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Research Article

Theorems for Boyd-Wong-Type Contractions in Ordered Metric Spaces

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We give some fixed point results using an ICS mapping and involving Boyd-Wong-type contractions in partially ordered metric spaces. Our results generalize, extend, and unify several well-known comparable theorems in the literature. Also, we present some examples to support our results.

1. Introduction and Preliminaries

The Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. That is why, generalizations of the Banach principle have been heavily investigated by many authors. For instance, in 1977 Jaggi [2] proved the following theorem satisfying a contractive condition of rational type.

Theorem 1.1. *Let (X, d) be a complete metric space. Let $f : X \to X$ be a continuous mapping such that*

$$d(fx, fy) \leq \alpha \frac{d(x, fx)d(y, fy)}{d(x, y)} + \beta d(x, y) \tag{1.1}$$

for all distinct $x, y \in X$ where $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$. Then, f has a unique fixed point.

Existence of a fixed point for contraction-type mappings in partially ordered set has been considered by Ran and Reurings [3], and they applied their results to matrix equations. Later, Nieto and Rodríguez-López [4] studied some fixed point theorems for contractive mappings in partially ordered set and applied their main theorems to obtain a unique solution for a first-order ordinary differential equation. For more works on fixed point results in partially ordered metric spaces, we refer the reader to [5–29].

Note that, in the context of partially ordered metric spaces, the usual contractive condition is weakened, but at the expense that the operator is monotone.

Recently, Harjani et al. [17] proved the ordered version of Theorem 1.1 as follows.

Theorem 1.2 (see [17]). *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous and nondecreasing mapping such that*

$$d(fx, fy) \leq \alpha \frac{d(x, fx)d(y, fy)}{d(x, y)} + \beta d(x, y), \quad \text{for } x, y \in X,\ x \geq y,\ x \neq y, \tag{1.2}$$

where $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a unique fixed point.

Very recently, Luong and Thuan [21] generalized Theorem 1.2 as follows.

Theorem 1.3. *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that*

$$d(fx, fy) N(x, y) - \phi(N(x, y)) \tag{1.3}$$

for all distinct $x, y \in X$ with $y \leq x$ where $\phi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\phi(t) = 0$ if and only if $t = 0$, and

$$N(x, y) = \max\left\{ \frac{d(x, fx)d(y, fy)}{d(x, y)}, d(x, y) \right\}. \tag{1.4}$$

Also, assume either

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq f x_0$, then f has a fixed point.

In the sequel, we give the following definition (see e.g., [30]).

Definition 1.4. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be ICS if T is injective (also said, one to one), continuous, and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.

Throughout this paper, the letters $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all nonnegative real numbers and the set of all nonnegative integer numbers, respectively.

The purpose of this paper is to generalize the above results using an ICS mapping $T : X \to X$ and involving some generalized weak contractions of Boyd-Wong-type [31]. Also, some examples are presented to show that our results are effective.

2. Main Result

First, denote by Φ the set of functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying

(a) $\phi(t) < t$ for all $t > 0$,

(b) ϕ is upper semicontinuous from the right [i.e., for any sequence $\{t_n\}$ in $[0, \infty)$ such that $t_n \to t$, $t_n > t$ as $n \to \infty$, we have $\limsup_{n\to\infty}\phi(t_n) \le \phi(t)$].

Now we prove our first result.

Theorem 2.1. *Let $(X, \le)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a complete metric space. Let $f, T : X \to X$ be such that T is an ICS mapping and f a nondecreasing mapping satisfying*

$$d(Tfx, Tfy) \le \phi(M(x, y)) \tag{2.1}$$

for all distinct $x, y \in X$ with $x \le y$ where $\phi \in \Phi$ and

$$M(x, y) = \max\left\{\frac{d(Tx, Tfx)d(Ty, Tfy)}{d(Tx, Ty)}, d(Tx, Ty)\right\}. \tag{2.2}$$

Also, assume either

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \le fx_0$, then f has a fixed point.

