Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 260457, 11 pages
doi:10.1155/2012/260457

Research Article

Generalized Caratheodory Extension Theorem on Fuzzy Measure Space

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Received 23 April 2012; Revised 28 June 2012; Accepted 10 October 2012

Academic Editor: Alberto D'Onofrio

Lattice-valued fuzzy measures are lattice-valued set functions which assign the bottom element of the lattice to the empty set and the top element of the lattice to the entire universe, satisfying the additive properties and the property of monotonicity. In this paper, we use the lattice-valued fuzzy measures and outer measure definitions and generalize the Caratheodory extension theorem for lattice-valued fuzzy measures.

1. Introduction

Recently studies including the fuzzy convergence [1], fuzzy soft multiset theory [2], lattices of fuzzy objects [3], on fuzzy soft sets [4], fuzzy sets, fuzzy S-open and S-closed mappings [5], the intuitionistic fuzzy normed space of coefficients [6], set-valued fixed point theorem for generalized contractive mapping on fuzzy metric spaces [7], the centre of the space of Banach lattice-valued continuous functions on the generalized Alexandroff duplicate [8], (L, M)-fuzzy σ-algebras [9], fuzzy number-valued fuzzy measure and fuzzy number-valued fuzzy measure space [10–12], construction of a lattice on the completion space of an algebra and an isomorphism to its Caratheodory extension [13], fuzzy sets [14, 15], generalized σ-algebras and generalized fuzzy measures [16], generalized fuzzy sets [17–19], common fixed points theorems for commutating mappings in fuzzy metric spaces [20], and fuzzy measure theory [21] have been investigated.

The well-known Caratheodory extension theorem in classical measure theory is very important [22, 23]. In a graduate course in real analysis, students learn the Caratheodory extension theorem, which shows how to extended an algebra to a σ-algebra, and a finitely additive measure on the algebra to a countable additive measure on the σ-algebra [13]. In this paper, first we give new definition for lattice-valued-fuzzy measure on $[-\infty, \infty]$, which is more general than that of [24]. Using this new definition, we provide new proof of Caratheodory extension theorem for lattice valued-fuzzy measure. In related literature, not many studies have been explored including Caratheodory extension theorem on lattice-valued fuzzy measure. In [25], Sahin used the definitions given in [19] and generalized Caratheodory extension theorem for fuzzy sets. In [24], lattice-valued fuzzy measure and fuzzy integral were studied on $[0, \infty]$. However, no study has been done related to Caratheodory extension theorem for lattice-valued fuzzy measure. This provides the motivation for present paper where we provide the proof of generalized Caratheodory extension theorem for lattice-valued fuzzy measure space.

The outline of the paper is as follows. In the next section, basic definitions of lattice theory, lattice σ-algebra, are given. In Section 3, definitions for lattice-valued fuzzy σ-algebra, and lattice-valued fuzzy outer measure are given, and some necessary theorems for our main theorem (generalized caratheodory extension theorem) related to lattice-valued fuzzy outer measure and main theorem of this paper are given.

2. Preliminaries

In this section, we shall briefly review the well-known facts about lattice theory [26, 27], purpose an extension lattice, and investigate its properties. $(L, \wedge, \vee)$ or simply L under closed operations $\wedge, \vee$ is called a lattice. For two lattices L and L^*, a bijection from L to L^*, which preserves lattice operations is called a lattice isomorphism, or simply an isomorphism. If there is an isomorphism from L to L^*, then L is called a lattice isomorphic with L^*, and we write $L \cong L^*$. We write $x \leq y$ if $x \wedge y = x$ or, equivalently, if $x \wedge y = y$. L is called complete, if any subset A of L includes the supremum $\vee A$ and infimum $\wedge A$, with respect to the above order. A complete lattice L includes the maximum and minimum elements, which are denoted L_1 and L_0.

Throughout this paper, X will be denoted the entire set and L is a lattice of any subset sets of X.

Definition 2.1 (see [28]). If a lattice L satisfies the following conditions, then it is called a lattice σ-algebra.

(i) For all $f \in L$, $f^c \in L$.

(ii) If $f_n \in L$ for $n = 1, 2, 3, \ldots$, then $\bigvee_{n=1}^{\infty} f_n \in L$.

It is denoted $\sigma(L)$ as the lattice σ-algebra generated by L.

Definition 2.2 (see [29]). A lattice-valued set μ_E is called lattice-valued m^*-measurable if for every $\mu_A \leq \mu_X$,

$$m^*(\mu_A) = m^*(\mu_A \wedge \mu_E) + m^*\left(\mu_A \wedge \mu_E^C\right). \tag{2.1}$$

This is equivalent to requiring only $m^*(\mu_A) \geq m^*(\mu_A \wedge \mu_E) + m^*(\mu_A \wedge \mu_E^C)$, since the converse inequality is obvious from the subadditive property of m^*.

Also, $M = \{\mu_E : \mu_E \text{ is } m^*\text{-measurable}\}$ is a class of all lattice-valued measurable sets.

