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Research Article

FDM for Elliptic Equations with Bitsadze-Samarskii-Dirichlet Conditions

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A numerical method is proposed for solving nonlocal boundary value problem for the multidimensional elliptic partial differential equation with the Bitsadze-Samarskii-Dirichlet condition. The first and second-orders of accuracy stable difference schemes for the approximate solution of this nonlocal boundary value problem are presented. The stability estimates, coercivity, and almost coercivity inequalities for solution of these schemes are established. The theoretical statements for the solutions of these nonlocal elliptic problems are supported by results of numerical examples.

1. Introduction

Many problems in fluid mechanics, dynamics, elasticity, and other areas of engineering, physics, and biological systems lead to partial differential equations of elliptic type. The role played by coercive inequalities in the study of local boundary-value problems for elliptic and parabolic differential equations is well known (see, e.g., [1, 2]).

In the present paper, we consider the Bitsadze-Samarskii type nonlocal boundary value problem

$$\begin{gathered} -\frac{d^2u(t)}{dt^2} + Au(t) = f(t), \quad (0 < t < 1), \\ u'(0) = \varphi, \\ u'(1) = \beta u'(\lambda) + \psi, \quad \left|\beta\right| \le 1, \ 0 \le \lambda < 1 \end{gathered} \tag{1.1}$$

for elliptic differential equations in a Hilbert space H with self-adjoint positive definite operator A. It is known (see, e.g., [3–11]) that various nonlocal boundary value problems for elliptic equations can be reduced to the boundary value problem (1.1). The simply nonlocal boundary value problem was presented and investigated for the first time by Bitsadze and Samarskii [12]. Further, methods of solutions of Bitsadze-Samarskii nonlocal boundary value problems for elliptic differential equations have been studied extensively by many researchers (see [13–21] and the references given therein).

A function $u(t)$ is called a solution of problem (1.1) if the following conditions are satisfied.

(i) $u(t)$ is twice continuously differentiable on the segment $[0,1]$. Derivatives at the endpoints of the segment are understood as the appropriate unilaterial derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0,1]$, and the function $Au(t)$ is continuous on $[0,1]$.

(iii) $u(t)$ satisfies the equation and nonlocal boundary condition in (1.1).

Let Ω be the open unit cube in $R^n(x = (x_1,\dots,x_n) : 0 < x_k < 1,\ 1 \le k \le n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In present paper, we are interested in studying the stable difference schemes for the numerical solution of the following nonlocal boundary value problem for the multidimensional elliptic equation

$$\begin{gathered} -u_{tt} - \sum_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + \delta u = f(t,x), \quad 0 < t < 1, \\ x = (x_1,\dots,x_n) \in \Omega, \\ u_t(0,x) = \varphi(x), \\ u_t(1,x) = \beta u_t(\lambda,x) + \psi(x), \quad x \in \overline{\Omega},\ |\beta| \le 1,\ 0 \le \lambda < 1, \\ u(t,x) = 0, \quad 0 \le t \le 1,\ x \in S,\ S = \partial\overline{\Omega}. \end{gathered} \tag{1.2}$$

Here $\psi(x), \varphi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ are given smooth functions, δ is a large positive constant and $a_r(x) \ge a > 0$.

In the present paper, the first and second-orders of accuracy difference schemes are presented for the approximate solution of problem (1.2). The stability and coercive stability estimates for the solution of these difference schemes are obtained. A numerical method is proposed for solving nonlocal boundary value problem for the multidimensional elliptic partial differential equation with the Bitsadze-Samarskii-Dirichlet condition. A procedure of modified Gauss elimination method is used for solving these difference schemes in the case of two-dimensional elliptic partial differential equations.

2. Difference Schemes: Well-Posedness

The discretization of problem (1.2) is carried out in two steps. In the first step let us define the grid sets as follows:

$$\begin{gathered}\widetilde{\Omega}_h = \{x : x = x_m = (h_1 m_1, \dots, h_n m_n),\ m = (m_1, \dots, m_n), \\ 0 \le m_r \le N_r,\ h_r N_r = L,\ r = 1, \dots, n\} \\ \Omega_h = \widetilde{\Omega}_h \cap \Omega, \qquad S_h = \widetilde{\Omega}_h \cap S.\end{gathered} \tag{2.1}$$

We introduce the Hilbert space $L_{2h} = L_2(\widetilde{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \dots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_2(\widetilde{\Omega}_h)} = \left(\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n\right)^{1/2} \tag{2.2}$$

and the Hilbert space $W_2^2(\widetilde{\Omega}_h)$ defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\begin{aligned}\left\|\varphi^h\right\|_{W_2^2(\widetilde{\Omega}_h)} = {}& \left(\sum_{x \in \widetilde{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n\right)^{1/2} \\ & + \left(\sum_{x \in \widetilde{\Omega}_h} \sum_{r=1}^{n} \left|\varphi^h_{x_r, m_r}\right|^2 h_1 \cdots h_n\right)^{1/2} + \left(\sum_{x \in \widetilde{\Omega}_h} \sum_{r=1}^{n} \left|\varphi^h_{x_r \overline{x}_r, m_r}\right|^2 h_1 \cdots h_n\right)^{1/2}.\end{aligned} \tag{2.3}$$

Finally, we introduce the Banach spaces $C([0,1]_\tau, L_{2h})$ and $C^\alpha([0,1]_\tau, L_{2h})$ of grid abstract function $\{\varphi_k^h(x)\}_1^{N-1}$ defined on $[0,1]_\tau$ with values L_{2h}, equipped with the following norms:

$$\begin{aligned}\left\|\left\{\varphi_k^h\right\}_1^{N-1}\right\|_{C([0,1]_\tau, L_{2h})} &= \max_{1 \le k \le N-1} \left\|\varphi_k^h\right\|_{L_{2h}}, \\ \left\|\left\{\varphi_k^h\right\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau, L_{2h})} &= \max_{1 \le k \le N-1} \left\|\varphi_k^h\right\|_{L_{2h}} + \sup_{1 \le k < k+r \le N-1} \frac{\left\|\varphi_{k+r} - \varphi_k\right\|_{L_{2h}}}{(r\tau)^\alpha}.\end{aligned} \tag{2.4}$$

To the differential operator A generated by the problem (1.2) we assign the difference operator A_h^x by the formula

$$A_h^x u^h = -\sum_{r=1}^{n} \left(a_r(x) u^h_{\overline{x}_r}\right)_{x_r, m_r} + \delta u^h_{x_r} \tag{2.5}$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\widetilde{\Omega}_h)$. With the help of

A_h^x, we arrive at the nonlocal boundary value problem for an infinite system of the following ordinary differential equations:

$$
\begin{gathered}
-\frac{d^2u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x), \quad 0<t<1,\ x\in\Omega_h, \\
u_t^h(0,x) = \varphi^h(x), \quad u_t^h(1,x) = \beta u_t^h(\lambda,x) + \psi^h(x), \quad x\in\widetilde{\Omega}_h.
\end{gathered} \tag{2.6}
$$