Proof. Given $x_1 = fx_0$, define a sequence $\{x_n\}$ in X as follows:

$$x_n = fx_{n-1} \quad \text{for } n \ge 1. \tag{2.3}$$

Since f is a nondecreasing mapping, together with $x_0 \le x_1 = fx_0$, we have $x_1 = fx_0 \le fx_1 = x_2$. Inductively, we obtain

$$x_0 \le x_1 \le x_2 \le \cdots \le x_{n-1} \le x_n \le x_{n+1} \le \cdots. \tag{2.4}$$

Assume that there exists n_0 such that $x_{n_0} = x_{n_0+1}$. Since $x_{n_0} = x_{n_0+1} = fx_{n_0}$, then f has a fixed point which ends the proof.

Suppose that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. Thus, by (2.4) we have

$$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n < x_{n+1} < \cdots. \tag{2.5}$$

Since (2.5) holds, then condition (2.1) implies that

$$d(Tx_n, Tx_{n+1}) = d\big(Tfx_{n-1}, Tfx_n\big) \le \phi(M(x_{n-1}, x_n)), \tag{2.6}$$

where,

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{ \frac{d(Tx_{n-1}, Tfx_{n-1})d(Tx_n, Tfx_n)}{d(Tx_{n-1}, Tx_n)}, d(Tx_{n-1}, Tx_n) \right\} \\ &= \max\{d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\}. \end{aligned} \tag{2.7}$$

Suppose that $M(x_{n-1}, x_n) = d(Tx_n, Tx_{n+1})$ for some $n \ge 1$. Then, inequality (2.6) turns into

$$d(Tx_n, Tx_{n+1}) \le \phi(d(Tx_n, Tx_{n+1})). \tag{2.8}$$

Regarding (2.5) and the property (a) of ϕ, we get

$$d(Tx_n, Tx_{n+1}) \le \phi(d(Tx_n, Tx_{n+1})) < d(Tx_n, Tx_{n+1}), \tag{2.9}$$

which is a contradiction. Thus, $M(x_{n-1}, x_n) = d(Tx_{n-1}, Tx_n)$ for all $n \ge 1$. Therefore, the inequality (2.6) yields that

$$d(Tx_n, Tx_{n+1}) \le \phi(d(Tx_{n-1}, Tx_n)) < d(Tx_{n-1}, Tx_n). \tag{2.10}$$

Consequently, the sequence $\{d(Tx_{n-1}, Tx_n)\}$ of positive real numbers is decreasing and bounded below. So, there exists $L \ge 0$ such that $\lim_{n\to\infty} d\,(Tx_{n-1}, Tx_n) = L$.

We claim that $L = 0$. Suppose to the contrary that $L > 0$. Letting $\limsup_{n\to\infty}$ in (2.10) and using the fact that ϕ is upper semicontinuous from the right, we get

$$L = \limsup_{n\to\infty} d(Tx_n, Tx_{n+1}) \le \limsup_{n\to\infty} \phi(d(Tx_{n-1}, Tx_n)) \le \phi(L) < L, \tag{2.11}$$

which is a contradiction. Hence, we conclude that $L = 0$, that is,

$$\lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = 0. \tag{2.12}$$

We prove that the sequence $\{Tx_n\}$ is Cauchy in X. Suppose, to the contrary, that $\{Tx_n\}$ is not a Cauchy sequence. So, there exists $\varepsilon > 0$ such that

$$d\big(Tx_{m(k)}, Tx_{n(k)}\big) \ge \varepsilon, \tag{2.13}$$

where $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ are subsequences of $\{Tx_n\}$ with

$$n(k) > m(k) \ge k. \tag{2.14}$$

Moreover, $n(k)$ is chosen as the smallest integer satisfying (2.13). Thus, we have

$$d(Tx_{m(k)}, Tx_{n(k)-1}) < \varepsilon. \tag{2.15}$$

By the triangle inequality, we get

$$\begin{aligned} \varepsilon \le d(Tx_{m(k)}, Tx_{n(k)}) &\le d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)}) \\ &< d(Tx_{n(k)}, Tx_{n(k)-1}) + \varepsilon. \end{aligned} \tag{2.16}$$