Theorem 2.3 (see [29]). *Let μ_{E_1} and μ_{E_2} be measureable lattice-valued sets. Then,*

$$\begin{gathered} m^*\left(\mu_{E_1} \wedge \mu_{E_1}^C\right) = 0, \\ m^*\left(\mu_{E_1} \vee \mu_{E_2}\right) = m^*\left(\mu_{E_1}\right) + m^*\left(\mu_{E_2} \wedge \mu_{E_1}^C\right). \end{gathered} \tag{2.2}$$

3. Main Results

Throughout this paper, we will consider lattices as complete lattices, X will denote space, and μ is a membership function of any fuzzy set X.

Definition 3.1. If $m : \sigma(L) \to R \cup \{\infty\}$ satisfies the following properties, then m is called a lattice measure on the lattice σ-algebra $\sigma(L)$.

(i) $m(\emptyset) = L_0$.

(ii) For all $f, g \in \sigma(L)$ such that $m(f), m(g) \geq L_0 : f \leq g \Rightarrow m(f) \leq m(g)$.

(iii) For all $f, g \in \sigma(L) : m(f \vee g) + m(f \wedge g) = m(f) + m(g)$.

(iv) $f_n \subset \sigma(L)$, $n \in N$ such that $f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$, then $m(\bigvee_{n=1}^{\infty} f_n) = \lim_{n\to\infty} m(f_n)$.

Definition 3.2. Let m_1 and m_2 be lattice measures defined on the same lattice σ-algebra $\sigma(L)$. If one of them is finite, the set function $m(E) = m_1(E) - m_2(E)$, $E \in \sigma(L)$ is well defined and countable additive $\sigma(L)$.

Definition 3.3. If a family $\sigma(L)$ of membership functions on X satisfies the following conditions, then it is called a lattice fuzzy σ-algebra.

(i) For all $\alpha \in L$, $\alpha \in \sigma(L)$, (α constant).

(ii) For all $\mu \in \sigma(L)$, $\mu^C = 1 - \mu \in \sigma(L)$.

(iii) If $(\mu_n) \in \sigma(L)$, $\sup(\mu_n) \in \sigma(L)$ for all $n \in N$.

Definition 3.4. If $m : \sigma(L) \to R \cup \{\infty\}$ satisfies the following properties, then m is called a lattice-valued fuzzy measure.

(i) $m(\emptyset) = L_0$.

(ii) For all $\mu_1, \mu_2 \in \sigma(L)$ such that $m(\mu_1), m(\mu_2) \geq L_0 : \mu_1 \leq \mu_2 \Rightarrow m(\mu_1) \leq m(\mu_2)$.

(iii) For all $\mu_1, \mu_2 \in \sigma(L) : m(\mu_1 \vee \mu_2) + m(\mu_1 \wedge \mu_2) = m(\mu_1) + m(\mu_2)$.

(iv) $(\mu_n) \in \sigma(L)$, $n \in N$ such that $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots$; $\sup(\mu_n) = \mu \quad \Rightarrow \quad m(\mu_n) = \lim_{n\to\infty} m(\mu_n)$.

Definition 3.5. With a lattice-valued fuzzy outer measure m^* having the following properties, we mean an extended lattice-valued set function defined on L^X:

(i) $m^*(\emptyset) = L_0$,

(ii) $m^*(\mu_1) \leq m^*(\mu_2)$ for $\mu_1 \leq \mu_2$,

(iii) $m^*(\bigvee_{n=1}^{\infty} \mu_{E_n}) \leq (\bigvee_{n=1}^{\infty} m^*(\mu_{E_n}))$.

Example 3.6. Suppose

$$m^* = \begin{cases} L_0, & \mu_E = \emptyset, \\ L_1, & \mu_E \neq \emptyset. \end{cases} \tag{3.1}$$

L_0 is infimum of sets of lattice family, and L_1 is supremum of sets of lattice family.

If X has at least two member, then m^* is a lattice-valued fuzzy outer measure which is not lattice-valued fuzzy measure on L^X.

Proposition 3.7. *Let F be a class of fuzzy sublattice sets of X containing L_0 such that for every $\mu_A \leq \mu_X$, there exists a sequence $(\mu_{B_n})_{n=1}^{\infty}$ from F such that $\mu_A \leq (\mu_{B_n})_{n=1}^{\infty}$. Let ψ be an extended lattice-valued function on F such that $\psi(\emptyset) = L_0$ and $\psi(\mu_A) \geq L_0$ for $\mu_A \in F$. Then, m^* is defined on L^X by*

$$m^*(\mu_A) = \inf\{\psi(\mu_{B_n})_{n=1}^{\infty} : \mu_{B_n} \in F, \mu_A \leq \mu_{B_n}\}, \tag{3.2}$$

and m^* is a lattice fuzzy outer measure.

Proof. (i) $m^*(\emptyset) = L_0$ is obvious.

(ii) If $\mu_{A_1} \leq \mu_{A_2}$ and $\mu_{A_2} \leq (\mu_{B_n})_{n=1}^{\infty}$, then $\mu_{A_1} \leq (\mu_{B_n})_{n=1}^{\infty}$. This means that $m^*(\mu_{A_1}) \leq m(\mu_{A_2})$.