In the second step, we replaced problem (2.6) by the first-order of accuracy difference scheme as follows:

$$
\begin{gathered}
-\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = f_k^h(x), \\
f_k^h(x) = f^h(t_k,x), \quad t_k = k\tau,\ 1\le k\le N-1,\ N\tau=1,\ x\in\Omega_h, \\
\frac{u_1^h(x) - u_0^h(x)}{\tau} = \varphi^h(x), \quad x\in\widetilde{\Omega}_h, \\
\frac{u_N^h(x) - u_{N-1}^h(x)}{\tau} = \beta\frac{u_{[\lambda/\tau]+1}^h(x) - u_{[\lambda/\tau]}^h(x)}{\tau} + \psi^h(x), \quad x\in\widetilde{\Omega}_h
\end{gathered} \tag{2.7}
$$

and the second order of accuracy difference scheme as follows:

$$
\begin{gathered}
-\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = f_k^h(x), \\
f_k^h(x) = f^h(t_k,x), \quad t_k = k\tau,\ 1\le k\le N-1,\ N\tau=1,\ x\in\Omega_h, \\
\frac{-3u_0^h(x) + 4u_1^h(x) - u_2^h(x)}{2\tau} = \varphi^h(x), \quad x\in\widetilde{\Omega}_h, \\
\frac{u_{N-2}^h(x) - 4u_{N-1}^h(x) + 3u_N^h(x)}{2\tau} = \beta\Bigg[\frac{u_{[\lambda/\tau]-1}^h(x) - 4u_{[\lambda/\tau]}^h(x) + 3u_{[\lambda/\tau]+1}^h(x)}{2\tau} \\
+ \frac{u_{[\lambda/\tau]+1}^h(x) - 2u_{[\lambda/\tau]}^h(x) + u_{[\lambda/\tau]-1}^h(x)}{\tau}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)\Bigg] \\
+ \psi^h(x), \quad x\in\widetilde{\Omega}_h.
\end{gathered} \tag{2.8}
$$

Now, we will study the well-posedness of (2.7) and (2.8). We have the following theorem on stability of (2.7) and (2.8).

Theorem 2.1. *Let τ and $|h|$ be sufficiently small positive numbers. Then the solutions of difference schemes* (2.7) *and* (2.8) *satisfy the following stability estimate:*

$$\max_{1\le k\le N}\left\|u_k^h\right\|_{L_{2h}} \le M_1\left[\left\|\varphi^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}\right], \tag{2.9}$$

where M_1 does not depend on $\tau,\ h,\ \psi^h(x),\ \varphi^h(x)$ and $f_k^h(x),\ 1\le k\le N-1$.

Proof. The proof of (2.9) is based on the following formula:

$$\begin{aligned} u_k^h(x) = {} & \left(I-R^{2N}\right)^{-1}\Bigg\{\left(R^k-R^{2N-k}\right)u_0^h(x)+\left(R^{N-k}-R^{N+k}\right)u_N^h(x) \\ & -\left(R^{N-k}-R^{N+k}\right)\left(2I+\tau B_h^x\right)^{-1}\left(B_h^x\right)^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)f_i^h(x)\tau\Bigg\} \\ & +\left(2I+\tau B_h^x\right)^{-1}\left(B_h^x\right)^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|-1}-R^{k+i-1}\right)f_i^h(x)\tau \quad \text{for } k=1,\ldots,N-1, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} u_0^h(x) = {} & P_\tau\left(I+\tau B_h^x\right)\left(2I+\tau B_h^x\right)^{-1}\left(B_h^x\right)^{-1} \\ & \times\Bigg[(I+R)R^{N-2}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)f_i^h(x)\tau \\ & \quad +(I+R)\left[I+R^{2N-2}-\beta\left(R^{N-[\lambda/\tau]-1}+R^{N+[\lambda/\tau]}\right)\right] \\ & \quad \times\sum_{i=1}^{N-1}R^{i-1}f_i^h(x)\tau-\beta\left(R^N+R^{N-1}\right)\sum_{i=1}^{[\lambda/\tau]-1}R^{[\lambda/\tau]-i}f_i^h(x)\tau \\ & \quad +\beta\left(R^{N-2}+R^{N-1}\right)\sum_{i=[\lambda/\tau]+1}^{N-1}R^{i-[\lambda/\tau]}f_i^h(x)\tau \\ & \quad +\beta\left(R^N+R^{N-1}\right)\sum_{i=1}^{N-1}R^{[\lambda/\tau]+i}f_i^h(x)\tau-\beta\left(R^{N-1}+R^N\right)f_{[\lambda/\tau]}^h(x)\tau\Bigg] \\ & -P_\tau(I-R)^{-1}\left[I+R^{2N-1}-\beta\left(R^{N-[\lambda/\tau]-1}+R^{N+[\lambda/\tau]}\right)\right]\varphi^h(x)\tau \\ & +P_\tau(I-R)^{-1}\left(R^{N-1}+R^N\right)\psi^h(x)\tau, \end{aligned}$$

$$
\begin{aligned}
u_N^h(x) = {} & P_\tau\left(I+\tau B_h^x\right)\left(2I+\tau B_h^x\right)^{-1}\left(B_h^x\right)^{-1} \\
& \times\Bigg[\left[R^{-1}(I+R)-\beta\left(R^{N-[\lambda/\tau]-1}+R^{N+[\lambda/\tau]-1}\right)\right]\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)f_i^h(x)\tau \\
& \quad -\beta\left(I+R^{2N-1}\right)\sum_{i=1}^{[\lambda/\tau]-1}R^{[\lambda/\tau]-i}f_i^h(x)\tau+\beta\left(I+R^{2N-1}\right)R^{-1}\sum_{i=[\lambda/\tau]+1}^{N-1}R^{i-[\lambda/\tau]}f_i^h(x)\tau \\
& \quad +\beta\left(I+R^{2N-1}\right)\sum_{i=1}^{N-1}R^{[\lambda/\tau]+i}f_i^h(x)\tau-\beta\left(I+R^{2N-1}\right)f_{[\lambda/\tau]}^h(x)\tau \\
& \quad +(I+R)\left[R^N+R^{N-1}-\beta\left(R^{[\lambda/\tau]}+R^{2N-[\lambda/\tau]-1}\right)\right]\sum_{i=1}^{N-1}R^{i-1}f_i^h(x)\tau\Bigg] \\
& -P_\tau(I-R)^{-1}\left[R^N+R^{2N-1}-\beta\left(R^{[\lambda/\tau]}+R^{2N-[\lambda/\tau]-1}\right)\right]\varphi^h(x)\tau \\
& +P_\tau(I-R)^{-1}\left(I+R^{2N-1}\right)\psi^h(x)\tau, \\
P_\tau = {} & \left[I-R^{2N-2}-\beta\left(R^{N-[\lambda/\tau]-1}+R^{N+[\lambda/\tau]-1}\right)\right]^{-1},
\end{aligned}
\tag{2.11}
$$