Letting $k \to \infty$ in above inequality and using (2.12), we get that

$$\lim_{k\to\infty} d(Tx_{m(k)}, Tx_{n(k)}) = \varepsilon. \tag{2.17}$$

By a triangle inequality, we have

$$\begin{aligned} d(Tx_{m(k)}, Tx_{n(k)}) &\le d(Tx_{m(k)}, Tx_{m(k)-1}) + d(Tx_{m(k)-1}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)}), \\ d(Tx_{m(k)-1}, Tx_{n(k)-1}) &\le d(Tx_{m(k)-1}, Tx_{m(k)}) + d(Tx_{m(k)}, Tx_{n(k)}) + d(Tx_{n(k)}, Tx_{n(k)-1}). \end{aligned} \tag{2.18}$$

Using (2.12), (2.17) and letting $k \to \infty$ in (2.18), we get

$$\lim_{k\to\infty} d(Tx_{m(k)-1}, Tx_{n(k)-1}) = \varepsilon. \tag{2.19}$$

Regarding $m(k) < n(k)$, we have $x_{m(k)-1} < x_{n(k)-1}$. From (2.1) we have

$$\begin{aligned} d(Tx_{m(k)}, Tx_{n(k)}) &= d(Tfx_{m(k)-1}, Tfx_{n(k)-1}) \\ &\le \phi(M(x_{m(k)-1}, x_{n(k)-1})), \end{aligned} \tag{2.20}$$

where,

$$\begin{aligned} &M(x_{m(k)-1}, x_{n(k)-1}) \\ &\quad = \max\left\{ \frac{d(Tx_{m(k)-1}, Tfx_{m(k)-1})d(Tx_{n(k)-1}, Tfx_{n(k)-1})}{d(Tx_{m(k)-1}, Tx_{n(k)-1})}, d(Tx_{m(k)-1}, Tx_{n(k)-1}) \right\} \\ &\quad = \max\left\{ \frac{d(Tx_{m(k)-1}, Tx_{m(k)})d(Tx_{n(k)-1}, Tx_{n(k)})}{d(Tx_{m(k)-1}, Tx_{n(k)-1})}, d(Tx_{m(k)-1}, Tx_{n(k)-1}) \right\}. \end{aligned} \tag{2.21}$$

Letting $k \to \infty$ in (2.20) (and hence in (2.21)), and using (2.12), (2.17) and (2.19), we obtain

$$\varepsilon \le \phi(\max\{0, \varepsilon\}) = \phi(\varepsilon) < \varepsilon, \tag{2.22}$$

which is a contradiction. Thus, $\{Tx_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, there exists $z \in X$ such that $\lim_{n\to\infty} Tx_n = z$. Since T is an ICS mapping, there exists $x \in X$ such that

$$\lim_{n\to+\infty} x_n = x. \tag{2.23}$$

But T is continuous, hence

$$z = \lim_{n\to+\infty} Tx_n = Tx. \tag{2.24}$$

We will show that x is a fixed point of f.

Assume that (i) holds. Then by continuity of f, we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} fx_{n-1} = f\left(\lim_{n\to\infty} x_{n-1}\right) = fx. \tag{2.25}$$

Suppose that (ii) holds. Since $\{x_n\}$ is a nondecreasing sequence and $\lim_{n\to\infty} x_n = x$ then $x = \sup\{x_n\}$. Hence, $x_n \leq x$ for all $n \in \mathbb{N}$. Regarding that f is a nondecreasing mapping, we conclude that $fx_n \leq fx$, or equivalently,

$$x_n \leq x_{n+1} \leq fx, \quad \forall n \in \mathbb{N} \tag{2.26}$$

and as $x = \sup\{x_n\}$, we get $x \leq fx$.