(iii) Let $\mu_{E_n} \leq \mu_X$ for each natural number n. Then, $m^*(\mu_{E_n}) = \infty$ for some n. $m^*(\bigvee_{n=1}^{\infty}\mu_{E_n}) \leq (\bigvee_{n=1}^{\infty} m^*(\mu_{E_n}))$.

The following theorem is an extension of the above proposition. □

Theorem 3.8. *The class B of m^* lattice-valued fuzzy measurable sets is a σ-algebra. Also, $\overline{m}$ the restriction m^* of to B is a lattice valued fuzzy measure.*

Proof. It follows from extension of the proposition. □

Now, we shall generalize the well-known Caratheodory extension theorem in classical measure theory for lattice-valued fuzzy measure.

Theorem 3.9 (Generalized Caratheodory Extension Theorem). *Let m be a lattice valued fuzzy measure on a σ-algebra $(L) \leq L^X$. Suppose for $\mu_E \leq \mu_X$, $m^*(\mu_E) = \inf\{m(\bigvee_{n=1}^{\infty}\mu_{E_n}) : \mu_{E_n} \in \sigma(L),\ \mu_E \leq \bigvee_{n=1}^{\infty}\mu_{E_n}\}$.*

Then, the following properties are hold.

(i) *m^* is a lattice-valued fuzzy outer measure.*

(ii) *$\mu_E \in \sigma(L)$ implies $m(\mu_E) = m^*(\mu_E)$.*

(iii) *$\mu_E \in \sigma(L)$ implies μ_E is m^* lattice fuzzy measurable.*

(iv) *The restriction $\overline{m}$ of m^* to the m^*-lattice-valued fuzzy measurable sets in an extension of m to a lattice-valued fuzzy measure on a fuzzy σ-algebra containing (L).*

(v) *If m is lattice-valued fuzzy σ-finite, then $\overline{m}$ is the only lattice fuzzy measure (on the smallest fuzzy σ- algebra containing $\sigma(L)$ that is an extension of m).*

Proof. (i) It follows from Proposition 3.7.

(ii) Since m^* is a lattice-valued fuzzy outer measure, we have

$$m^*(\mu_E) \leq m(\mu_E). \tag{3.3}$$

For given $\varepsilon > 0$, there exists $(\mu_{E_n}; n = 1, 2, \ldots)$ such that $\bigvee_{n=1}^{\infty}(m(\mu_{E_n})) \leq m^*(\mu_E) + \varepsilon$ [29]. Since $\mu_E = \mu_E \wedge (\bigvee_{n=1}^{\infty}\mu_{E_n}) = \bigvee_{n=1}^{\infty}(\mu_E \wedge \mu_{E_n})$ and by the monotonicity and σ-additivity of m, we have

$m(\mu_E) \leq \bigvee_{n=1}^{\infty} m(\mu_E \wedge \mu_{E_n}) \leq \bigvee_{n=1}^{\infty} m(\mu_{E_n}) \leq m^*(\mu_E) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that

$$m(\mu_E) \leq m^*(\mu_E). \tag{3.4}$$

From (3.3) and (3.4), $m(\mu_E) = m^*(\mu_E)$ is obtained.

(iii) Let $\mu_E \in \sigma(L)$. In order to prove μ_E is lattice fuzzy measurable, it suffices to show that

$$m^*(\mu_A) \geq m^*(\mu_A \wedge \mu_E) + m^*(\mu_A \wedge \mu_E^c), \quad \text{for } \mu_A \leq \mu_E. \tag{3.5}$$

For given $\varepsilon > 0$, there exists $\mu_{A_n} \in \sigma(L)$, $1 \leq n < \infty$ such that

$$\bigvee_{n=1}^{\infty} m(\mu_{A_n}) \leq m^*(\mu_A) + \varepsilon, \quad \mu_A \leq \bigvee_{n=1}^{\infty} (\mu_{A_n}). \tag{3.6}$$

Now,

$$\begin{aligned} \mu_A \wedge \mu_E &\leq \bigvee_{n=1}^{\infty} (\mu_{A_n} \wedge \mu_E), \\ \mu_A \wedge \mu_E^c &\leq \bigvee_{n=1}^{\infty} (\mu_{A_n} \wedge \mu_E^c). \end{aligned} \tag{3.7}$$

Therefore,

$$\begin{aligned} m^*(\mu_A \wedge \mu_E) &\leq \bigvee_{n=1}^{\infty} m(\mu_{A_n} \wedge \mu_E), \\ m^*(\mu_A \wedge \mu_E^c) &\leq \bigvee_{n=1}^{\infty} m(\mu_{A_n} \wedge \mu_E^c). \end{aligned} \tag{3.8}$$

From inequalities (3.6) and (3.8), the inequality (3.5) follows.

(iv) Let $\overline{m}$ be the restriction of m^* to the m^* lattice-valued measurable sets, when we write $\overline{m} = m^*/\sigma(\overline{L})$. Now, we must show that $\sigma(\overline{L})$ is a lattice fuzzy σ-algebra containing $\sigma(L)$ and $\overline{m}$ is a lattice-valued fuzzy measure on $\sigma(L)$. We show it step by stepin the following.