for (2.7), and

$$
\begin{aligned}
u_0^h(x) = {} & D_\tau\left(I+\tau B_h^x\right)\left(2I+\tau B_h^x\right)^{-1}\left(B_h^x\right)^{-1} \\
& \times\Bigg\{(I+R)\left(4R-I-R^2\right)(I-3R)R^{N-4}\left(I-R^{2N}\right)\sum_{i=1}^{N-2}\left(R^{N-i}-R^{N+i}\right)f_i^h(x)\tau \\
& \quad -(I+R)\left(4R-I-R^2\right)\left(I-R^{2N}\right) \\
& \quad \times\Bigg[3I-R-R^{2N-2}(I-3R) \\
& \qquad -\beta\Bigg[R^{N-[\lambda/\tau]-1}\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right) \\
& \qquad\quad -R^{N+[\lambda/\tau]-1}\left(3I-R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\Bigg]\Bigg] \\
& \quad \times\sum_{i=2}^{N-1}R^{i-2}f_i^h(x)\tau-\beta(I+R)R^{N-2}\left(4R-I-R^2\right)\left(I-R^{2N}\right) \\
& \quad \times\Bigg[\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\sum_{i=1}^{[\lambda/\tau]-1}R^{[\lambda/\tau]-i-1}f_i^h(x)\tau \\
& \qquad +\left(3I-R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\sum_{i=[\lambda/\tau]+2}^{N-1}R^{i-[\lambda/\tau]-1}f_i^h(x)\tau \\
& \qquad -\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\sum_{i=1}^{N-1}R^{[\lambda/\tau]+i-1}f_i^h(x)\tau\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&-\beta\left(I-R^{2N}\right)(I+R)R^{N-2}\left(4R-R^2-I\right)\left(4+4\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)\right)\\
&\times\left(f^h_{[\lambda/\tau]+1}(x)\tau-f^h_{[\lambda/\tau]}(x)\tau\right)-(I+R)(4I-R)\left(I-R^{2N}\right)\\
&\times\Bigg[3I-R-R^{2N-2}(I-3R)\\
&\qquad-\beta\Bigg(R^{N-[\lambda/\tau]-1}\left(3I-R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\\
&\qquad\qquad-R^{N+[\lambda/\tau]-1}\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\Bigg)\Bigg]\\
&\times f^h_1(x)\tau-(I+R)\mathrm{R}^{N-1}\left(4R-R^2-I\right)\left(I-R^{2N}\right)\left(4I+R^{2N-3}(I-3R)\right)\\
&\times f^h_{N-1}(x)\tau\Bigg\}\\
&-D_\tau\left(I-R^{2N}\right)(I-R)^{-1}(I+R)R^{N-2}\left(4R-I-R^2\right)2\tau\psi^h(x)\\
&+D_\tau\left(I-R^{2N}\right)(I-R)^{-1}\\
&\times\Bigg[3I-R-R^{2N-2}(I-3R)\\
&\qquad-\beta\Bigg[R^{N-[\lambda/\tau]-1}\left(3I-R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\\
&\qquad-R^{N+[\lambda/\tau]-1}\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\Bigg]\Bigg]\;2\tau\varphi^h(x),
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
u^h_N(x)&=D_\tau\left(I+\tau B^x_h\right)\left(2I+\tau B^x_h\right)^{-1}\left(B^x_h\right)^{-1}\\
&\times\Bigg\{\left(I-R^{2N}\right)\Bigg[(I+R)\left(4R-I-R^2\right)R^{-2}(R-3I)\\
&\qquad+\beta\Bigg[(R-3I)\left(3I-R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)R^{N-[\lambda/\tau]-1}\\
&\qquad\qquad-(3R-I)\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)R^{N+[\lambda/\tau]-1}\Bigg]\Bigg]\\
&\quad\times\sum_{i=1}^{N-2}\left(R^{N-i}-R^{N+i}\right)f^h_i(x)\tau+(I+R)\left(I-R^{2N}\right)\left(4R-I-R^2\right)\\
&\quad\times\Bigg[(I+R)R^{N-2}\left(4R-I-R^2\right)\\
&\qquad-\beta\Bigg[R^{[\lambda/\tau]-1}\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\\
&\qquad\qquad-R^{2N-[\lambda/\tau]-1}\left(3I-R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\Bigg]\Bigg]\sum_{i=2}^{N-1}R^{i-2}f^h_i(x)\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\beta\left(I-R^{2N}\right)\left(R-3I+R^{2N-2}(I-3R)\right)\\
&\times\left[\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\sum_{i=1}^{[\lambda/\tau]-1}R^{[\lambda/\tau]-i-1}f_i^h(x)\tau\right.\\
&\quad+\left(3I-R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\sum_{i=[\lambda/\tau]+2}^{N-1}R^{i-[\lambda/\tau]-1}f_i^h(x)\tau\\
&\quad\left.-\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\sum_{i=1}^{N-1}R^{[\lambda/\tau]+i-1}f_i^h(x)\tau\right]\\
&+\beta\left(R-3I+R^{2N-2}(3I-R)\right)\left(I-R^{2N}\right)\\
&\times\left(4+4\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\left(f_{[\lambda/\tau]+1}^h(x)\tau-f_{[\lambda/\tau]}^h(x)\tau\right)\\
&+(I+R)\left(I-R^{2N}\right)\\
&\times\left[(I+R)R^{N-2}\left(I-4R+R^2\right)\right.\\
&\quad-\beta\left[R^{[\lambda/\tau]-1}\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\right.\\
&\qquad\left.\left.-R^{2N-[\lambda/\tau]-1}\left(3I-R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\right]\right]f_1^h(x)\tau\\
&+R\left(I-R^{2N-1}\right)\left[R-4I+R^{2N-4}\left(I+R^2-3R\right)\left(3I-R-R^2(4I-R)\right)\right.\\
&\qquad\qquad-\beta\left[R^{N-[\lambda/\tau]-1}\left(3I-R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\right.\\
&\qquad\qquad\quad\times\left(3R-I+R^{2N}(3I-R)\right)\\
&\qquad\qquad\quad-R^{N+[\lambda/\tau]-1}\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\\
&\qquad\qquad\quad\left.\left.\times\left(3I-R+R^{2N}(I-3R)\right)\right]\right]f_{N-1}^h(x)\tau\Big\}\\
&+D_\tau\left(I-R^{2N}\right)(I-R)^{-1}\left(R-3I+R^{2N}(I-3R)\right)2\tau\varphi^h(x)\\
&-D_\tau\left(I-R^{2N}\right)(I-R)^{-1}\\
&\times\left[R^{N-2}(I+R)\left(R^2-4R+I\right)\right.\\
&\quad-\beta\left[R^{[\lambda/\tau]-1}\left(I-3R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\right.\\
&\quad\left.\left.-R^{2N-[\lambda/\tau]-1}\left(3I-R+2\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\right]\right]2\tau\psi^h(x),
\end{aligned}
$$

