Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 865496, 14 pages
doi:10.1155/2011/865496

Research Article

A New Roper-Suffridge Extension Operator on a Reinhardt Domain

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Received 5 July 2011; Revised 28 September 2011; Accepted 5 October 2011

Academic Editor: Sung Guen Kim

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We introduce a new Roper-Suffridge extension operator on the following Reinhardt domain $\Omega_{n,p_2,\ldots,p_n} = \{z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^n |z_j|^{p_j} < 1\}$given by $F(z) = (f(z_1) + f'(z_1)\sum_{j=2}^n a_j z_j^{p_j}, (f'(z_1))^{1/p_2} z_2, \ldots, (f'(z_1))^{1/p_n} z_n)$, where f is a normalized locally biholomorphic function on the unit disc D, p_j are positive integer, a_j are complex constants, and $j = 2, \ldots, n$. Some conditions for a_j are found under which the operator preserves almost starlike mappings of order α and starlike mappings of order α, respectively. In particular, our results reduce to many well-known results when all $a_j = 0$.

1. Introduction

In 1995, Roper and Suffridge [1] introduced an extension operator. This operator is defined as follows:

$$\Phi_n(f)(z) = \left(f(z_1), \sqrt{f'(z_1)} z_0'\right)', \tag{1.1}$$

where f is a normalized locally biholomorphic function on the unit disk D in $\mathbb{C}$, $z = (z_1, z_0')'$ belonging to the unit ball $\mathcal{B}^n$ in $\mathbb{C}^n$, $z_0 = (z_2, \ldots, z_n)' \in \mathbb{C}^{n-1}$ and the branch of the square root is chosen such that $\sqrt{f'(0)} = 1$.

It is well known that the Roper-Suffridge extension operator has the following remarkable properties:

(i) if f is a normalized convex function on D, then $\Phi_n(f)$ is a normalized convex mapping on $\mathcal{B}^n$;

(ii) if f is a normalized starlike function on D, then $\Phi_n(f)$ is a normalized starlike mapping on $\mathcal{B}^n$;

(iii) if f is a normalized Bloch function on D, then $\Phi_n(f)$ is a normalized Bloch mapping on $\mathcal{B}^n$.

The above result (i) was proved by Roper and Suffridge [1] and the result (ii) and (iii) was proved by Graham and Kohr [2, 3]. Until now, it is difficult to construct the concrete convex mappings, starlike mappings, and Bloch mappings on $\mathcal{B}^n$. By making use of the Roper-Suffridge extension operator, we may easily give many concrete examples about these mappings. This is one important reason why people are interested in this extension operator.

In 2005, Muir [4] modified the Roper-Suffridge extension operator as follows:

$$F(z) = \left(f(z_1) + f'(z_1)P(z_0), \sqrt{f'(z_1)}z_0'\right)', \tag{1.2}$$

where $P(z_0)$ is a homogeneous polynomial of degree 2 with respect to z_0, and $f,\ z_1$, and z_0 are defined as above. They proved that this operator preserves starlikeness and convexity if and only if $\|P\| \leqslant 1/4$ and $\|P\| \leqslant 1/2$, respectively. The modified operator plays a key role to study the extreme points of convex mappings on $\mathcal{B}^n$ (see [5, 6]). Later, Kohr [7] and Muir [8] used the Loewner chain to study the modified Roper-Suffridge extension operator. Recently, the modified Roper-Suffridge extension operator on the unit ball is also studied by Wang and Liu [9] and Feng and Yu [10].

On the other hand, people also considered the generalized Roper-Suffridge extension operator on the general Reinhardt domains. For example, Gong and Liu [11, 12] induced the definition of ε starlike mappings and obtained that the operator

$$\Phi_{n,(1/p)}(f)(z) = \left(f(z_1), \left(f'(z_1)\right)^{1/p} z_0'\right)' \tag{1.3}$$

maps the ε starlike functions on D to the ε starlike mappings on the Reinhardt domain $\Omega_{n,p} = \{z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^{n}|z_j|^p < 1\}$, where $p \geqslant 1,\ f,\ z_1$, and z_0 are defined as above. When $\varepsilon = 0$ and $\varepsilon = 1$, $\Phi_{n,(1/p)}$ maps the starlike function and convex function on D to the starlike mapping and the convex mapping on $\Omega_{n,p}$, respectively.

Furthermore, Gong and Liu [13] proved that the operator

$$\Phi_{n,(1/p_2),\ldots,1/p_n}(f)(z) = \left(f(z_1), \left(f'(z_1)\right)^{1/p_2} z_2, \ldots, \left(f'(z_1)\right)^{1/p_n} z_n\right)' \tag{1.4}$$

maps the ε starlike functions on D to the ε starlike mappings on the domain $\Omega_{n,p_2,\ldots,p_n} = \{z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^{n}|z_j|^{p_j} < 1\}$, where $p_j \geqslant 1,\ j = 2,\ldots,n,\ f,\ z_1$, and z_0 are defined as above. Liu and Liu [14] proved that this operator preserves starlikeness of order α on the domain $\Omega_{n,p_2,\ldots,p_n}$. On the other hand, Feng and Liu [15] proved that this operator preserves almost starlikeness of order α on the domain $\Omega_{n,p_2,\ldots,p_n}$.