Step 1. If $\mu_A, \mu_B \in \sigma(\overline{L})$, then $\mu_A \vee \mu_B \in \sigma(\overline{L})$. It also implies that

$$m^*(\mu_E) = m^*(\mu_E \wedge \mu_B) + m^*(\mu_E \wedge \mu_B^c). \tag{3.9}$$

If we write $\mu_E \wedge \mu_A$ instead of μ_E in (3.9),

$$m^*(\mu_E \wedge \mu_A) = m^*(\mu_E \wedge \mu_A \wedge \mu_B) + m^*(\mu_E \wedge \mu_A \wedge \mu_B^c) \tag{3.10}$$

is obtained. Now, if we write $\mu_E \wedge \mu_A^c$ instead of μ_E in (3.9),

$$m^*(\mu_E \wedge \mu_A^c) = m^*(\mu_E \wedge \mu_A^c \wedge \mu_B) + m^*(\mu_E \wedge \mu_A^c \wedge \mu_B^c) \tag{3.11}$$

is obtained. If we aggregate with (3.10) and (3.11); we have

$$\begin{aligned} m^*(\mu_E) = {} & m^*(\mu_E \wedge \mu_A \wedge \mu_B) + m^*(\mu_E \wedge \mu_A \wedge \mu_B^c) + m^*(\mu_E \wedge \mu_A^c \wedge \mu_B) \\ & + m^*(\mu_E \wedge \mu_A^c \wedge \mu_B^c). \end{aligned} \tag{3.12}$$

If we write $\mu_E \wedge (\mu_A \vee \mu_B)$ instead of μ_E in (3.12), then we get

$$\begin{aligned} m^*(\mu_E \wedge (\mu_A \vee \mu_B)) &= m^*(\mu_E \wedge (\mu_A \vee \mu_B) \wedge \mu_A \wedge \mu_B) + m^*(\mu_E \wedge (\mu_A \vee \mu_B) \wedge (\mu_A^c \wedge \mu_B)) \\ &\quad + m^*(\mu_E \wedge (\mu_A \vee \mu_B) \wedge \mu_A \wedge \mu_B^c) + m^*(\mu_E \wedge (\mu_A \vee \mu_B) \wedge \mu_A^c \wedge \mu_B^c) \\ &= m^*(\mu_E \wedge \mu_A \wedge \mu_B) + m^*(\mu_E \wedge \mu_A^c \wedge \mu_B) + m^*(\mu_E \wedge \mu_A \wedge \mu_B^c) + m^*(L_0) \\ &= m^*(\mu_E \wedge \mu_A \wedge \mu_B) + m^*(\mu_E \wedge \mu_A^c \wedge \mu_B) + m^*(\mu_E \wedge \mu_A \wedge \mu_B^c). \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we obtain

$$m^*(\mu_E) = m^*(\mu_E \wedge (\mu_A \vee \mu_B)) + m^*(\mu_E \wedge (\mu_A \vee \mu_B)^c). \tag{3.14}$$

Step 2. If $\mu_A \in \sigma(\overline{L})$, then $\mu_A^C \in \sigma(\overline{L})$. If we write μ_A^c instead of μ_A in the equality

$$m^*(\mu_E) = m^*(\mu_E \wedge \mu_A) + m^*(\mu_E \wedge \mu_A^c), \tag{3.15}$$

we have

$$\begin{aligned} m^*(\mu_E) &= m^*(\mu_E \wedge \mu_A^c) + m^*(\mu_E \wedge (\mu_A^c)^c); \; (\mu_A^c)^c = \mu_A \\ &= m^*(\mu_E \wedge \mu_A^c) + m^*(\mu_E \wedge \mu_A) = m^*(\mu_E). \end{aligned} \tag{3.16}$$

Therefore, it follows that $\mu_A^C \in \sigma(\overline{L})$. Therefore, we showed that $\sigma(\overline{L})$ is the algebra of lattice sets.

Step 3. Let $\mu_A, \mu_B \in \sigma(L)$ and $\mu_A \wedge \mu_B = \emptyset$, From (3.13), we have

$$\begin{aligned} m^*(\mu_E \wedge (\mu_A \vee \mu_B)) &= m^*(\mu_E \wedge \mu_A^c \wedge \mu_B) + m^*(\mu_E \wedge \mu_A \wedge \mu_B^c) \\ &= m^*(\mu_E \wedge \mu_B) + m^*(\mu_E \wedge \mu_A). \end{aligned} \tag{3.17}$$

Step 4. $\sigma(\overline{L})$ is a lattice σ-algebra.

