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Research Article

Limit Circle/Limit Point Criteria for Second-Order Sublinear Differential Equations with Damping Term

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The purpose of the present paper is to establish some new criteria for the classification of the sublinear differential equation as of the nonlinear limit circle type or of the nonlinear limit point type. The criteria presented here generalize some known results in the literature.

1. Introduction

In 1910, Weyl [1] published his now classical paper on eigenvalue problems for second-order linear differential equations of the form

$$\left(a(t)y'\right)' + r(t)y = \theta y, \quad \theta \in C. \tag{1.1}$$

He classified this equation to be of the limit circle type if each solution $y(t)$ is square integrable (denoted by $y(t) \in L^2$), that is,

$$\int_0^\infty y^2(t)dt < \infty, \tag{1.2}$$

and to be of the limit point type if at least one solution $y(t)$ does not belong to L^2, that is,

$$\int_0^\infty y^2(t)dt = \infty. \tag{1.3}$$

Weyl showed that the linear equation (1.1) always has at least one square integrable solution if Im $\theta \neq 0$. Thus, for second-order linear equations with Im $\theta \neq 0$, the problem reduces to whether (1.1) has one (limit point type) or two (limit circle type) linearly independent square integrable solutions. This is known as the Weyl Alternative. Weyl also proved that if (1.1) is of the limit circle type for some $\theta_0 \in C$, then it is of the limit circle type for all $\theta \in C$. In particular, this is true for $\theta = 0$, that is, if we can show the following equation

$$\left(a(t)y'\right)' + r(t)y = 0, \tag{1.4}$$

is of limit circle type, then (1.1) is of the limit circle type for all values of θ. There has been considerable interest in this problem over the years (see [1–10] and references cited therein). The analogous problem for nonlinear equations is relatively new and not as extensively studied as the linear cases. For a survey of known results on the linear and nonlinear problems as well as their relationships to other properties of solutions such as boundedness, oscillation, and convergence to zero, we refer the reader to the recent monograph [10]. In this paper, we will discuss the equation with damping term

$$\left(a(t)y'\right)' + b(t)y' + r(t)y^{\gamma} = 0, \tag{1.5}$$

where $a,\ r : R_+ \mapsto R$ and $b : R_+ \mapsto R_+$ are continuous, $a',\ r' \epsilon AC_{\text{loc}}(R_+),\ a'',\ r'' \epsilon L^2_{\text{loc}}(R_+)$, $a(t) > 0$, $r(t) > 0$, and $0 < \gamma \leq 1$, $\gamma = \text{odd}/\text{odd}$, say $\gamma = (2M-1)/(2N-1)$, M and N are positive integers, we can write $\gamma = 2k-1$, where $k = (M+N-1)/(2N-1)$. When $b(t) \equiv 0$, then (1.5) turns into the following equation

$$\left(a(t)y'\right)' + r(t)y^{\gamma} = 0, \tag{1.6}$$

which is widely researched by many authors (see [10] and references cited therein).

Definition 1.1 (see [2]). A nontrivial solution $y(t)$ of (1.5) is said to be of the nonlinear limit circle type if

$$\int_0^{\infty} y^{\gamma+1}(t)dt < \infty, \tag{1.7}$$

and it is of the nonlinear limit point type otherwise, that is, there exists a nontrivial solution $y(t)$ satisfying

$$\int_0^{\infty} y^{\gamma+1}(t)dt = \infty. \tag{1.8}$$

Equation (1.5) is said to be of the nonlinear limit circle type if all its solutions satisfy (1.7), and it is said to be of the nonlinear limit point type if there is at least one solution satisfying (1.8).

In this paper, we will give sufficient and necessary conditions to guarantee the nonlinear limit circle type or nonlinear limit point type for (1.5).

2. Main Results

To simplify notations, let $\alpha = 1/2(k+1)$ and $\beta = (2k+1)/2(k+1)$. We define

$$s = \int_0^t \left[\frac{r^\alpha(u)}{a^\beta(u)}\right] du, \qquad x(s) = y(t). \tag{2.1}$$

Then (1.5) becomes

$$\ddot{x} + A(t)\dot{x} + B(t)x^\gamma = 0, \tag{2.2}$$

where

$$A(t) = \frac{b(t)r^\alpha(t)}{a^\beta(t)} + \frac{\alpha(a(t)r(t))'}{a^\alpha(t)r^{\alpha+1}(t)'}, \qquad B(t) = (a(t)r(t))^{\beta-\alpha}. \tag{2.3}$$

Note that in the transformation (2.1), k is no longer an integer here. In fact, $1/2 < k \leq 1$.