In contrast to the modified Roper-Suffridge extension operator in the unit ball, it is natural to ask if we can modify the Roper-Suffridge extension operator on the Reinhardt domains. In this paper, we will introduce the following modified operator:

$$F(z) = \left(f(z_1) + f'(z_1)\sum_{j=2}^{n} a_j z_j^{p_j}, \left(f'(z_1)\right)^{1/p_2} z_2, \ldots, \left(f'(z_1)\right)^{1/p_n} z_n \right)' \tag{1.5}$$

on the Reinhardt domain $\Omega_{n,p_2,\ldots,p_n}$. We will give some sufficient conditions for a_j under which the above Roper-Suffridge operator preserves an almost starlike mappings of order α and starlike mappings of order α, respectively.

In the following, we give some notation and definitions. Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \ldots, z_n)'$ with the Euclidean inner product $\langle z, w\rangle = \sum_{i=1}^{n} z_i \overline{w}_i$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$, where $z, w \in \mathbb{C}^n$ and the symbol "$'$" means transpose. The unit ball of $\mathbb{C}^n$ is the set $\mathcal{B}^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$, and the unit sphere is denoted by $\partial\mathcal{B}^n = \{z \in \mathbb{C}^n : \|z\| = 1\}$. In the case of one complex variable, $\mathcal{B}^1$ is the unit disk, usually denoted by D. Let Ω be a domain in $\mathbb{C}^n$. Denote $H(\Omega)$ by the space of all holomorphic mappings from Ω into $\mathbb{C}^n$. A mapping $f \in H(\mathcal{B}^n)$ is called normalized if $f(0) = 0$ and $J_f(0) = I_n$, where $J_f(0)$ is the complex Jacobian matrix of f at the origin and I_n is the identity operator on $\mathbb{C}^n$. A mapping $f \in H(\mathcal{B}^n)$ is said to be locally biholomorphic if $\det J_f(z) \neq 0$ for every $z \in \mathcal{B}^n$. A normalized mapping $f \in H(\mathcal{B}^n)$ is said to be convex if $\lambda\omega_1 + (1-\lambda)\omega_2 \in f(\mathcal{B}^n)$ for arbitrary $\omega_1, \omega_2 \in f(\mathcal{B}^n)$ and $0 \leqslant \lambda \leqslant 1$. A normalized mapping $f \in H(\mathcal{B}^n)$ is said to be starlike with respect to the origin if $\lambda f(\mathcal{B}^n) \subset f(\mathcal{B}^n)$, $0 \leqslant \lambda \leqslant 1$. A normalized mapping $f \in H(\mathcal{B}^n)$ is said to be ε starlike if there exists a positive number ε, $0 \leqslant \varepsilon \leqslant 1$, such that $f(\mathcal{B}^n)$ is starlike with respect to every point in $\varepsilon f(\mathcal{B}^n)$.

A domain Ω is called a Reinhardt domain if $(e^{i\theta_1} z_1, e^{i\theta_2} z_2, \ldots, e^{i\theta_n} z_n)' \in \Omega$ holds for any $z = (z_1, z_2, \ldots, z_n)' \in \Omega$ and $\theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R}$. A domain Ω is called a circular domain if $e^{i\theta} z \in \Omega$ holds for any $z \in \Omega$ and $\theta \in \mathbb{R}$. The Minkowski functional $\rho(z)$ of the Reinhardt domain

$$\Omega_{n,p_2,\ldots,p_n} = \left\{ z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^{n} \left|z_j\right|^{p_j} < 1 \right\}, \quad p_j \geqslant 1,\ j = 2, \ldots, n \tag{1.6}$$

is defined as

$$\rho(z) = \inf\left\{ t > 0, \frac{z}{t} \in \Omega_{n,p_2,\ldots,p_n} \right\}, \quad z \in \mathbb{C}^n. \tag{1.7}$$

Then, the Minkowski functional $\rho(z)$ is a norm of $\mathbb{C}^n$ and $\Omega_{n,p_2,\cdots,p_n}$ is the unit ball in the Banach space $\mathbb{C}^n$ with respect to this norm. The Minkowski functional $\rho(z)$ is C^1 on $\overline{\Omega}_{n,p_2,\ldots,p_n}$

except for a lower-dimensional manifold Ω_0. Moreover, we give the following properties of the Minkowski functional $\rho(z)$ (see [16]):

$$\begin{aligned}
2\frac{\partial\rho}{\partial z}(z)z &= \rho(z), \quad \forall z \in \mathbb{C}^n \setminus \Omega_0, \\
2\frac{\partial\rho}{\partial z}(z)z &= 1, \quad \forall z \in \partial\Omega_{n,p_2,\ldots,p_n} \setminus \Omega_0, \\
\frac{\partial\rho}{\partial z}(\lambda z) &= \frac{\partial\rho}{\partial z}(z), \quad \forall \lambda \in [0,\infty),\ z \in \mathbb{C}^n \setminus \Omega_0, \\
\frac{\partial\rho}{\partial z}\left(e^{i\theta}z\right) &= e^{-i\theta}\frac{\partial\rho}{\partial z}(z), \quad \forall z \in \mathbb{C}^n \setminus \Omega_0,\ \theta \in \mathbb{R}.
\end{aligned} \tag{1.8}$$