From the previous step, we have for every family of (for each disjoint lattice sets) $(\mu_{B_n}), n = 1, 2, \ldots,$

$$m^*\left(\mu_E \wedge \left(\bigvee_{n=1}^{k} \mu_{B_n}\right)\right) = \bigvee_{n=1}^{k} m^*(\mu_E \wedge \mu_{B_n}). \tag{3.18}$$

Let $\mu_A = \bigvee_{n=1}^{\infty} \mu_{A_n}$ and $\mu_{A_n} \in \sigma(L)$. Then, $\mu_A = \bigvee_{n=1}^{\infty} \mu_{B_n}$, $\mu_{B_n} = (\mu_{A_n} \wedge (\bigvee_{x=1}^{n-1} \mu_{A_x})^c)$, and $\mu_{B_i} \wedge \mu_{B_j} = \emptyset$ for $i \neq j$. Therefore, we obtain following inequality:

$$m^*(\mu_E) \geq m^*\left(\mu_E \wedge \left(\bigvee_{n=1}^{\infty} \mu_{B_n}\right)\right) + m^*\left(\mu_E \wedge \left(\bigvee_{n=1}^{\infty} \mu_{B_n}\right)^c\right). \tag{3.19}$$

Hence, m^* is a lattice σ-semiadditive.

Since $\sigma(\overline{L})$ is an algebra, $\bigvee_{n=1}^{k} \mu_{B_n} \in \sigma(\overline{L})$ for all $n \in N$. The following inequality is satisfied for all n:

$$m^*(\mu_E) \geq m^*\left(\mu_E \wedge \left(\bigvee_{n=1}^{k} \mu_{B_n}\right)\right) + m^*\left(\mu_E \wedge \left(\bigvee_{n=1}^{k} \mu_{B_n}\right)^c\right). \tag{3.20}$$

From the inequality $\mu_E \wedge (\bigvee_{n=1}^{\infty} \mu_{B_n})^c \leq \mu_E \wedge (\bigvee_{n=1}^{\infty} \mu_{B_n})^c$ and monotonicity of lattice-valued fuzzy measure and (3.20), we have

$$m^*(\mu_E) \geq \bigvee_{j=1}^{n} m^*\left(\mu_E \wedge \mu_{B_j}\right) + m^*(\mu_E \wedge \mu_A^c). \tag{3.21}$$

Then, taking the limit of both sides, we get

$$m^*(\mu_E) \geq \bigvee_{j=1}^{\infty} m^*\left(\mu_E \wedge \mu_{B_j}\right) + m^*(\mu_E \wedge \mu_A^c). \tag{3.22}$$

Using the semiadditivity, we have,

$$m^*(\mu_E \wedge \mu_A) = m^*\left(\bigvee_{j=1}^{\infty}\left(\mu_E \wedge \mu_{B_j}\right)\right) = m^*\left(\mu_E \wedge \left(\bigvee_{j=1}^{\infty} \mu_{B_j}\right)\right) \leq m^*\left(\mu_E \wedge \mu_{B_j}\right). \tag{3.23}$$

From (3.22), we have

$$m^*(\mu_E) \geq m^*(\mu_E \wedge \mu_A) + m^*(\mu_E \wedge \mu_A^c). \tag{3.24}$$

Hence, $\mu_A \in \sigma(\overline{L})$. This shows that $\sigma(\overline{L})$ is a lattice fuzzy σ-algebra.

Step 5. $\overline{m} = m^*/\sigma(\overline{L})$ is a lattice fuzzy measure, where we only need to show lattice is σ-additive.

Let $\mu_E = \bigvee_{j=1}^{\infty}\mu_{A_j}$. From (3.22), we have

$$m^*\left(\bigvee_{j=1}^{\infty}\mu_{A_j}\right) \geq \bigvee_{j=1}^{\infty} m^*\left(\mu_{A_j}\right). \tag{3.25}$$

Step 6. We have $\sigma(\overline{L}) \supset \sigma(L)$.

Let $\mu_A \in \sigma(L)$ and $\mu_E \leq \mu_A$. Then, we must show the following inequality:

$$m^*(\mu_E) \geq m^*(\mu_E \wedge \mu_A) + m^*(\mu_E \wedge \mu_A^c). \tag{3.26}$$

If $\mu_E \in \sigma(L)$, then $\mu_E \wedge \mu_A$ and $\mu_E \wedge \mu_A^c$ are different and both of them belong to $\sigma(L)$, (3.26) is obvious and since $m^* = m$, hence additive.

With $\mu_E \leq \mu_X$ and given $\varepsilon > 0$, $\sigma(L)$, there is μ_{E_j} which contains $\sigma(L)$ such that we have

$$m^*(\mu_E) + \varepsilon > \bigvee_{j=1}^{\infty} m\left(\mu_{E_j}\right). \tag{3.27}$$

Now, from the equality

$$\mu_{E_j} = \left(\mu_{E_j} \wedge \mu_A\right) \vee \left(\mu_{E_j} \wedge \mu_A^c\right) \tag{3.28}$$

and from the Definition 2.1 and Theorem 2.3, we have the following equality:

$$m\left(\mu_{E_j}\right) = m\left(\mu_{E_j} \wedge \mu_A\right) + m\left(\mu_{E_j} \wedge \mu_A^c\right). \tag{3.29}$$

Therefore, we obtain the following:

$$\begin{aligned} \mu_E \wedge \mu_A &\leq \bigvee_{j=1}^{\infty}\left(\mu_{E_j} \wedge \mu_A\right), \\ \mu_E \wedge \mu_A^c &\leq \bigvee_{j=1}^{\infty}\left(\mu_{E_j} \wedge \mu_A^c\right). \end{aligned} \tag{3.30}$$