We begin with a boundedness result.

Theorem 2.1. *If the condition*

$$\int_0^\infty \frac{(a(t)r(t))'_-}{a(t)r(t)} dt < \infty \tag{2.4}$$

holds, then each solution of (1.5) *is bounded.*

Proof. We rewrite (1.5) as the system

$$y' = w, \qquad w' = \frac{-a'(t)w - r(t)y^\gamma - b(t)w}{a(t)}, \tag{2.5}$$

and we define

$$V(t) = \frac{a(t)w^2}{2r(t)} + \frac{y^{\gamma+1}(t)}{\gamma+1}, \tag{2.6}$$

then we have

$$\begin{aligned}
V'(t) &= \frac{a(t)w}{r(t)}w' + \left[\frac{a(t)}{2r(t)}\right]' w^2 + y^\gamma(t)w \\
&= -\frac{a'(t)}{r(t)}w^2 - \frac{b(t)}{r(t)}w^2 + \left[\frac{a(t)}{2r(t)}\right]' w^2 \\
&= -\frac{a'(t)}{r(t)}w^2 - \frac{b(t)}{r(t)}w^2 + \frac{a'(t)}{2r(t)}w^2 - \frac{a(t)r'(t)}{2r^2(t)}w^2
\end{aligned}$$

$$
\begin{aligned}
&= -\left[\frac{a'(t)}{2r(t)} + \frac{a(t)r'(t)}{2r^2(t)}\right]w^2 - \frac{b(t)}{r(t)}w^2 \\
&= -\frac{(a(t)r(t))'}{2r^2(t)}w^2 - \frac{b(t)}{r(t)}w^2 \\
&\le -\frac{(a(t)r(t))'}{2r^2(t)}w^2 \le \frac{(a(t)r(t))'_-}{2r^2(t)}w^2 \\
&\le \frac{(a(t)r(t))'_-}{2r^2(t)}w^2 + \frac{(a(t)r(t))'_-}{a(t)r(t)}\frac{y^{\gamma+1}(t)}{\gamma+1} \\
&= \frac{(a(t)r(t))'_-}{a(t)r(t)}V(t).
\end{aligned}
\tag{2.7}
$$

Gronwall' inequality and condition (2.4) imply that $V(t)$ is bounded, so $y(t)$ is bounded. □

To prove our main limit-circle result, we write (2.2) as the system

$$\dot{x} = z - A(t)x, \qquad \dot{z} = \dot{A}(t)x - B(t)x^\gamma. \tag{2.8}$$

Theorem 2.2. *Assume that condition* (2.4) *holds and*

$$\int_0^\infty \left| \left[\frac{\alpha(a(t)r(t))'}{a^{1/2}(t)r^{3/2}(t)}\right]' + \left(\frac{\alpha}{2} - \alpha^2\right)\frac{\left[(a(t)r(t))'\right]^2}{a^{3/2}(t)r^{5/2}(t)} + \frac{\left(b(t)r^\alpha(t)/a^\beta(t)\right)'}{(a(t)r(t))^{k/(2(k+1))}}\right| dt < \infty, \tag{2.9}$$

and condition

$$B'(t) \le (\gamma+1)A(t)B(t), \tag{2.10}$$

is satisfied. If

$$\int_0^\infty \frac{1}{B(t)}dt < \infty, \tag{2.11}$$

then (1.5) *is of the nonlinear limit circle type, that is, each solution* $y(t)$ *of* (1.5) *satisfies*

$$\int_0^\infty y^{\gamma+1}(t)dt < \infty. \tag{2.12}$$

Proof. Define

$$V(x,z,s) = \frac{z^2}{2} + B(t)\frac{x^{\gamma+1}}{\gamma+1} + \int_0^t \left[A(\xi)B(\xi) - \frac{\dot{B}(\xi)}{\gamma+1}\right]\frac{r^\alpha(\xi)}{a^\beta(\xi)}y^{\gamma+1}(\xi)d\xi. \tag{2.13}$$