$$
\begin{aligned}
D_\tau = {} & \left(I - R^{2N}\right)^{-1} \\
& \times \Bigg\{ \left[-(3I - R)^2 + (I - 3R)^2\right] \\
& \quad - \beta\Bigg[-R^{N+[\lambda/\tau]-3}(I - 3R)\left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I - R)\right) \\
& \qquad -R^{N-[\lambda/\tau]-1}(3I - R)\left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I - R)\right)\Bigg]\Bigg\}^{-1}, \\
& R = \left(I + \tau B_h^x\right)^{-1}, \\
& B_h^x = \frac{1}{2}\left(\tau A_h^x + \sqrt{4A_h^x + \tau^2\left(A_h^x\right)^2}\right)
\end{aligned}
\tag{2.13}
$$

for (2.8), and the symmetry properties of the difference operator A_h^x defined by the formula (2.5). □

Difference schemes (2.7) and (2.8) are ill-posed in $C([0,1]_\tau, L_{2h})$. We have the following theorem on almost coercive stability.

Theorem 2.2. *Let τ and $|h|$ be sufficiently small positive numbers. Then the solutions of difference schemes* (2.7) *and* (2.8) *satisfy the following almost coercive stability estimate:*

$$
\begin{aligned}
& \max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2}\right\|_{L_{2h}} + \max_{1\le k\le N-1}\left\|u_k^h\right\|_{W_{2h}^2} \\
& \le M_2\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^2} + \ln\frac{1}{\tau + |h|}\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}\right],
\end{aligned}
\tag{2.14}
$$

where M_2 does not depend on τ, h, $\psi^h(x)$, $\varphi^h(x)$, and $f_k^h(x)$, $1 \le k \le N-1$.

Proof. The proof of (2.14) is based on the formulas (2.11), (2.12), (2.13), the symmetry properties of the difference operator A_h^x defined by the formula (2.5) and on the following theorem on well-posedness of the elliptic difference problem. □

Theorem 2.3. *For the solutions of the elliptic difference problem*

$$
\begin{aligned}
A_h^x u^h(x) = w^h(x), \quad x \in \Omega_h, \\
u^h(x) = 0, \quad x \in S_h
\end{aligned}
\tag{2.15}
$$

the following coercivity inequality holds (see [21]):

$$
\sum_{r=1}^{n}\left\|\left(u^h\right)_{x_r\overline{x}_r,\, m_r}\right\|_{L_{2h}} \le M\left\|w^h\right\|_{L_{2h}},
\tag{2.16}
$$

where M does not depend on h and $w^h(x)$.

Theorem 2.4. *Let $\varphi^h(x)=\psi^h(x)=0$. Then the difference problems* (2.7) *and* (2.8) *are well-posed in Hölder spaces $C^\alpha([0,1]_\tau, L_{2h})$ and the following coercivity inequality holds:*

$$\begin{aligned}
&\max_{1\le k\le N-1}\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,\, L_{2h})}+\left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,\, W_{2h}^2)}\\
&\quad\le\frac{M_3}{\alpha(1-\alpha)}\max_{1\le k\le N-1}\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau, L_{2h})}.
\end{aligned}\tag{2.17}$$

Here M_3 does not depend on τ, h, and $f_k^h(x)$, $1\le k\le N-1$.

Proof. The proof of (2.17) is based on the formula

$$\begin{aligned}
A_h^x u_0^h(x)-f_1^h(x)={}&P_\tau\left(I+\tau B_h^x\right)\left(2I+\tau B_h^x\right)^{-1}(I+R)\\
&\times\Bigg[R^{N-1}\sum_{i=1}^{N-1}\tau B_h^x R^{N-i}\left(f_i^h-f_{N-1}^h\right)\\
&\quad-R^{N-1}\sum_{i=1}^{N-1}\tau B_h^x R^{N+i}\left(f_i^h(x)-f_1^h(x)\right)\\
&\quad-\beta R^N\sum_{i=1}^{[\lambda/\tau]-1}\tau B_h^x R^{[\lambda/\tau]-i}\left(f_i^h(x)-f_{[\lambda/\tau]}^h(x)\right)\\
&\quad+\left[I+R^{2N-2}-\beta\left(R^{N-[\lambda/\tau]-1}+R^{N+[\lambda/\tau]}\right)\right]\\
&\quad\times\sum_{i=1}^{N-1}\tau B_h^x R^i\left(f_i^h(x)-f_1^h(x)\right)\\
&\quad+\beta R^N\sum_{i=[\lambda/\tau]+1}^{N-1}\tau B_h^x R^{i-[\lambda/\tau]}\left(f_i^h(x)-f_{[\lambda/\tau]}^h(x)\right)\\
&\quad+\beta R^N\sum_{i=1}^{N-1}\tau B_h^x R^{[\lambda/\tau]+i}\left(f_i^h(x)-f_1^h(x)\right)\\
&\quad+R^{N-1}\left(I-R^{N-1}\right)f_{N-1}^h(x)\\
&\quad+\beta\left[R^{N+[\lambda/\tau]-1}-R^{2N-[\lambda/\tau]-1}-\tau B_h^x R^N\right]f_{[\lambda/\tau]}^h(x)\\
&\quad+\Bigg[R^{2N-2}-R^{2N-1}+R^{2N-2}-R^{3N-3}+R^{3N-2}-R^{N-1}\\
&\qquad+\beta\Big(R^{2N-[\lambda/\tau]-2}+R^{2N+[\lambda/\tau]-1}\\
&\qquad\quad+R^{2N+[\lambda/\tau]}-R^{N+[\lambda/\tau]-1}\Big)\Bigg]f_1^h(x)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&- P_\tau (I-R)^{-1}\tau\Big[I+R^{2N-1}-\beta\Big(R^{N-[\lambda/\tau]-1}+R^{N+[\lambda/\tau]}\Big)\Big]A_h^x\varphi^h \\
&+ P_\tau (I-R)^{-1}\tau\Big(R^{N-1}+R^{N}\Big)A_h^x\psi^h,
\end{aligned}
$$