To this end, we construct a new sequence $\{y_n\}$ as follows:

$$y_0 = x, \qquad y_n = fy_{n-1}, \quad \forall n \geq 1. \tag{2.27}$$

Since $x \leq fx$, so we have $y_0 \leq fy_0 = y_1$ and hence similarly we may find that $\{y_n\}$ is a nondecreasing sequence. By repeating the discussion above, one can conclude that $\{Ty_n\}$ is a Cauchy sequence. Thus, $\{Ty_n\}$ converges and since T is an ICS mapping, so there exists $y \in X$ such that $\lim_{n\to\infty} y_n = y$. The mapping T is continuous, hence

$$\lim_{n\to+\infty} Ty_n = Ty. \tag{2.28}$$

By (ii), we have $y = \sup\{y_n\}$ and so we have $y_n \leq y$. From (2.26), we get

$$x_n < x = y_0 \leq fx = fy_0 \leq y_n \leq y, \quad \forall n \in \mathbb{N}. \tag{2.29}$$

If $x = y$, then the proof ends.

Suppose that $x \neq y$ and since the mapping T is one to one, we have $Tx \neq Ty$, so $d(Tx, Ty) > 0$. On account of (2.29), the expression (2.1) implies that

$$d(Tx_{n+1}, Ty_{n+1}) = d(Tfx_n, Tfy_n) \leq \phi(M(x_n, y_n)), \tag{2.30}$$

where,

$$M(x_n, y_n) = \max\left\{\frac{d(Tx_n, Tfx_n)d(Ty_n, Tfy_n)}{d(Tx_n, Ty_n)}, d(Tx_n, Ty_n)\right\}$$
$$= \max\left\{\frac{d(Tx_n, Tx_{n+1})d(Ty_n, Ty_{n+1})}{d(Tx_n, Ty_n)}, d(Tx_n, Ty_n)\right\}. \quad (2.31)$$

Letting $n \to \infty$ in (2.30) and using (2.24), (2.28), we obtain

$$d(Tx, Ty) \leq \phi(d(Tx, Ty)) < d(Tx, Ty), \quad (2.32)$$

which is a contradiction. So $x = y$ and we have $x \leq fx \leq x$, then $fx = x$. □

Corollary 2.2. *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a complete metric space. Let $f, T : X \to X$ be such that T is an ICS mapping and f is a nondecreasing mapping satisfying*

$$d(Tfx, Tfy) \leq kM(x, y), \quad (2.33)$$

for all distinct $x, y \in X$ with $x \leq y$. Also, assume either

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point.

Proof. Take $\phi(t) = (1 - k)t$ for all $t \in [0, \infty)$ in Theorem 2.1. □

Corollary 2.3. *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a complete metric space. Let $f, T : X \to X$ be such that T is an ICS mapping and f is nondecreasing mapping with*

$$d(Tfx, Tfy) \leq \alpha\frac{d(Tx, Tfx)d(Ty, Tfy)}{d(Tx, Ty)} + \beta d(Tx, Ty), \quad (2.34)$$

for all distinct $x, y \in X$ with $x \leq y$ where $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$. Also, assume either

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point.

Proof. Take $k = \alpha + \beta$ for all $t \in [0, \infty)$ in Corollary 2.2. Indeed,

$$
\begin{aligned}
d(Tfx, Tfy) &\leq \alpha \frac{d(Tx, Tfx)d(Ty, Tfy)}{d(Tx, Ty)} + \beta d(Tx, Ty) \\
&\leq (\alpha + \beta) \max\left\{ \frac{d(Tx, Tfx)d(Ty, Tfy)}{d(Tx, Ty)}, d(Tx, Ty) \right\},
\end{aligned}
\tag{2.35}
$$

and this completes the proof. □

Theorem 2.4. *In addition to hypotheses of Theorem 2.1, assume that*

$$
\text{for every } x, y \in X \text{ there exists } z \in X \text{ that is comparable to } x \text{ and } y, \tag{2.36}
$$

then f has a unique fixed point.

Proof. Suppose, to the contrary, that x and y are fixed points of f where $x \neq y$. By (2.36), there exists a point $z \in X$ which is comparable to x and y. Without loss of generality, we choose $z \leq x$. We construct a sequence $\{z_n\}$ as follows:

$$
z_0 = z, \qquad z_n = f z_{n-1}, \quad \forall n \geq 1. \tag{2.37}
$$

Since f is nondecreasing, $z \leq x$ implies $z_1 = fz_0 = fz \leq fx = x$. By induction, we get $z_n \leq x$.