Using the monotonicity and semiadditivity, we obtain

$$\begin{aligned} m^*(\mu_E \wedge \mu_A) &\leq \bigvee_{j=1}^{\infty} m\left(\mu_{E_j} \wedge \mu_A\right), \\ m^*(\mu_E \wedge \mu_A^c) &\leq \bigvee_{j=1}^{\infty} m\left(\mu_{E_j} \wedge \mu_A^c\right). \end{aligned} \tag{3.31}$$

Using the sum of the inequalities (3.31),

$$m^*(\mu_E \wedge \mu_A) + m^*\left(\mu_{E_j} \wedge \mu_A^c\right) \leq \bigvee_{j=1}^{\infty} m^*\left(\mu_{E_j}\right) < m^*(\mu_E) + \varepsilon \tag{3.32}$$

is obtained. For arbitrary $\varepsilon > 0$, (3.26) is proven. Therefore, (iv) it is obtained as required.

(v) Let $\sigma(\overline{L})$ be the smallest σ-algebra which contain the $\sigma(L)$ and let m_1 be a lattice fuzzy measure on $\sigma(\overline{L})$. Then, $m_1(\mu_E) = m(\mu_E)$ for all $\mu_E \in \sigma(L)$. We must show that

$$m_1(\mu_A) = \overline{m}(\mu_A). \tag{3.33}$$

Since m is a finite σ-lattice fuzzy measure, we can write

$$X = \bigvee_{n=1}^{\infty} \mu_{E_n}, \quad \mu_{E_n} \in \sigma(L), \quad n \neq k, \quad \mu_{E_n} \wedge \mu_{E_k} = \emptyset, \quad m(\mu_{E_n}) < \infty; \quad 1 \leq n < \infty. \tag{3.34}$$

If $\mu_A \in \sigma(\overline{L})$, then we have

$$\overline{m}(\mu_B) = \bigvee_{n=1}^{\infty} \overline{m}(\mu_A \wedge \mu_{E_n}), \qquad m_1(\mu_A) = \bigvee_{n=1}^{\infty} m_1(\mu_A \wedge \mu_{E_n}). \tag{3.35}$$

To prove the inequality (3.33), it suffices to show that

$$m_1(\mu_A) = \overline{m}(\mu_A), \quad \mu_A \in \sigma\left(\overline{L}\right), \quad \overline{m}(\mu_A) < \infty. \tag{3.36}$$

Let $\mu_A \in \sigma(\overline{L})$, $\overline{m}(\mu_A) < \infty$, and $\varepsilon > 0$ arbitrary. Then, we have

$$\mu_A \leq \bigvee_{n=1}^{\infty} \mu_{E_n}, \quad \text{for } \mu_{E_n} \in \sigma(L), \quad 1 \leq n < \infty, \tag{3.37}$$

$$\overline{m}\left(\bigvee_{n=1}^{\infty} \mu_{E_n}\right) \leq \bigvee_{n=1}^{\infty} m(\mu_{E_n}) < \overline{m}(\mu_A) + \varepsilon. \tag{3.38}$$

Since $m_1(\mu_A) \leq m_1(\bigvee_{n=1}^{\infty} \mu_{E_n}) \leq \bigvee_{n=1}^{\infty} m_1(\mu_{E_n}) = \bigvee_{n=1}^{\infty} m(\mu_{E_n})$ and from (3.38), we get

$$m_1(\mu_A) \leq \overline{m}(\mu_A). \tag{3.39}$$

Also, from (3.38), we can write $\mu_F = \vee_{n=1}^{\infty}\mu_{E_n} \in \sigma(\overline{L})$ for the sets μ_{E_n}. Therefore, μ_F is m^* lattice fuzzy measurable. From the inequality $\mu_A \leq \mu_F$ and (3.38),

$$\overline{m}(\mu_F) = \overline{m}(\mu_A) + \overline{m}(\mu_F - \mu_A),$$
$$\overline{m}(\mu_F - \mu_A) = \overline{m}(\mu_F) - \overline{m}(\mu_A) < \varepsilon \tag{3.40}$$

are obtained.

From the equalities $m(\mu_E) = \overline{m}(\mu_E)$ and $m_1(\mu_F) = \overline{m}(\mu_F)$ for all $\mu_A \in \sigma(L)$, we can write

$$\begin{aligned} m(\mu_A) \leq m(\mu_F) = m_1(\mu_F) &= m_1(\mu_A) + m_1(\mu_F - \mu_A) \\ &\leq m_1(\mu_A) + \overline{m}(\mu_F - \mu_A). \end{aligned} \tag{3.41}$$

Therefore, from (3.41),

$$\overline{m}(\mu_A) \leq m_1(\mu_A) \tag{3.42}$$

is obtained.

Finally from the inequalities (3.41) and (3.39), hence the proof is completed. □

An Application of Generalized Caratheodory Extension Theorem

An application of generalized Caratheodory extension theorem is in the following. This application is essentially related to option (v)th of the generalized Caratheodory extension theorem.