By direct calculation, we have

$$\begin{aligned}\dot{V} &= z\dot{z} + \dot{B}(t)\frac{x^{\gamma+1}}{\gamma+1} + B(t)x^{\gamma}\dot{x} + \left[A(t)B(t) - \frac{\dot{B}(t)}{\gamma+1}\right]y^{\gamma+1}(t)\\ &= z\left(\dot{A}(t)x - B(t)x^{\gamma}\right) + \dot{B}(t)\frac{x^{\gamma+1}}{\gamma+1} + B(t)x^{\gamma}(z - A(t)x)\\ &\quad + \left[A(t)B(t) - \frac{\dot{B}(t)}{\gamma+1}\right]y^{\gamma+1}(t) = \dot{A}(t)xz.\end{aligned} \tag{2.14}$$

Since condition (2.4) and Theorem 2.1 are satisfied, the solution of (1.5) $y(t) = x(s)$ is bounded, that is, there exists a constant $K_1 \geq 0$ such that $|x(t)| \leq K_1$. So

$$\begin{aligned}\left|\dot{A}(t)xz\right| &= \left[\frac{\left|\dot{A}(t)\right||x|^{(1-\gamma)/2}}{[a(t)r(t)]^{(\beta-\alpha)/2}}\right] \times \left[(a(t)r(t))^{(\beta-\alpha)/2}|x|^{(1+\gamma)/2}|z|\right]\\ &\leq K_1^{(1-\gamma)/2}\left[\frac{\left|\dot{A}(t)\right|}{[a(t)r(t)]^{(\beta-\alpha)/2}}\right] \times \frac{\left[(a(t)r(t))^{\beta-\alpha}x^{1+\gamma} + z^2\right]}{2}\\ &\leq K_1^{(1-\gamma)/2}\left[\frac{\left|\dot{A}(t)\right|}{[a(t)r(t)]^{(\beta-\alpha)/2}}\right] \times \left[\frac{Bx^{1+\gamma}}{1+\gamma} + \frac{z^2}{2}\right]\\ &\leq K_1^{(1-\gamma)/2}\left[\frac{\left|\dot{A}(t)\right|}{[a(t)r(t)]^{(\beta-\alpha)/2}}\right] \times V(t).\end{aligned} \tag{2.15}$$