$$
\begin{aligned}
A_x^h u_N^h(x) - f_{N-1}^h(x) = {} & P_\tau\left(I+\tau B_h^x\right)\left(2I+\tau B_h^x\right)^{-1} \\
& \times \Bigg\{ \Big[R(I+R)-\beta\Big(R^{N-[\lambda/\tau]+1}+R^{N+[\lambda/\tau]+1}\Big)\Big] \\
& \quad \times \sum_{i=1}^{N-1}\tau B_h^x R^{N-i}\Big(f_i^h(x)-f_{N-1}^h(x)\Big) \\
& \quad - \Big[R(I+R)-\beta\Big(R^{N-[\lambda/\tau]+1}+R^{N+[\lambda/\tau]+1}\Big)\Big] \\
& \quad \times \sum_{i=1}^{N-1}\tau B_h^x R^{N+i}\Big(f_i^h(x)-f_1^h(x)\Big) \\
& \quad - \beta R^2\Big(I+R^{2N-1}\Big)\sum_{i=1}^{[\lambda/\tau]-1}\tau B_h^x R^{[\lambda/\tau]-i}\Big(f_i^h(x)-f_{[\lambda/\tau]}^h(x)\Big) \\
& \quad + \beta R\Big(I+R^{2N-1}\Big)\sum_{i=[\lambda/\tau]+1}^{N-1}\tau B_h^x R^{i-[\lambda/\tau]}\Big(f_i^h(x)-f_{[\lambda/\tau]}^h(x)\Big) \\
& \quad + \beta R^2\Big(I+R^{2N-1}\Big)\sum_{i=1}^{N-1}\tau B_h^x R^{[\lambda/\tau]+i}\Big(f_i^h(x)-f_1^h(x)\Big) \\
& \quad - \beta\Big(I+R^{2N-1}\Big) \\
& \quad \times \Big[R\Big(I-R^{[\lambda/\tau]-1}\Big)-\Big(I-R^{N-[\lambda/\tau]-1}\Big)-\tau B_h^x R\Big] \\
& \quad \times f_{[\lambda/\tau]}^h(x) \\
& \quad + \Big[(I+R)\Big(R^{2N-2}-R^{N}-I+R\Big) \\
& \qquad - \beta\Big(R^{N-[\lambda/\tau]+1} \\
& \qquad\quad + R^{N+[\lambda/\tau]+1}-R^{2N-[\lambda/\tau]} \\
& \qquad\quad - R^{2N+[\lambda/\tau]}-R^{N-[\lambda/\tau]-1} \\
& \qquad\quad + R^{N+[\lambda/\tau]-1}-R^{N-[\lambda/\tau]}+R^{N+[\lambda/\tau]}\Big)\Big] \\
& \quad \times f_{N-1}^h(x) + \Big[(I+R)\Big(R^{N}-R^{2N-1}\Big) \\
& \qquad + \beta\Big(R^{N+[\lambda/\tau]}+R^{N+[\lambda/\tau]+1}-R^{N+[\lambda/\tau]+2} \\
& \qquad\quad + R^{2N+[\lambda/\tau]+1}+R^{2N+[\lambda/\tau]+1}-R^{3N+[\lambda/\tau]} \\
& \qquad\quad - R^{3N+[\lambda/\tau]+1}+R^{[\lambda/\tau]+1}-R^{2N-[\lambda/\tau]} \\
& \qquad\quad + R^{3N-[\lambda/\tau]-1}\Big)\Big] f_1^h(x) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ P_\tau (I-R)^{-1} R\tau \left(I + R^{2N-1}\right) A_x^h \psi \\
&- P_\tau (I-R)^{-1} R\tau \left[R^{N-1} + R^N - \beta\left(R^{[\lambda/\tau]} + R^{2N-[\lambda/\tau]-1}\right)\right] A_x^h \varphi
\end{aligned}
\tag{2.18}
$$

for (2.7) and

$$
\begin{aligned}
A_h^x u_0^h(x) - f_1^h(x) = {}& D_\tau \left(I + \tau B_h^x\right)\left(2I + \tau B_h^x\right)^{-1} \\
&\times \Bigg\{ (I+R)\left(4R - I - R^2\right)(I-3R)R^{N-3}\left(I - R^{2N}\right) \\
&\quad \times \left[\sum_{i=1}^{N-2} B_h^x \tau R^{N-i}\left(f_i^h(x) - f_{N-1}^h(x)\right) \right. \\
&\qquad \left. - \sum_{i=1}^{N-2} B_h^x \tau R^{N+i}\left(f_i^h(x) - f_1^h(x)\right)\right] \\
&\quad - (I+R)\left(4R - I - R^2\right)\left(I - R^{2N}\right) \\
&\quad \times \Bigg[3I - R - R^{2N-2}(I-3R) \\
&\qquad - \beta\Bigg[R^{N-[\lambda/\tau]-1}\left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right) \\
&\qquad\quad - R^{N+[\lambda/\tau]-1}\left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\Bigg]\Bigg] \\
&\quad \times \sum_{i=2}^{N-1} B_h^x \tau R^{i-1}\left(f_i^h(x) - f_1^h(x)\right) \\
&\quad - \beta(I+R)R^{N-2}\left(4R - I - R^2\right)\left(I - R^{2N}\right) \\
&\quad \times \Bigg[\left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right) \\
&\qquad \times \sum_{i=1}^{[\lambda/\tau]-1} B_h^x \tau R^{[\lambda/\tau]-i}\left(f_i^h(x) - f_{[\lambda/\tau]}^h(x)\right) \\
&\qquad + \left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right) \\
&\qquad \times \sum_{i=[\lambda/\tau]+2}^{N-1} B_h^x \tau R^{i-[\lambda/\tau]}\left(f_i^h(x) - f_{[\lambda/\tau]}^h(x)\right)
\end{aligned}
$$

$$
\begin{aligned}
& - \left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau} \right] \right)(I - R) \right) \\
& \times \sum_{i=1}^{N-1} B_h^x \tau R^{[\lambda/\tau]+i} \left(f_i^h(x) - f_1^h(x) \right) \Bigg] \\
& - \beta \left(I - R^{2N} \right)(I + R) R^{N-1} \left(4R - R^2 - I \right) \\
& \times \left(4 + 4\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau} \right] \right) \right) \tau B_h^x f_{[\lambda/\tau]+1}^h(x) \\
& + \beta \left(I - R^{2N} \right)(I + R) R^{N-2} \left(4R - R^2 - I \right) \\
& \times \Bigg[2R \left(R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau} \right] \right) \right) \\
& \quad + R^{[\lambda/\tau]-1} \left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau} \right] \right)(I - R) \right) + R^{N-[\lambda/\tau]-2} \\
& \quad \times \left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau} \right] \right)(I - R) \right) \Bigg] f_{[\lambda/\tau]}^h(x) \\
& \times \Bigg[(I + R)\left(I - R^{2N} \right) \\
& \quad \times \Bigg[(I - 3R)\Big(3R - I - R^{2N-5}(I - R) \\
& \qquad \qquad \times \left(R^2 - R^{2N}(I + R)\left(R^2 - 4R + I \right) \right) \Big) \\
& \qquad - (3I - R) R^{N-1} \left(R^2 - 4R + I \right) - \beta \left(4R - I - R^2 \right) \\
& \qquad \times \Bigg(\left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau} \right] \right)(I - R) \right) R^{N-[\lambda/\tau]-3} \\
& \qquad \quad - \left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau} \right] \right)(I - R) \right) \\
& \qquad \quad \times R^{N+[\lambda/\tau]-3} \Bigg) \Bigg] \Bigg] f_1^h(x) \\
& + (I + R)\left(4R - I - R^2 \right)\left(I - R^{2N} \right) R^{N-2} \\
& \times \left(R - 3I - R^{N-2}(I - 3R)\left(I + R^N (I - R) \right) \right) f_{N-1}^h(x) \Bigg\} \\
& - D_\tau \left(I - R^{2N} \right)(I - R)^{-1}(I + R) R^{N-2} \left(4R - I - R^2 \right) 2\tau A_h^x \psi^h(x) \\
& + D_\tau \left(I - R^{2N} \right)(I - R)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Bigg[3I - R - R^{2N-2}(I-3R)\\
&\quad - \beta\Bigg[R^{N-[\lambda/\tau]}\left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\\
&\qquad - R^{N+[\lambda/\tau]-1}\left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\Bigg]\Bigg] 2\tau A_h^x \varphi^h(x),
\end{aligned}
$$