Example 3.10. Show that the lattice-valued fuzzy σ-finiteness assumption is essential in generalized Caratheodory extension theorem for the uniqueness of the extension of m on the smallest fuzzy σ-algebra containing $\sigma(L)$.

In this example, let we assume $\sigma(\overline{L})$ is the smallest fuzzy σ-algebra containing $\sigma(L)$. And let $\sigma(\overline{L})$ be the smallest fuzzy σ-algebra containing $\sigma(L)$. Otherwise, let L be lattice family such that $L = (L_0, L_1]$ and

$$\sigma(L) = \left\{ \bigvee_{i=1}^{\infty} (L_{0_i}, L_{1_i}] : (L_{0_i}, L_{1_i}] \subset (L_0, L_1] \right\}. \tag{3.43}$$

For $\mu_A \in \sigma(L), m(\mu_A) = \infty$ if $\mu_A \neq \emptyset$, and $m(\mu_A) = L_0$ if $\mu_A = \emptyset$.

After all these, solution of application is clearly in the generalized Caratheodory extension theorem at property (v).

References

[1] A. H. Q. A. Al-Tai, "On the fuzzy convergence," *Journal of Applied Mathematics*, vol. 2011, Article ID 147130, 8 pages, 2011.

[2] S. Alkhazaleh and A. R. Salleh, "Fuzzy soft multiset theory," *Abstract and Applied Analysis*, vol. 2012, Article ID 350603, 20 pages, 2012.

[3] A. A. L. Sangalli, "Lattices of fuzzy objects," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 4, pp. 759–766, 1996.

[4] B. Ahmad and A. Kharal, "On fuzzy soft sets," *Advances in Fuzzy Systems*, vol. 2009, Article ID 586507, 6 pages, 2009.

[5] B. Ahmad and A. Kharal, "Fuzzy sets, fuzzy S-open and S-closed mappings," *Advances in Fuzzy Systems*, vol. 2009, Article ID 303042, 5 pages, 2009.

[6] B. T. Bilalov, S. Farahani, and F. A. Guliyeva, "The intuitionistic fuzzy normed space of coefficients," *Abstract and Applied Analysis*, vol. 2012, Article ID 969313, 11 pages, 2012.

[7] S. K. Elegan and D. Baleanu, "Set-valued fixed-point theorem for generalized contractive mapping on fuzzy metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 832807, 7 pages, 2012.

[8] F. Polat, "The centre of the spaces of Banach lattice-valued continuous functions on the generalized Alexandroff duplicate," *Abstract and Applied Analysis*, vol. 2011, Article ID 604931, 5 pages, 2011.

[9] F.-G. Shi, "(L, M)-fuzzy σ-algebras," *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 356581, 11 pages, 2010.

[10] G. Zhang, "Some theorems on fuzzy number valued fuzzy measure and fuzzy number valued fuzzy measure space," in *Proceedings of the 5th Conference of fuzzy Mathematics and system society of China*, pp. 78–81, Chengdu, China, 1990.

[11] G. Q. Zhang, "Fuzzy number valued fuzzy measure and Fuzzy number valued fuzzy integral on the $\mathcal{L}$-fuzzy sets," *Chinese Quaterly Journal of Mathematics*, vol. 6, no. 2, 1991.

[12] G. Q. Zhang, "Fuzzy number-valued fuzzy measure and fuzzy number-valued fuzzy integral on the fuzzy set," *Fuzzy Sets and Systems*, vol. 49, no. 3, pp. 357–376, 1992.

[13] J. Tanaka and P. F. Mcloughlin, "Construction of a lattice on the completion space of an algebra and an isomorphism to its Carathéodory extension," *Far East Journal of Mathematical Sciences*, vol. 43, no. 2, pp. 153–164, 2010.

[14] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.

[15] L. A. Zadeh, "Calculus of fuzzy restrictions," in *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, L. A. Zadeh, K. S. Fu, K. Tanaka, and M. Shimura, Eds., pp. 1–39, Academic Press, New York, NY, USA, 1975.

[16] M. Sahin, *Generalized σ-algebras and Generalizedfuzzy Measures [Ph.D. thesis]*, Trabzon, Turkey, 2004.

[17] M. Şahin, "Generalized fuzzy sets," *Proceedings of the Jangjeon Mathematical Society*, vol. 8, no. 2, pp. 153–158, 2005.

[18] M. Sahin and Z. Yapar, "Measure in generalized fuzzy sets," *Far East Journal of Mathematical Sciences*, vol. 24, no. 2, pp. 217–226, 2007.

[19] N. Nakajima, "Generalized fuzzy sets," *Fuzzy Sets and Systems*, vol. 32, no. 3, pp. 307–314, 1989.

[20] F. Zheng and X. Lian, "Common fixed point theorems for commutating mappings in fuzzy metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 729758, 5 pages, 2012.

[21] Z. Y. Wang and G. J. Klir, *Fuzzy Measure Theory*, Springer, 1993.