Definition 1.1 (see [17]). Suppose that Ω is a bounded starlike circular domain in $\mathbb{C}^n$. Its Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let $0 \leqslant \alpha < 1$. We say that a normalized locally biholomorphic mapping $f \in H(\Omega)$ is an almost starlike mapping of order α if the following condition holds:

$$\Re \frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_f^{-1}(z)f(z) \geqslant \alpha, \quad z \in \Omega \setminus \{0\}. \tag{1.9}$$

When $\Omega = \mathcal{B}^n$, its Minkowski functional $\rho(z) = \|z\|$, the above inequality becomes

$$\Re \overline{z'} J_f^{-1}(z)f(z) \geqslant \alpha\|z\|^2, \quad z \in \mathcal{B}^n. \tag{1.10}$$

In particular, when $\alpha = 0$, f reduces to a starlike mapping on Ω.

Definition 1.2 (see [18]). Suppose that $\Omega \in \mathbb{C}^n$ is a bounded starlike circular domain. Its Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let $0 < \alpha < 1$. We say that a normalized locally biholomorphic mapping $f \in H(\Omega)$ is a starlike mapping of order α if the following condition holds:

$$\left|\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_f^{-1}(z)f(z) - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \quad z \in \Omega \setminus \{0\}. \tag{1.11}$$

When $\Omega = \mathcal{B}^n$, the above inequality reduces to

$$\left|\frac{1}{\|z\|^2}\overline{z'} J_f^{-1}(z)f(z) - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \quad z \in \mathcal{B}^n \setminus \{0\}. \tag{1.12}$$

2. Some Lemmas

In order to prove the main results, we need the following three lemmas.

Lemma 2.1 (see [19]). *Let p be a holomorphic function on D. If $\Re p(z) > 0$ and $p(0) > 0$, then*

$$\left|p'(z)\right| \leqslant \frac{2\Re p(z)}{1-|z|^2}. \tag{2.1}$$

Lemma 2.2 (see [19]). *Let f be a normalized biholomorphic function on D. Then,*

$$\left|\left(1-|z|^2\right)\frac{f''(z)}{f'(z)} - 2\overline{z}\right| \leqslant 4 \tag{2.2}$$

holds for all $z \in D$.

Lemma 2.3 (see [20]). *If $\rho(z)$ is a Minkowski function of the domain $\Omega_{n,p_2,\ldots,p_n}$, $z \neq 0$, then*

$$\begin{aligned}
\frac{\partial\rho}{\partial z_1}(z) &= \frac{\overline{z}_1}{\rho(z)\left[2\left|z_1/\rho(z)\right|^2 + \sum_{j=2}^{n} p_j\left|z_j/\rho(z)\right|^{p_j}\right]},\\
\frac{\partial\rho}{\partial z_j}(z) &= \frac{p_j\overline{z}_j\left|z_j/\rho(z)\right|^{p_j-2}}{2\rho(z)\left[2\left|z_1/\rho(z)\right|^2 + \sum_{j=2}^{n} p_j\left|z_j/\rho(z)\right|^{p_j}\right]}, \quad j = 2,\ldots,n.
\end{aligned} \tag{2.3}$$

3. Main Results

Theorem 3.1. *Let $0 \leqslant \alpha < 1$ and let f be an almost starlike function of order α on the unit disc D. If complex numbers a_j satisfy the condition $|a_j| \leqslant (1-\alpha)/4$, $j = 2,\ldots,n$, then*

$$F(z) = \left(f(z_1) + f'(z_1)\sum_{j=2}^{n} a_j z_j^{p_j}, \left(f'(z_1)\right)^{1/p_2} z_2, \ldots, \left(f'(z_1)\right)^{1/p_n} z_n\right)' \tag{3.1}$$

is an almost starlike mapping of order α on the domain $\Omega_{n,p_2,\ldots,p_n}$, where p_j are positive integer and $p_j \geqslant 2$; the branches are chosen such that $\left(f'(z_1)\right)^{1/p_j}\big|_{z_1=0} = 1$.

Proof. By the definition of almost starlike mapping of order α, we need only to prove that the following inequality:

$$\Re\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z) \geqslant \alpha \tag{3.2}$$

holds for all $z \in \Omega_{n,p_2,\ldots,p_n}$ and $z \neq 0$.