$$
\begin{aligned}
&A_h^x R u_N^h(x) - f_{N-1}^h(x)\\
&\quad = D_\tau \left(I + \tau B_h^x\right)\left(2I + \tau B_h^x\right)^{-1}\\
&\quad\times \Bigg\{ \left(I - R^{2N}\right)\\
&\qquad \times \Bigg[(I+R)\left(4R - I - R^2\right)(R - 3I)\\
&\qquad\quad + \beta\Bigg[(R-3I)\left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\\
&\qquad\qquad \times R^{N-[\lambda/\tau]+1} - (3R - I)\\
&\qquad\qquad \times \left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right) R^{N+[\lambda/\tau]+1}\Bigg]\Bigg]\\
&\quad \times \left(\sum_{i=1}^{N-2} B_h^x \tau R^{N-i}\left(f_i^h(x) - f_{N-1}^h(x)\right) - \sum_{i=1}^{N-2} B_h^x \tau R^{N+i}\left(f_i^h(x) - f_1^h(x)\right)\right)\\
&\quad + (I+R)\left(I - R^{2N}\right)\left(4R - I - R^2\right)\\
&\quad \times \Bigg[(I+R)R^{N-2}\left(-4R + I + R^2\right)\\
&\qquad - \beta\Bigg[R^{[\lambda/\tau]-1}\left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\\
&\qquad\quad - R^{2N-[\lambda/\tau]-1}\left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\Bigg]\Bigg]\\
&\quad \times \sum_{i=2}^{N-1} B_h^x \tau R^i \left(f_i^h(x) - f_1^h(x)\right)\\
&\quad + \beta R\left(R - 3I + R^{2N-2}(I - 3R)\right)\left(I - R^{2N}\right)\\
&\quad \times \Bigg[\left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\\
&\qquad \times \sum_{i=1}^{[\lambda/\tau]-1} B_h^x \tau R^{[\lambda/\tau]-i}\left(f_i^h(x) - f_{[\lambda/\tau]}^h(x)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right) \\
&\quad \times \sum_{i=[\lambda/\tau]+2}^{N-1} B_h^x \tau R^{[\lambda/\tau]+i}\left(f_i^h(x) - f_1^h(x)\right) \\
&\quad + \left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right) \\
&\quad \times \sum_{i=1}^{N-1} B_h^x \tau R^{[\lambda/\tau]+i}\left(f_i^h(x) - f_1^h(x)\right)\Bigg] \\
&+ \beta R^2\left(R - 3I + R^{2N-2}(I-3R)\right)\left(I - R^{2N}\right)\left(4 + 4\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)\right) \\
&\times f_{[\lambda/\tau]-1}^h(x) - \beta R\left(I - R^{2N}\right) \\
&\times \Bigg[\left(I - R + R^{N-[\lambda/\tau]}\right)\left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right) \\
&\quad + R^{[\lambda/\tau]-1}\left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)\Bigg] \\
&\times f_{[\lambda/\tau]}^h(x) + \left(I - R^{2N}\right) \\
&\times \Bigg[(I+R)\left(4R - R^2 - I\right)R^{N+1} \\
&\quad \times (R - 3I)\left(I - R^{N-2}\right) \\
&\quad + \beta\Bigg[\left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right) \\
&\qquad \times R^{2N-[\lambda/\tau]}\left(R^2(3I-R) + R^{N-2}\left(R^3 - 4R^2 + I\right)\right) \\
&\qquad - \left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)R^{[\lambda/\tau]} \\
&\qquad \times \left(R^2(3I-R) + (I-3R)R^{N-3}\left(R + R^N(I-R)\right)\right)\Bigg]\Bigg] \\
&\times f_1^h(x) + \left(I - R^{2N}\right) \\
&\times \Bigg[(I-3R)\left(4R - R^2 - I\right)\left(3I - R + R^{2N-2}\right) \\
&\quad - \beta\Bigg[\left(3I - R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)R^{N-[\lambda/\tau]-1}(3I-R) \\
&\qquad \times \left(R + I + R^N\left(I - R + R^{N-2}(I-R) + R^{3N-2}(2I-R)\right)\right) \\
&\qquad - \left(I - 3R + 2\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)(I-R)\right)R^{N+[\lambda/\tau]-1}(3R-I)
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big(I - R - R^{2N-3} \\
&\qquad \times\Big(1 + R - R^{N}\Big(3I - R^{2}\Big) + R^{2N}\Big(R^{3} - 4R^{2} + I\Big)\Big)\Big)\Big]\Big] \\
&\times f^{h}_{N-1}(x)\Big\} \\
&+ D_{\tau}\Big(I - R^{2N}\Big)(I - R)^{-1}R\Big(R - 3I + R^{2N-2}(I - 3R)\Big)2\tau A^{x}_{h}\varphi^{h}(x) \\
&- D_{\tau}\Big(I - R^{2N}\Big)(I - R)^{-1}R \\
&\times \Big[R^{N-2}(I + R)\Big(R^{2} - 4R + I\Big) \\
&\quad - \beta\Big[R^{[\lambda/\tau]-1}\Big(I - 3R + 2\Big(\frac{\lambda}{\tau} - \Big[\frac{\lambda}{\tau}\Big]\Big)(I - R)\Big) \\
&\qquad - R^{2N-[\lambda/\tau]-1}\Big(3I - R + 2\Big(\frac{\lambda}{\tau} - \Big[\frac{\lambda}{\tau}\Big]\Big)(I - R)\Big)\Big]\Big]2\tau A^{x}_{h}\varphi^{h}(x)
\end{aligned}
\tag{2.19}
$$

for (2.8), the symmetry properties of the difference operator A^{x}_{h} defined by the formula (2.5) and on Theorem 2.3 on well-posedness of the elliptic difference problem. □