[22] A. Mukherjea and K. Pothoven, *Real and Functional Analysis, Part A Real Analysis*, vol. 27, Plenum Press, New York, NY, USA, 1984.

[23] A. W. Knapp, *Advanced Real Analysis: Along with a Companion Volume Basic Real Analysis*, 2005.

[24] L. Xuecheng and Z. Guangquan, "Lattice-valued fuzzy measure and lattice-valued fuzzy integral," *Fuzzy Sets and Systems*, vol. 62, no. 3, pp. 319–332, 1994.

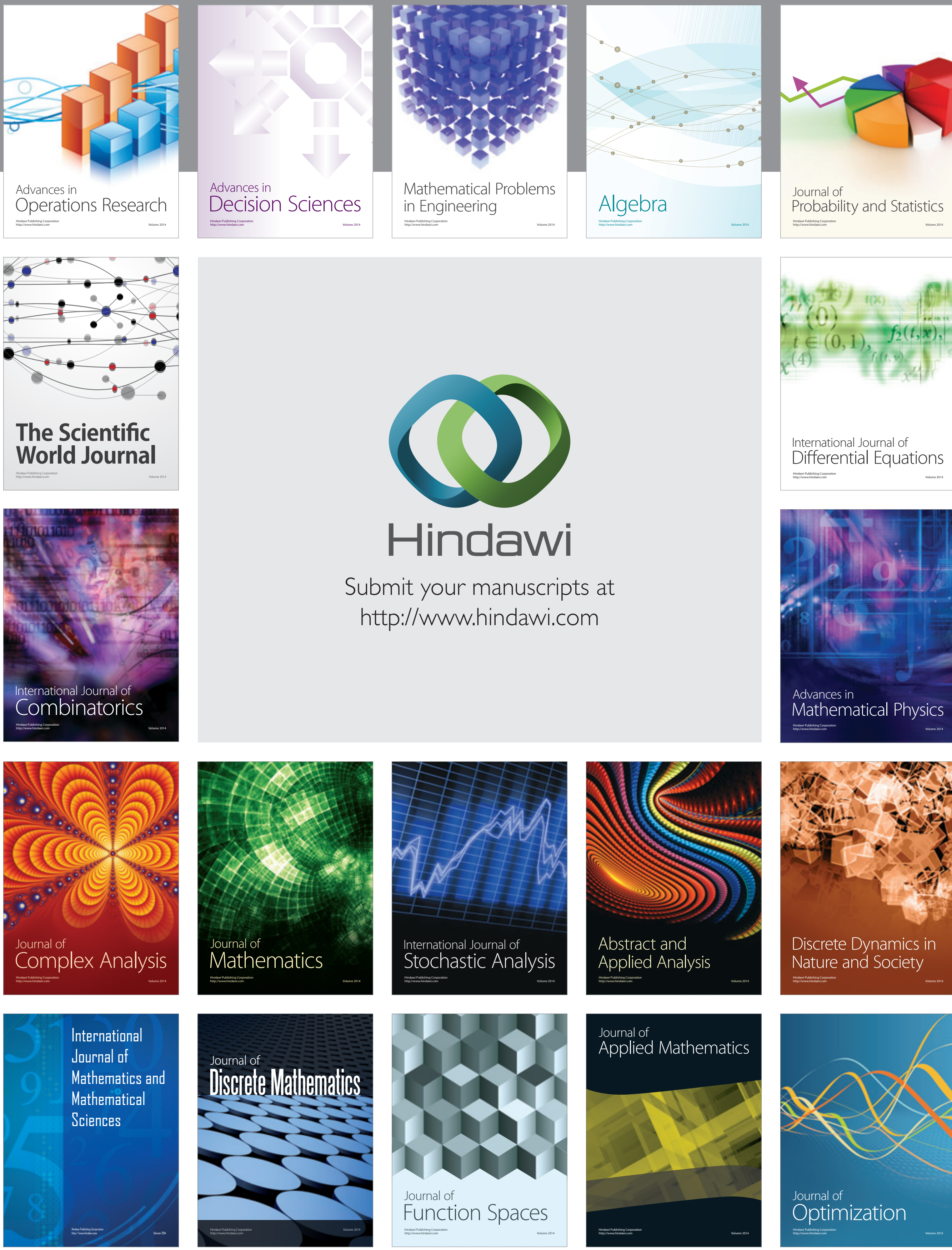
[25] M. Şahin, "On Carathedory extension theorem on fuzzy measure spaces," *Far East Journal of Mathematical Sciences*, vol. 26, no. 2, pp. 311–317, 2007.

[26] G. Birkhoff, *Lattice Theory*, AMS Colloquim Publications, Providence, RI, USA, 3rd edition, 1967.

[27] G. Grätzer, *General Lattice Theory*, 1978.

[28] J. Tanaka, "Hahn decomposition theorem of signed fuzzy measure," *Advances in Fuzzy Sets and Systems*, vol. 3, no. 3, pp. 315–323, 2008.

[29] M. Sahin and M. Demir, "Lattice valued caratheodory extension theorem," *Archives des Sciences*, vol. 65, no. 7, pp. 89–106, 2012.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