The case of $z_0 = 0$ is trivial. So, we need only consider that $z = (z_1, z_0')' \in \overline{\Omega}_{n,p_2,\ldots,p_n}$, $z_0 \neq 0$. Let $z = \zeta u = |\zeta| e^{i\theta} u$, $u \in \partial\Omega_{n,p_2,\ldots,p_n}$, and $\zeta \in \overline{D} \setminus \{0\}$, then we have

$$\begin{aligned}
&\Re \frac{2}{\rho(z)} \frac{\partial\rho}{\partial z}(z) J_F^{-1}(z) F(z) \geqslant \alpha \\
&\Longleftrightarrow \Re \frac{2}{\rho(|\zeta| e^{i\theta} u)} \frac{\partial\rho}{\partial z}\left(|\zeta| e^{i\theta} u\right) J_F^{-1}\left(|\zeta| e^{i\theta} u\right) F\left(|\zeta| e^{i\theta} u\right) \geqslant \alpha \\
&\Longleftrightarrow \Re \frac{2}{|\zeta|} \frac{e^{-i\theta}\partial\rho}{\partial z}(u) J_F^{-1}\left(|\zeta| e^{i\theta} u\right) F\left(|\zeta| e^{i\theta} u\right) \geqslant \alpha \\
&\Longleftrightarrow \Re \frac{2\partial\rho}{\partial z}(u) \frac{J_F^{-1}(\zeta u) F(\zeta u)}{\zeta} \geqslant \alpha.
\end{aligned} \tag{3.3}$$

For a fixed u, the expression $\Re(2\partial\rho/\partial z)(u)(J_F^{-1}(\zeta u)F(\zeta u)/\zeta) - \alpha$ is the real part of a holomorphic function with respect to ζ, so it is a harmonic function. By the minimum of harmonic function principle, we know that it attains its minimum on $|\zeta| = 1$, so we need only to prove for all $z \in \partial\Omega_{n,p_2,\ldots,p_n}$ and $z_0 \neq 0$. Hence, $\rho(z) = 1$ and inequality (3.2) becomes

$$\Re \frac{2\partial\rho}{\partial z}(z) J_F^{-1}(z) F(z) \geqslant \alpha, \quad z \in \partial\Omega_{n,p_2,\ldots,p_n}, \ z_0 \neq 0. \tag{3.4}$$

In the following, we will prove inequality (3.4).

Since

$$F(z) = \left(f(z_1) + f'(z_1) \sum_{j=2}^{n} a_j z_j^{p_j}, \left(f'(z_1)\right)^{1/p_2} z_2, \ldots, \left(f'(z_1)\right)^{1/p_n} z_n \right)', \tag{3.5}$$

we have

$$J_F(z) = \begin{pmatrix} f'(z_1) + f''(z_1) \sum_{j=2}^{n} a_j z_j^{p_j} & a_2 p_2 f'(z_1) z_2^{p_2-1} & \cdots & a_n p_n f'(z_1) z_n^{p_n-1} \\ \frac{1}{p_2}\left(f'(z_1)\right)^{(1/p_2)-1} f''(z_1) z_2 & \left(f'(z_1)\right)^{1/p_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{p_n}\left(f'(z_1)\right)^{(1/p_n)-1} f''(z_1) z_n & 0 & \cdots & \left(f'(z_1)\right)^{1/p_n} \end{pmatrix}. \tag{3.6}$$

Suppose that $J_F^{-1}(z)F(z) = A = (x_1, x_2, \dots, x_n)'$, then $F(z) = J_F(z)A$; that is,

$$
\begin{aligned}
x_1\left[f'(z_1)+f''(z_1)\sum_{j=2}^{n}a_j z_j^{p_j}\right]+f'(z_1)\sum_{j=2}^{n}a_j p_j x_j z_j^{p_j-1} &= f(z_1)+f'(z_1)\sum_{j=2}^{n}a_j z_j^{p_j}, \\
x_1\frac{f''(z_1)}{p_2 f'(z_1)}z_2 + x_2 &= z_2, \\
&\vdots \\
x_1\frac{f''(z_1)}{p_n f'(z_n)}z_n + x_n &= z_n.
\end{aligned}
\tag{3.7}
$$

Some computation shows that

$$
\begin{aligned}
x_1 &= \frac{f(z_1)}{f'(z_1)} - \sum_{j=2}^{n}a_j(p_j-1)z_j^{p_j}, \\
x_2 &= \left[1-\frac{f(z_1)f''(z_1)}{p_2(f'(z_1))^2}+\frac{f''(z_1)}{p_2 f'(z_1)}\sum_{j=2}^{n}a_j(p_j-1)z_j^{p_j}\right]z_2, \\
&\vdots \\
x_n &= \left[1-\frac{f(z_1)f''(z_1)}{p_n(f'(z_1))^2}+\frac{f''(z_1)}{p_n f'(z_1)}\sum_{j=2}^{n}a_j(p_j-1)z_j^{p_j}\right]z_n.
\end{aligned}
\tag{3.8}
$$

From Lemma 2.3, we obtain

$$
\begin{aligned}
\frac{\partial\rho}{\partial z_1}(z) &= \frac{\overline{z}_1}{\rho(z)\left[2|z_1/\rho(z)|^2+\sum_{j=2}^{n}p_j|z_j/\rho(z)|^{p_j}\right]} = \frac{\overline{z}_1}{2|z_1|^2+\sum_{j=2}^{n}p_j|z_j|^{p_j}}, \\
\frac{\partial\rho}{\partial z_j}(z) &= \frac{p_j\overline{z}_j|z_j|^{p_j-2}}{2\rho(z)\left[2|z_1/\rho(z)|^2+\sum_{j=2}^{n}p_j|z_j|^{p_j}\right]} = \frac{p_j\overline{z}_j|z_j|^{p_j-2}}{2\left[2|z_1|^2+\sum_{j=2}^{n}p_j|z_j|^{p_j}\right]}.
\end{aligned}
\tag{3.9}
$$

