_{i+1}(\beta, m)$ and define

$$\widehat{m}(x) = \begin{cases} m(x) & \text{if } u(x) \neq 0, \\ m(x) + \epsilon & \text{if } u(x) = 0, \end{cases} \tag{4.9}$$

where $\epsilon > 0$ is chosen such that $\Gamma_i(\beta, m) < \Gamma_{i+1}(\beta, \widehat{m})$, which is possible by the continuous dependence of the eigenvalues with respect to the weight. We have

$$\Delta^2 u + 2\beta \cdot \nabla(\Delta u) + |\beta|^2 \Delta u = \Gamma_i(\beta, m) m u = \Gamma_i(\beta, m) \widehat{m} u, \tag{4.10}$$

which shows that $\Gamma_i(\beta, m)$ is an eigenvalue for the weight $\widehat{m}$, that is, $\Gamma_i(\beta, m) = \Gamma_l(\beta, \widehat{m})$ for some $l \in \mathbb{N}$. Let us choose the largest l such that this equality holds. It follows from $\Gamma_i(\beta, m) < \Gamma_{i+1}(\beta, \widehat{m})$ that $l < i+1$. Moreover, the monotone dependence, $\Gamma_i(\beta, \widehat{m}) \leq \Gamma_i(\beta, m)$, implies $l \geq i$. Then we conclude that $l = i$. Hence, we have $\Gamma_i(\beta, \widehat{m}) = \Gamma_i(\beta, m)$. □

Proof of Corollary 4.3. Suppose that (P_m) has nontrivial solution, that is, $1 \in \sigma_3(\Delta^2, m)$. From the inequality $\Gamma_k(\beta, 1) \underset{\neq}{\leq} m \underset{\neq}{\leq} \Gamma_{k+1}(\beta, 1)$ and the strict monotonicity, we deduce

$$\begin{aligned} \Gamma_k(\beta, \Gamma_k(\beta, 1)) &> \Gamma_k(\beta, m), \\ \Gamma_{k+1}(\beta, m) &> \Gamma_{k+1}(\beta, \Gamma_{k+1}(\beta, 1)). \end{aligned} \tag{4.11}$$

Since

$$\Gamma_{k+1}(\beta, \Gamma_{k+1}(\beta, 1)) = \Gamma_k(\beta, \Gamma_k(\beta, 1)) = 1 \tag{4.12}$$

we deduce that

$$\Gamma_k(\beta, m) < 1 < \Gamma_{k+1}(\beta, m) \tag{4.13}$$

which is a contradiction. Hence, the proof is complete. □

Acknowledgment

The authors would like to thank the anonymous referees for their constructive suggestions.

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