3. Numerical Analysis

We have not been able to obtain a sharp estimate for the constants figuring in the stability inequality. Therefore, we will give the following results of numerical experiments of the Bitsadze-Samarskii-Dirichlet problem:

$$
\begin{aligned}
&-\frac{\partial^{2}u(t,x)}{\partial t^{2}} - \frac{\partial^{2}u(t,x)}{\partial x^{2}} + u = f(t,x), \quad 0 < t < 1,\ 0 < x < 1, \\
&f(t,x) = \exp(-\pi t)\sin(\pi x), \quad 0 < t < 1,\ 0 < x < 1, \\
&u_{t}(0,x) = -\pi\sin(\pi x), \quad 0 \le x \le 1, \\
&u_{t}(1,x) = u_{t}\Big(\frac{1}{2},x\Big) + \pi\sin(\pi x)\Big(\exp\Big(-\frac{\pi}{2}\Big) - \exp(-\pi)\Big), \quad 0 \le x \le 1, \\
&u(t,0) = u(t,1) = 0, \quad 0 \le t \le 1
\end{aligned}
\tag{3.1}
$$

for the two-dimensional elliptic equation.

The exact solution of this problem is

$$u(t,x)=\exp(-\pi t)\sin(\pi x). \tag{3.2}$$

For the approximate solution of problem (3.1), we consider the set $[0,1]_\tau\times[0,1]_h$ of a family of grid points depending on small parameters τ and h as follows:

$$\begin{aligned}[0,1]_\tau\times[0,1]_h=\{(t_k,x_n): t_k=k\tau,\ 0\le k\le N,\ N\tau=1,\\ x_n=nh,\ 0\le n\le M,\ Mh=1\}.\end{aligned} \tag{3.3}$$

Applying (2.7), we present the following first-order of accuracy difference scheme for the approximate solution of problem (3.1):

$$\begin{aligned}
&-\frac{u_{k+1}^n-2u_k^n+u_{k-1}^n}{\tau^2}-\frac{u_k^{n+1}-2u_k^n+u_k^{n-1}}{h^2}+u_k^n=\exp(-\pi t_k)\sin(\pi x_n), \quad 1\le k\le N-1,\\
&\qquad 1\le n\le M-1,\\
&\frac{u_1^n-u_0^n}{\tau}=-\pi\sin(\pi x_n), \quad 0\le n\le M,\\
&\frac{u_N^n-u_{N-1}^n}{\tau}=\beta\frac{u_{N/2}^n-u_{N/2-1}^n}{\tau}+\pi\sin(\pi x_n)\\
&\qquad\times\left(\exp\left(-\frac{\pi}{2}\right)-\exp(-\pi)\right), \quad 0\le n\le M,\\
&u_k^0=u_k^M=0, \quad 0\le k\le N.
\end{aligned} \tag{3.4}$$

We have $(N+1)\times(M+1)$ system of linear equations in (3.4) and we will write them in the following matrix form:

$$\begin{aligned}Au_{n+1}+Bu_n+Cu_{n-1}=D\varphi_n, \quad 1\le n\le M-1,\\ u_0=u_M=\tilde{0}.\end{aligned} \tag{3.5}$$

Here,

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & a & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & a & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & a & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & a & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} -1 & 1 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \\ c & b & c & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & c & b & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & b & c & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & c & b & c \\ 0 & 0 & 0 & \cdot & 1 & -1 & \cdot & 0 & -1 & 1 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.6}$$

$$C = A, \qquad D = [I]_{(N+1)\times(N+1)}, \qquad u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \cdot \\ u_s^{N-1} \\ u_s^N \end{bmatrix}_{(N+1)\times 1},$$

where $s = n-1, n, n+1$ and

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdot \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$a = \frac{1}{h^2}, \qquad b = -\frac{2}{\tau^2} - \frac{2}{h^2} - 1, \qquad c = \frac{1}{\tau^2}, \tag{3.7}$$

$$\varphi_n^k = \begin{cases} -\tau\pi \sin(\pi x_n), & k = 0, \\ -\exp(-\pi t_k)\sin(\pi x_n), & 1 \le k \le N-1, \\ \tau\pi \sin(\pi x_n)\left(\exp\left(-\frac{\pi}{2}\right) - \exp(-\pi)\right), & k = N. \end{cases}$$

We seek a solution of the matrix equation in following form:

$$\begin{gathered} u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 1, \\ u_M = \widetilde{0}, \end{gathered} \tag{3.8}$$

where $\alpha_n (n = 1, \ldots, M)$ are $(N+1) \times (N+1)$ square matrices and $\beta_n (n = 1, \ldots, M)$ are $(N+1) \times 1$ column matrices defined by (see, [17]) as follows:

$$\begin{aligned} \alpha_{n+1} &= -(B + C\alpha_n)^{-1} A, \\ \beta_{n+1} &= (B + C\alpha_n)^{-1} (D\varphi_n - C\beta_n), \quad n = 1, \ldots, M-1, \end{aligned} \tag{3.9}$$

where

$$\alpha_1 = [0]_{(N+1)\times(N+1)}, \qquad \beta_1 = [0]_{(N+1)\times 1}. \tag{3.10}$$

Now, applying (2.8) for N even number, we can present the following second-order of accuracy difference scheme:

$$\begin{aligned} -\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\tau^2} - \frac{u_k^{n+1} - 2u_k^n + u_k^{n-1}}{h^2} + u_k^n &= -\exp(-\pi t_k) \sin(\pi x_n), \quad 1 \le k \le N-1, \\ & \qquad 1 \le h \le M-1, \\ \frac{-3u_0^n + 4u_1^n - u_2^n}{2\tau} &= -\pi \sin(\pi x_n), \quad 0 \le n \le M, \\ \frac{u_{N-2}^n(x) - 4u_{N-1}^n(x) + 3u_N^n(x)}{2\tau} &= \frac{u_{N/2-2}^n - 4u_{N/2-1}^n + 3u_{N/2}^n}{2\tau} \\ &\quad + \pi \sin(\pi x_n)\left(\exp\left(-\frac{\pi}{2}\right) - \exp(-\pi)\right), \quad 0 \le n \le M, \\ u_k^0 = u_k^M &= 0, \quad 0 \le k \le N \end{aligned} \tag{3.11}$$

for the approximate solution of problem (3.1).