In terms of (3.8) and (3.9), we obtain

$$
\frac{2\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z) = \frac{G(z)}{2|z_1|^2+\sum_{j=2}^{n}p_j|z_j|^{p_j}},
\tag{3.10}
$$

where

$$G(z) = 2\overline{z}_1\left[\frac{f(z_1)}{f'(z_1)} - \sum_{j=2}^{n} a_j(p_j-1)z_j^{p_j}\right] + \sum_{j=2}^{n} p_j|z_j|^{p_j}\left[1 - \frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2} + \frac{f''(z_1)}{p_j f'(z_1)}\sum_{k=2}^{n} a_k(p_k-1)z_k^{p_k}\right]$$
$$= 2|z_1|^2\frac{f(z_1)}{z_1 f'(z_1)} + \sum_{j=2}^{n} p_j|z_j|^{p_j}\left[1 - \frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\right] + \sum_{k=2}^{n} a_k(p_k-1)z_k^{p_k}\left[\frac{f''(z_1)}{f'(z_1)}\sum_{j=2}^{n}|z_j|^{p_j} - 2\overline{z}_1\right]. \tag{3.11}$$

By making use of the equality $|z_1|^2 + \sum_{j=2}^{n}|z_j|^{p_j} = 1$, we then get

$$G(z) = 2|z_1|^2\frac{f(z_1)}{z_1 f'(z_1)} + \sum_{j=2}^{n} p_j|z_j|^{p_j}\left[1 - \frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\right] + \sum_{j=2}^{n} a_j(p_j-1)z_j^{p_j}\left[\frac{f''(z_1)}{f'(z_1)}\left(1-|z_1|^2\right) - 2\overline{z_1}\right]. \tag{3.12}$$

Let $h(z_1) = (f(z_1)/z_1 f'(z_1)) - \alpha$. Notice that f is an almost starlike function of order α on the unit disc; hence, $\Re h(z_1) > 0$ and $h(0) = 1-\alpha > 0$. By Lemma 2.1, we can obtain that

$$\left|h'(z)\right| \leqslant \frac{2\Re h(z)}{1-|z|^2}. \tag{3.13}$$

Furthermore, we get

$$\frac{f(z_1)f''(z_1)}{\left[f'(z_1)\right]^2} = 1 - \alpha - h(z_1) - z_1 h'(z_1). \tag{3.14}$$

Substituting (3.14) into (3.12), we have

$$G(z) = 2|z_1|^2(h(z_1)+\alpha) + \sum_{j=2}^{n} p_j|z_j|^{p_j}\left(1 - \frac{1}{p_j} + \frac{\alpha}{p_j} + \frac{h(z_1)}{p_j} + \frac{z_1}{p_j}h'(z_1)\right) + \sum_{j=2}^{n} a_j(p_j-1)z_j^{p_j}\left[\frac{f''(z_1)}{f'(z_1)}\left(1-|z_1|^2\right) - 2\overline{z}_1\right]$$
$$= h(z_1)\left(2|z_1|^2 + \sum_{j=2}^{n}|z_j|^{p_j}\right) + 2\alpha|z_1|^2 + \sum_{j=2}^{n}(p_j-1+\alpha)|z_j|^{p_j} + \sum_{j=2}^{n} z_1|z_j|^{p_j}h'(z_1) + \sum_{j=2}^{n} a_j(p_j-1)z_j^{p_j}\left[\frac{f''(z_1)}{f'(z_1)}\left(1-|z_1|^2\right) - 2\overline{z}_1\right]. \tag{3.15}$$

Hence,

$$\begin{aligned}\Re G(z) \geqslant{}& \left(2|z_1|^2 + \sum_{j=2}^{n}|z_j|^{p_j}\right)\Re h(z_1) + 2\alpha|z_1|^2 + \sum_{j=2}^{n}(p_j - 1 + \alpha)|z_j|^{p_j} \\ &- \sum_{j=2}^{n}|z_j|^{p_j}|z_1 h'(z_1)| - \sum_{j=2}^{n}|a_j|(p_j - 1)|z_j|^{p_j}\left|\frac{f''(z_1)}{f'(z_1)}\left(1 - |z_1|^2\right) - 2\overline{z}_1\right|.\end{aligned} \tag{3.16}$$

By Lemma 2.2 and (3.13), we can get that

$$\begin{aligned}\Re G(z) \geqslant{}& \left(1 + |z_1|^2\right)\Re h(z_1) + 2\alpha|z_1|^2 + \sum_{j=2}^{n}(p_j - 1 + \alpha)|z_j|^{p_j} - \left(1 - |z_1|^2\right)\frac{2|z_1|\Re h(z_1)}{1 - |z_1|^2} \\ &- 4\sum_{j=2}^{n}|a_j|(p_j - 1)|z_j|^{p_j} \\ ={}& \left(1 + |z_1|^2\right)\Re h(z_1) + 2\alpha|z_1|^2 + \sum_{j=2}^{n}(p_j - 1 + \alpha)|z_j|^{p_j} - 2|z_1|\Re h(z_1) \\ &- 4\sum_{j=2}^{n}|a_j|(p_j - 1)|z_j|^{p_j} \\ ={}& (1 - |z_1|)^2\Re h(z_1) + 2\alpha|z_1|^2 + \sum_{j=2}^{n}|z_j|^{p_j}\left[\alpha + (1 - |4a_j|)(p_j - 1)\right].\end{aligned} \tag{3.17}$$