If $x = z_{N_0}$ for some $N_0 \geq 1$ then $z_n = fz_{n-1} = fx = x$ for all $n \geq N_0 - 1$. So $\lim_{n \to \infty} z_n = x$. Analogously, we get that $\lim_{n \to \infty} z_n = y$ which completes the proof.

Consider the other case, that is, $x \neq z_n$ for all $n = 0, 1, 2, \ldots$. Having in mind, T is one to one, so $d(Tx, Tz_n) > 0$ for any $n \geq 0$. Then, by (2.1), we observe that

$$
d(Tz_n, Tx) = d(Tfz_{n-1}, tfx) \leq \phi(M(z_{n-1}, x)), \tag{2.38}
$$

where,

$$
\begin{aligned}
M(z_{n-1}, x) &= \max\left\{ \frac{d(Tx, Tfx)d(Tz_{n-1}, Tfz_{n-1})}{d(Tx, Tz_{n-1})}, d(Tx, Tz_{n-1}) \right\} \\
&= \max\left\{ \frac{d(Tx, Tx)d(Tz_{n-1}, Tz_n)}{d(Tz_{n-1}, Tx)}, d(Tz_{n-1}, Tx) \right\} \\
&= d(Tz_{n-1}, Tx).
\end{aligned}
\tag{2.39}
$$

Thus,

$$
d(Tz_n, Tx) \leq \phi(d(Tz_{n-1}, Tx)) < d(Tz_{n-1}, Tx). \tag{2.40}
$$

Consequently, $\{d(Tz_n, Tx)\}$ is a decreasing sequence of positive real numbers which is bounded below. So, there exists $L \geq 0$ such that $\lim_{n \to \infty} d(Tx, Tz_n) = L$.

We claim that $L = 0$. Suppose, to the contrary, that $L > 0$. Taking $\limsup_{n\to\infty}$ and using a property of ϕ, we get

$$L \leq \phi(L) < L, \tag{2.41}$$

which is a contradiction. Hence, we conclude that $L = 0$, that is,

$$\lim_{n\to\infty} d(Tx, Tz_n) = 0. \tag{2.42}$$

Analogously, repeating the same work we find that

$$\lim_{n\to\infty} d(Ty, Tz_n) = 0. \tag{2.43}$$

By uniqueness of limit of $\{Tz_n\}$, we deduce that $Tx = Ty$ and since T is one to one, we have $x = y$, which is a contradiction. This ends the proof. □

Recently, Jachymski [32] in his interesting paper showed the equivalence between several generalized contractions on (ordered) metric spaces. Since the key of his study is [32, Lemma 1], then we will combine this lemma with Theorems 2.1 and 2.4 to deduce the following.

Corollary 2.5. *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a complete metric space. Let $f, T : X \to X$ be such that T is an ICS mapping and f a nondecreasing mapping satisfying*

$$\psi(d(Tfx, Tfy)) \leq \psi(M(x, y)) - \phi(M(x, y)) \tag{2.44}$$

for all distinct $x, y \in X$ with $x \leq y$, where $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and nondecreasing functions such that $\psi^{-1}(\{0\}) = \phi^{-1}(\{0\}) = \{0\}$. Also, suppose that (2.36) *holds and assume either*

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point. Moreover, if for any $x, y \in X$, there is $z \in X$ that is comparable to x and y, then f has a unique fixed point.

Proof. By [32, Lemma 1], (ii) ⇒ (viii), so there exists a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$ and $d(Tfx, Tfy) \leq \phi(M(x, y))$ for all distinct $x, y \in X$ with $x \leq y$. Therefore, Theorems 2.1 and 2.4 give, respectively, existence and uniqueness of the fixed point of f, which completes the proof. □

The following remarks are in order.

(i) Corollary 2.5 corresponds to Theorems 2.1 and 2.4 of Luong and Thuan [21] by taking $Tx = x$ and $\psi(t) = t$ for all $x \in X$ and $t \geq 0$.