So, again we have $(N+1) \times (M+1)$ system of linear equations in (3.11) and we will write them in the following matrix form:

$$\begin{aligned} Au_{n+1} + Bu_n + Cu_{n-1} &= D\varphi_n, \quad 1 \le n \le M-1, \\ u_0 = u_M &= \tilde{0}, \end{aligned} \tag{3.12}$$

where

$$A=\begin{bmatrix}0&0&0&\cdot&0&0&0&\cdot&0&0&0\\0&a&0&\cdot&0&0&0&\cdot&0&0&0\\0&0&a&\cdot&0&0&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdot&0&0&0&\cdot&a&0&0\\0&0&0&\cdot&0&0&0&\cdot&0&a&0\\0&0&0&\cdot&0&0&0&\cdot&0&0&0\end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix}-3&4&-1&\cdot&0&0&0&\cdot&0&0&0\\b&c&b&\cdot&0&0&0&\cdot&0&0&0\\0&b&c&\cdot&0&0&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdot&0&0&0&\cdot&c&b&0\\0&0&0&\cdot&0&0&0&\cdot&b&c&b\\0&0&0&\cdot&-1&4&-3&\cdot&1&-4&3\end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.13}$$

$$C=A,\qquad D=[I]_{(N+1)\times(N+1)},$$

$$u_s=\begin{bmatrix}u_s^0\\u_s^1\\\cdot\\u_s^{N-1}\\u_s^N\end{bmatrix}_{(N+1)\times 1},$$

where $s=n-1,n,n+1$ and

$$\varphi_n^k=\begin{bmatrix}\varphi_n^0\\\varphi_n^1\\\cdot\\\varphi_n^{N-1}\\\varphi_n^N\end{bmatrix}_{(N+1)\times 1}. \tag{3.14}$$

Here,

$$a=\frac{1}{h^2},\qquad b=\frac{1}{\tau^2},\qquad c=-\frac{2}{h^2}-\frac{2}{\tau^2}-1, \tag{3.15}$$

$$\varphi_n^k=\begin{cases}-2\tau\pi\sin(\pi x_n), & k=0,\\ -\exp(-\pi t_k)\sin(\pi x_n), & 1\le k\le N-1,\\ 2\tau\pi\sin(\pi x_n)\left(\exp\left(-\frac{\pi}{2}\right)-\exp(-\pi)\right), & k=N.\end{cases} \tag{3.16}$$

So, we have the second-order difference equation with respect to n with matrix coefficients. To solve this difference equation, we use the same algorithm (3.8) and (3.9).

Now, we will give the results of the numerical experiments.

Table 1: Comparison of the errors of difference schemes.

Difference schemes	$N = M = 20$	$N = M = 40$	$N = M = 60$
First-order difference scheme (2.7)	0.05384197882300	0.02631633782987	0.01740639813932
Second-order difference scheme (2.8)	0.00631619894867	0.00164455475890	7.4144589892985e -004

The errors in numerical solutions are computed by

$$E_M^N = \max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|u(t_k,x_n)-u_n^k\right|^2 h\right)^{1/2} \tag{3.17}$$

for different values of M and N, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n). The results are shown in Table 1 for $N = M = 20$, $N = M = 40$, and $N = M = 60$.

Thus, second-order of accuracy difference scheme is more accurate compared with the first-order of accuracy difference scheme.

4. Conclusion

The first and second-orders of accuracy difference schemes for approximate solutions of the Bitsadze-Samarskii-Dirichlet type nonlocal boundary value problem for the multidimensional elliptic partial differential equation are presented. Theorems on the stability, almost coercive stability, and coercive stability estimates for the solution of these difference schemes are established. Numerical experiments are given.

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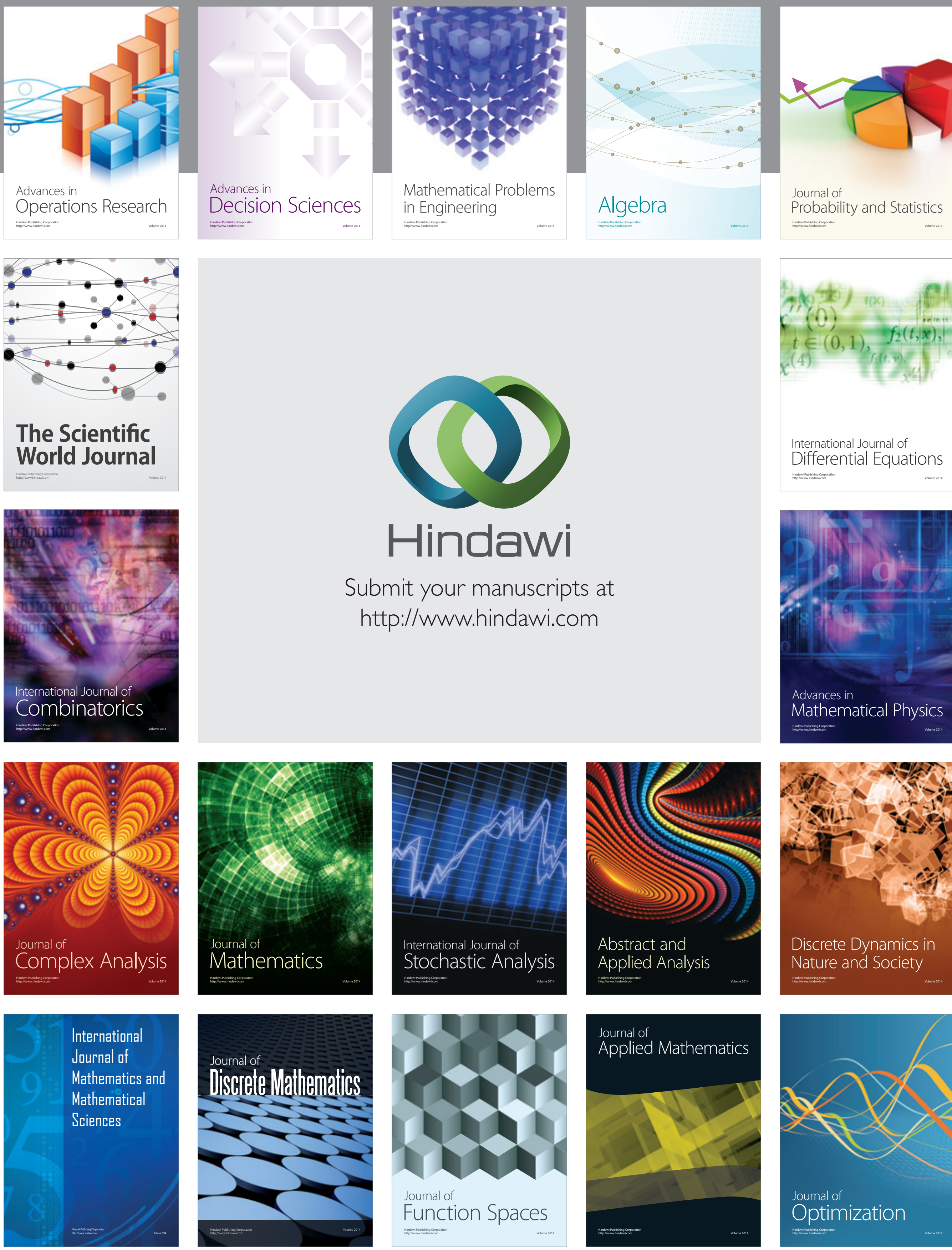
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