Hence, when $|a_j| \leqslant (1-\alpha)/4$, we have

$$\Re G(z) \geqslant (1 - |z_1|)^2\Re h(z_1) + 2\alpha|z_1|^2 + \alpha\sum_{j=2}^{n}p_j|z_j|^{p_j} \geqslant 2\alpha|z_1|^2 + \alpha\sum_{j=2}^{n}p_j|z_j|^{p_j}. \tag{3.18}$$

In terms of (3.10) and (3.18), we obtain

$$\Re\frac{2\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z) \geqslant \alpha, \tag{3.19}$$

which completes the proof of Theorem 3.1. □

Remark 3.2. When $a_2 = a_3 = \cdots = a_n = 0$, the result of Theorem 3.1 has been obtained by Liu and Liu [14].

Corollary 3.3. *Let f be a normalized biholomorphic starlike function on the unit disc D. If $|a_j| \leqslant 1/4$, $j = 2, \ldots, n$, then*

$$F(z) = \left(f(z_1) + f'(z_1)\sum_{j=2}^{n}a_j z_j^{p_j}, \left(f'(z_1)\right)^{1/p_2}z_2, \ldots, \left(f'(z_1)\right)^{1/p_n}z_n\right)' \tag{3.20}$$

is a normalized biholomorphic starlike mapping on the domain $\Omega_{n,p_2,\ldots,p_n}$*, where* p_j *are positive integer and* $p_j \geqslant 2$*; the branches are chosen such that* $(f'(z_1))^{1/p_j}|_{z_1=0} = 1$.

Theorem 3.4. *Let* $0 < \alpha < 1$ *and let* f *be a starlike function of order* α *on the unit disc* D*. If complex numbers* a_j *satisfy the condition* $|a_j| \leqslant (1-|2\alpha-1|)/8\alpha$, $j = 2,\ldots,n$*, then*

$$F(z) = \left(f(z_1) + f'(z_1)\sum_{j=2}^{n} a_j z_j^{p_j}, (f'(z_1))^{1/p_2} z_2, \ldots, (f'(z_1))^{1/p_n} z_n \right)' \tag{3.21}$$

is a starlike mapping of order α *on the domain* $\Omega_{n,p_2,\ldots,p_n}$*, where* p_j *are positive integer and* $p_j \geqslant 1$*; the branches are chosen such that* $(f'(z_1))^{1/p_j}|_{z_1=0} = 1$.

Proof. By the definition of starlike mapping of order α, we need only to prove that the following inequality:

$$\left| \frac{2}{\rho(z)} \frac{\partial\rho}{\partial z}(z) J_F^{-1}(z) F(z) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \tag{3.22}$$

holds for all $z \in \Omega_{n,p_2,\ldots,p_n}$ and $z_0 \neq 0$.

Similar to the proof of Theorem 3.1, we need only to prove that (3.22) holds for $\rho(z) = 1$ and $z_0 \neq 0$ according to the maximum modulus theorem for analytic functions. So, it is suffice to show that

$$\left| \frac{2\partial\rho}{\partial z}(z) J_F^{-1}(z) F(z) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}. \tag{3.23}$$

From the proof of Theorem 3.1, we can get

$$\begin{aligned} &\frac{\partial\rho}{\partial z}(z) J_F^{-1}(z) F(z) \\ &\quad = \frac{1}{2\left[2|z_1|^2 + \sum_{j=2}^{n} p_j |z_j|^{p_j}\right]} \left\{ 2|z_1|^2 \frac{f(z_1)}{z_1 f'(z_1)} + \sum_{j=2}^{n} p_j |z_j|^{p_j} \left[1 - \frac{f(z_1) f''(z_1)}{p_j (f'(z_1))^2} \right] \right. \\ &\qquad \left. + \sum_{j=2}^{n} a_j (p_j - 1) z_j^{p_j} \left[\frac{f''(z_1)}{f'(z_1)} \left(1 - |z_1|^2\right) - 2\overline{z}_1 \right] \right\}. \end{aligned} \tag{3.24}$$

Hence,

$$\frac{2\partial\rho}{\partial z}(z) J_F^{-1}(z) F(z) - \frac{1}{2\alpha} = \frac{H(z)}{2\alpha\left[2|z_1|^2 + \sum_{j=2}^{n} p_j |z_j|^{p_j}\right]}, \tag{3.25}$$

where

$$H(z) = 2|z_1|^2\left[2\alpha\frac{f(z_1)}{z_1 f'(z_1)} - 1\right] + 2\alpha\sum_{j=2}^{n} p_j |z_j|^{p_j}\left[1 - \frac{1}{2\alpha} - \frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\right]$$
$$+ 2\alpha\sum_{j=2}^{n} a_j(p_j - 1)z_j^{p_j}\left[\frac{f''(z_1)}{f'(z_1)}\left(1 - |z_1|^2\right) - 2\overline{z}_1\right]. \tag{3.26}$$