(ii) Corollary 2.3 corresponds to Theorems 2.2 and 2.3 of Harjani et al. [17] by taking $Tx = x$.

Now we give some examples illustrating our obtained results.

Example 2.6. Let $X = \{0, 1, 2, 8\}$ be endowed with the metric $d(x, y) = |x - y|$ for all $x, y \in X$ and the order $\leq_X$ given as follows:

$$x \leq_X y \Longleftrightarrow x = y \quad \text{or} \quad \left(x \leq y,\ x, y \in \{0, 1, 2\}\right). \tag{2.45}$$

Take the mappings $f, T : X \to X$ be given by

$$f = \begin{pmatrix} 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 & 2 & 8 \\ 0 & 1 & 8 & 2 \end{pmatrix}. \tag{2.46}$$

Set $\phi(t) = (1/2)t$. It is easy that f is nondecreasing with respect to $\leq_X$. First, X satisfies the property: if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$. Indeed, let $\{z_n\}$ be a nondecreasing sequence in X with respect to $\leq_X$ such that $z_n \to z \in X$ as $n \to +\infty$. We have $z_n \leq_X z_{n+1}$ for all $n \in \mathbb{N}$.

(i) If $z_0 = 2$, then $z_0 = 2 \leq_X z_1$. From the definition of $\leq_X$, we have $z_1 = 2$. By induction, we get $z_n = 2$ for all $n \in \mathbb{N}$ and $z = 2$. Then, $z_n \leq_X z$ for all $n \in \mathbb{N}$ and $z = \sup\{z_n\}$.

(ii) If $z_0 = 1$, then $z_0 = 1 \leq_X z_1$. From the definition of $\leq_X$, we have $z_1 \in \{1, 2\}$. By induction, we get $z_n \in \{1, 2\}$ for all $n \in \mathbb{N}$. Suppose that there exists $p \geq 1$ such that $z_p = 2$. From the definition of $\leq_X$, we get $z_n = z_p = 2$ for all $n \geq p$. Thus, we have $z = 2$ and $z_n \leq_X z$ for all $n \in \mathbb{N}$. Now, suppose that $z_n = 1$ for all $n \in \mathbb{N}$. In this case, we get $z = 1$ and $z_n \leq_X z$ for all $n \in \mathbb{N}$ and $z = \sup\{z_n\}$.

(iii) If $z_0 = 0$, then $z_0 = 5 \leq_X z_1$. From the definition of $\leq_X$, we have $z_1 \in \{0, 1, 2\}$. Repeating the same idea as previous case, we get that $z = \sup\{z_n\}$.

Thus, we proved that in all cases, we have $z = \sup\{z_n\}$. We will show that (2.1) holds for all $x, y \in X$ with $x \neq y$. The unique possibilities are $(x = 0, y = 1)$, $(x = 0, y = 2)$ and $(x = 1, y = 2)$.

Case 1. If $x = 0$ and $y = 1$, we have $d(Tf0, Tf1) = 0$, so (2.1) holds.

Case 2. If $x = 0$ and $y = 2$, we have $d(Tf0, Tf2) = 1$ and $M(0, 2) = 8$, so $d(Tf0, Tf2) \leq \phi(M(0, 2))$.

Case 3. If $x = 1$ and $y = 2$, we have $d(Tf1, Tf2) = 1$ and $M(1, 2) = 7$, so $d(Tf1, Tf2) \leq \phi(M(1, 2))$.

Also, it is obvious that T is an ICS mapping. All hypotheses of Theorem 2.1 are satisfied and $u = 0$ is a fixed point of T.

On the other hand, taking $x = 1$ and $y = 2$, we have

$$1 = d(f1, f2) > 1 - \phi(1) = 1 - \phi(N(1,2)), \tag{2.47}$$

for each ϕ given in Theorem 1.3. Hence, the main result of Luong and Thuan [21] is not applicable.