Let $h(z_1) = 2\alpha(f(z_1)/z_1 f'(z_1)) - 1$. Then, $|h(z_1)| < 1$ because f is a starlike function of order α on the unit disc D. By the Schwarz-Pick lemma, we obtain that

$$\left|h'(z_1)\right| \leqslant \frac{1 - |h(z_1)|^2}{1 - |z_1|^2}. \tag{3.27}$$

On the other hand, we can get

$$\frac{f(z_1)f''(z_1)}{\left[f'(z_1)\right]^2} = 1 - \frac{1}{2\alpha} - \frac{h(z_1)}{2\alpha} - \frac{z_1 h'(z_1)}{2\alpha}. \tag{3.28}$$

Substituting (3.28) into (3.26), we have

$$H(z) = 2|z_1|^2\left[2\alpha\frac{f(z_1)}{z_1 f'(z_1)} - 1\right] + 2\alpha\sum_{j=2}^{n} p_j |z_j|^{p_j}\left[1 - \frac{1}{2\alpha} - \left(\frac{1}{p_j} - \frac{1}{2\alpha p_j} - \frac{h(z_1)}{2\alpha p_j} - \frac{z_1 h'(z_1)}{2\alpha p_j}\right)\right]$$
$$+ 2\alpha\sum_{j=2}^{n} a_j(p_j - 1)z_j^{p_j}\left[\frac{f''(z_1)}{f'(z_1)}\left(1 - |z_1|^2\right) - 2\overline{z}_1\right]$$
$$= 2|z_1|^2 h(z_1) + h(z_1)\sum_{j=2}^{n}|z_j|^{p_j} + \sum_{j=2}^{n} z_1 h'(z_1)|z_j|^{p_j} + (2\alpha - 1)\sum_{j=2}^{n}(p_j - 1)|z_j|^{p_j}$$
$$+ 2\alpha\sum_{j=2}^{n} a_j(p_j - 1)z_j^{p_j}\left[\frac{f''(z_1)}{f'(z_1)}\left(1 - |z_1|^2\right) - 2\overline{z}_1\right]. \tag{3.29}$$

Hence,

$$|H(z)| \leqslant \left(1 + |z_1|^2\right)|h(z_1)| + \sum_{j=2}^{n}\left|z_1 h'(z_1)\right||z_j|^{p_j} + |2\alpha - 1|\sum_{j=2}^{n}(p_j - 1)|z_j|^{p_j}$$
$$+ 2\alpha\sum_{j=2}^{n}|a_j|(p_j - 1)|z_j|^{p_j}\left|\frac{f''(z_1)}{f'(z_1)}\left(1 - |z_1|^2\right) - 2\overline{z}_1\right|. \tag{3.30}$$

By Lemma 2.2 and (3.27), we have

$$\begin{aligned}
|H(z)| &\leqslant \left(1+|z_1|^2\right)|h(z_1)| + |z_1|\frac{1-|h(z_1)|^2}{1-|z_1|^2}\left(1-|z_1|^2\right) + |2\alpha-1|\sum_{j=2}^{n}(p_j-1)\left|z_j\right|^{p_j} \\
&\quad + 8\alpha\sum_{j=2}^{n}\left|a_j\right|(p_j-1)\left|z_j\right|^{p_j} \\
&\leqslant \left(1+|z_1|^2\right)(|h(z_1)|-1) + \left(1+|z_1|^2\right) + 2|z_1|(1-|h(z_1)|) \\
&\quad + |2\alpha-1|\sum_{j=2}^{n}(p_j-1)\left|z_j\right|^{p_j} + 8\alpha\sum_{j=2}^{n}\left|a_j\right|(p_j-1)\left|z_j\right|^{p_j} \\
&\leqslant \left(1+|z_1|^2\right) + (|h(z_1)|-1)(1-|z_1|)^2 + \sum_{j=2}^{n}\left(|2\alpha-1| + 8\alpha\left|a_j\right|\right)(p_j-1)\left|z_j\right|^{p_j}.
\end{aligned} \tag{3.31}$$

If $|a_j| \leqslant (1-|2\alpha-1|)/8\alpha$, then we obtain

$$\begin{aligned}
|H(z)| &< 1+|z_1|^2 + \left(|2\alpha-1| + 8\alpha\frac{1-|2\alpha-1|}{8\alpha}\right)\sum_{j=2}^{n}(p_j-1)\left|z_j\right|^{p_j} \\
&\leq 1+|z_1|^2 + \sum_{j=2}^{n}(p_j-1)\left|z_j\right|^{p_j} \\
&= 2|z_1|^2 + \sum_{j=2}^{n}p_j\left|z_j\right|^{p_j}.
\end{aligned} \tag{3.32}$$

The equality (3.25) and (3.32) show that

$$\left|\frac{2\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z) - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \tag{3.33}$$

which completes the proof of Theorem 3.4. □

4. Problem

In 2003, Gong and Liu [13] proved that the Roper-Suffridge extension operator

$$\Phi_{n,(1/p_2),\ldots,(1/p_n)}(f)(z) = \left(f(z_1), \left(f'(z_1)\right)^{1/p_2}z_2, \ldots, \left(f'(z_1)\right)^{1/p_n}z_n\right)' \tag{4.1}$$

does preserve convexity on $\Omega_n, p_2, \ldots, p_n$, which solved the open problem posed by Graham and Kohr [2]. Naturally, we will propose the following problem on the new Roper-Suffridge extension operator.

Problem 1. Let p_j be positive integer. Under what conditions for a_j such that if f is a convex function in the disc D, then the mapping defined by the new Roper-Suffridge extension operator

$$F(z) = \left(f(z_1) + f'(z_1)\sum_{j=2}^{n} a_j z_j^{p_j}, \left(f'(z_1)\right)^{1/p_2} z_2, \dots, \left(f'(z_1)\right)^{1/p_n} z_n \right)' \tag{4.2}$$

is a convex mapping in the Reinhardt domain $\Omega_n, p_2, \dots, p_n$?

Acknowledgments

The authors cordially thank the referees thorough reviewing with useful suggestions and comments made to the paper. The project was supported by the National Natural Science Foundation of China (Grant no. 11001246 and no. 11101139), the Natural Science Foundation of Zhejiang province (Grant no. Y6090694), and the Zhejiang Innovation Project (Grant no. T200905).

References

[1] K. A. Roper and T. J. Suffridge, "Convex mappings on the unit ball of $\mathbb{C}^n$," *Journal d'Analyse Mathématique*, vol. 65, pp. 333–347, 1995.
[2] I. Graham and G. Kohr, "Univalent mappings associated with the Roper-Suffridge extension operator," *Journal d'Analyse Mathématique*, vol. 81, pp. 331–342, 2000.
[3] I. Graham, G. Kohr, and M. Kohr, "Loewner chains and the Roper-Suffridge extension operator," *Journal of Mathematical Analysis and Applications*, vol. 247, no. 2, pp. 448–465, 2000.
[4] J. R. Muir, "A modification of the Roper-Suffridge extension operator," *Computational Methods and Function Theory*, vol. 5, no. 1, pp. 237–251, 2005.
[5] J. R. Muir and T. J. Suffridge, "A generalization of half-plane mappings to the ball in $\mathbb{C}^n$," *Transactions of the American Mathematical Society*, vol. 359, no. 4, pp. 1485–1498, 2007.
[6] J. R. Muir and T. J. Suffridge, "Extreme points for convex mappings of B^n," *Journal d'Analyse Mathématique*, vol. 98, pp. 169–182, 2006.
[7] G. Kohr, "Loewner chains and a modification of the Roper-Suffridge extension operator," *Mathematica*, vol. 48, no. 1, pp. 41–48, 2006.
[8] J. R. Muir, "A class of Loewner chain preserving extension operators," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 862–879, 2008.
[9] J. F. Wang and T. S. Liu, "A modified Roper-Suffridge extension operator for some holomorphic mappings," *Chinese Annals of Mathematics. Series A*, vol. 31, no. 4, pp. 487–496, 2010.
[10] S. X. Feng and L. Yu, "Modified Roper-Suffridge operator for some holomorphic mappings," *Frontiers of Mathematics in China*, vol. 6, no. 3, pp. 411–426, 2011.
[11] S. Gong and T. S. Liu, "On the Roper-Suffridge extension operator," *Journal d'Analyse Mathématique*, vol. 88, pp. 397–404, 2002.
[12] T. S. Liu and S. Gong, "The family of starlike mappings. I," *Chinese Annals of Mathematics. Series A*, vol. 23, no. 3, pp. 273–282, 2002.
[13] S. Gong and T. S. Liu, "The generalized Roper-Suffridge extension operator," *Journal of Mathematical Analysis and Applications*, vol. 284, no. 2, pp. 425–434, 2003.
[14] X. S. Liu and T. S. Liu, "The generalized Roper-Suffridge extension operator on a Reinhardt domain and the unit ball in a complex Hilbert space," *Chinese Annals of Mathematics. Series A*, vol. 26, no. 5, pp. 721–730, 2005.
[15] S. X. Feng and T. S. Liu, "The generalized Roper-Suffridge extension operator," *Acta Mathematica Scientia. Series B*, vol. 28, no. 1, pp. 63–80, 2008.

[16] T. S. Liu and G. B. Ren, "The growth theorem for starlike mappings on bounded starlike circular domains," *Chinese Annals of Mathematics. Series B*, vol. 19, no. 4, pp. 401–408, 1998.

[17] S. X. Feng and K. P. Lu, "The growth theorem for almost starlike mappings of order on bounded starlike circular domains," *Chinese Quarterly Journal of Mathematics. Shuxue Jikan*, vol. 15, no. 2, pp. 50–56, 2000.

[18] H. Liu, *Class of Starlike Mappings, Its Extensions and Subclasses in Several Complex Variables*, Doctoral Thesis, University of Science and Technology of China, 1999.

[19] C. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, Germany, 1975.

[20] W. J. Zhang and T. S. Liu, "On decomposition theorem of normalized biholomorphic convex mappings in Reinhardt domains," *Science in China*, vol. 46, no. 1, pp. 94–106, 2003.

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