Moreover, taking $x = 1$ and $y = 2$, we have

$$1 = d(f1, f2) > \alpha + \beta = \alpha \frac{d(1, f1)d(2, f2)}{d(1,2)} + \beta d(1,2), \tag{2.48}$$

for each $\alpha, \beta \geq 0$, $\alpha + \beta < 1$, so (1.2) fails. Then, we couldn't apply Theorem 1.1 (also, the same for Theorems 2.2 and 2.3 of Harjani et al. [17]).

Example 2.7. Let $X = \{0, 1, 2, \ldots\}$ with usual order. Define $T, f : X \to X$ by the formulas

$$Tx = x + 1, \qquad fx = \begin{cases} 0, & x = 0; \\ x - 1, & x \neq 0. \end{cases} \tag{2.49}$$

Let $d : X \times X \to \mathbb{R}^{+}$ be given by

$$d(x, y) = \begin{cases} 0, & x = y; \\ x + y, & x \neq y. \end{cases} \tag{2.50}$$

Define $\psi, \phi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = t^2$ and $\phi(t) = t$. Then

(1) $(X, d, \leq)$ is a complete ordered metric space;

(2) f is non-decreasing;

(3) f is continuous;

(4) T is an ICS mapping;

(5) $\psi(d(Tfx, Tfy)) \leq \psi(M(x, y)) - \phi(M(x, y))$ for any distinct $x, y \in X$ with $x \leq y$.

Proof. The proof of (1) and (2) is clear. To prove (3), let (x_n) be a sequence in X such that $x_n \to x \in X$. By the definition of the metric d, there exists $k \in \mathbb{N}$ such that $x_n = x$ for all $n \geq k$. So $fx_n = fx$ for all $n \geq k$. Hence $fx_n \to fx$. So f is continuous. To prove (4), it is clear that T is injective and continuous. Now, let (x_n) be any sequence in X such that (Tx_n) converges to some $x \in X$. Then there exists $k \in \mathbb{N}$ such that $Tx_n = x_n + 1 = x$ for all $n \geq k$. Thus $x_n = x - 1$

for all $n \geq k$. Hence $x_n \to x-1$. So T is an ICS mapping. To prove (5), given $x, y \in X$ with $x > y$. If $y \neq 0$, then

$$\begin{aligned} d(Tfx, Tfy) &= d(x,y) = x+y, \\ M(x,y) &= \max\left\{ \frac{d(x+1,x)d(y+1,y)}{d(x+1,y+1)}, x+y+2 \right\} \\ &= \max\left\{ \frac{(2x+1)(2y+1)}{x+y+2}, x+y+2 \right\} = x+y+2. \end{aligned} \tag{2.51}$$

Since

$$(x+y)^2 \leq (x+y+2)^2 - (x+y+2), \tag{2.52}$$

we have $\psi(d(Tfx, Tfy)) \leq \psi(M(x,y)) - \phi(M(x,y))$.

If $x > 0$ and $y = 0$, then

$$d(Tfx, Tf0) = \begin{cases} 0, & x = 1; \\ x+1, & x \neq 1, \end{cases} \tag{2.53}$$

and $M(x,0) = d(x+1,1) = x+2$. Since

$$(x+1)^2 \leq (x+2)^2 - (x+2), \tag{2.54}$$

we have $\psi(d(Tfx, Tfy)) \leq \psi(M(x,y)) - \phi(M(x,y))$. Then T, f, ψ and ϕ satisfy all conditions of Corollary 2.5, so f has a unique fixed point, which is $u = 0$. □

Finally, we give a simple example which shows that if T is not an ICS mapping then the conclusion of Theorem 2.1 fails.

Example 2.8. Let $X = \mathbb{R}$ with the usual metric and the usual ordering. Take $fx = x+2$ and $\phi(t) = t/(1+t)$ for all $x \in X$ and $t \geq 0$. The mapping f is nondecreasing and continuous. Also, $\phi \in \Phi$ and there exists $x_0 = 0$ such that $x_0 \leq fx_0$.

Let $T : X \to X$ be such that $Tx = 1$ for all $x \in X$, then T is not an ICS mapping. Obviously, the condition (2.1) holds. However, f has no fixed point.

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