Now

$$\dot{A}(t) = A'(t)\frac{dt}{ds} = A'(t)\frac{a^{\beta}(t)}{r^{\alpha}(t)},$$

$$\begin{aligned}\frac{A'(t)}{[a(t)r(t)]^{(\beta-\alpha)/2}} &= \frac{\left[\left(b(t)r^{\alpha}(t)/a^{\beta}(t)\right) + \left(\alpha(a(t)r(t))'/a^{\alpha}(t)r^{\alpha+1}(t)\right)\right]}{[a(t)r(t)]^{(\beta-\alpha)/2}}\\ &= \frac{\alpha(a(t)r(t))''}{a^{1/2}(t)r^{3/2}(t)}\\ &\quad - \frac{\left[\left(\alpha^2 a'(t)(a(t)r(t))'/(a(t)r(t))^{1+\alpha}\right) + (1+\alpha)\left(\alpha r'(t)(a(t)r(t))'/a^{\alpha}(t)r^{\alpha+2}(t)\right)\right]}{[a(t)r(t)]^{(\beta-\alpha)/2}}\\ &\quad + \frac{\left[b(t)r^{\alpha}(t)/a^{\beta}(t)\right]'}{[a(t)r(t)]^{(\beta-\alpha)/2}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha(a(t)r(t))''}{a^{1/2}(t)r^{3/2}(t)} - \frac{\alpha^2 a'(t)(a(t)r(t))'}{(a(t)r(t))^{3/2}} - (1+\alpha)\alpha\frac{r'(t)(a(t)r(t))'}{a^{1/2}(t)r^{5/2}(t)} \\
&\quad + \frac{\left[b(t)r^{\alpha}(t)/a^{\beta}(t)\right]'}{\left[a(t)r(t)\right]^{(\beta-\alpha)/2}} \\
&= \left[\frac{\alpha(a(t)r(t))'}{a^{1/2}(t)r^{3/2}(t)}\right]' + \frac{\alpha}{2}\frac{(a(t)r(t))'(a'(t)r(t)+a(t)r'(t))}{a^{3/2}(t)r^{5/2}(t)} - \frac{\alpha^2\left[(a(t)r(t))'\right]^2}{a^{3/2}(t)r^{5/2}(t)} \\
&\quad + \frac{\left[b(t)r^{\alpha}(t)/a^{\beta}(t)\right]'}{\left[a(t)r(t)\right]^{(\beta-\alpha)/2}} \\
&= \left[\frac{\alpha(a(t)r(t))'}{a^{1/2}(t)r^{3/2}(t)}\right]' + \frac{\alpha}{2}\frac{\left[(a(t)r(t))'\right]^2}{a^{3/2}(t)r^{5/2}(t)} - \frac{\alpha^2\left[(a(t)r(t))'\right]^2}{a^{3/2}(t)r^{5/2}(t)} + \frac{\left[b(t)r^{\alpha}(t)/a^{\beta}(t)\right]'}{\left[a(t)r(t)\right]^{(\beta-\alpha)/2}} \\
&= \left[\frac{\alpha(a(t)r(t))'}{a^{1/2}(t)r^{3/2}(t)}\right]' + \left(\frac{\alpha}{2}-\alpha^2\right)\frac{\left[(a(t)r(t))'\right]^2}{a^{3/2}(t)r^{5/2}(t)} + \frac{\left[b(t)r^{\alpha}(t)/a^{\beta}(t)\right]'}{\left[a(t)r(t)\right]^{(\beta-\alpha)/2}}.
\end{aligned}
\tag{2.16}
$$

Let $\tau(s)$ denote the inverse function of $s(t)$, we obtain that

$$
\begin{aligned}
\int_0^s \frac{\left|\dot{A}(\tau(v))\right|}{B^{1/2}(\tau(v))}dv &= \int_0^s \frac{\left|A'(\tau(v))a^{\beta}(\tau(v))/r^{\alpha}(\tau(v))\right|}{B^{1/2}(\tau(v))}dv \\
&= \int_0^t \left|\left[\frac{\alpha(a(u)r(u))'}{a^{1/2}(u)r^{3/2}(u)}\right]' + \left(\frac{\alpha}{2}-\alpha^2\right)\frac{\left[(a(u)r(u))'\right]^2}{a^{3/2}(u)r^{5/2}(u)} + \frac{\left(b(u)r^{\alpha}(u)/a^{\beta}(u)\right)'}{(a(u)r(u))^{k/(2(k+1))}}\right|du
\end{aligned}
\tag{2.17}
$$

is convergent by condition (2.9). Hence, integrating $\dot{V}(s)$, applying Gronwall's inequality, and using condition (2.9), we obtain that $V(s)$ is bounded, so

$$
B(t)\frac{y^{\gamma+1}}{\gamma+1} = (a(t)r(t))^{\beta-\alpha}\frac{y^{\gamma+1}}{\gamma+1} \le K_2 \tag{2.18}
$$

for some constant $K_2 > 0$. Condition (2.11) then implies that $y(t)$ is of the nonlinear limit circle type.

When $a(t) \equiv 1$, the (1.5) becomes

$$
y'' + b(t)y' + r(t)y^{\gamma} = 0. \tag{2.19}
$$

In this case, $A(t) = b(t)r^{\alpha}(t) + \alpha r'(t)/r^{\alpha+1}(t)$, $B(t) = r^{\beta-\alpha}(t)$. We get the following corollary. □

Corollary 2.3. *Assume condition* (2.4) *and*

$$\int_0^\infty \left| \left[\frac{\alpha r(t)'}{r^{3/2}(t)} \right]' + \left(\frac{\alpha}{2} - \alpha^2 \right) \frac{[r(t)']^2}{r^{5/2}(t)} + \frac{(b(t)r^\alpha(t))'}{r(t)^{k/(2(k+1))}} \right| dt < \infty, \tag{2.20}$$

and condition

$$\dot{B}(t) \le (\gamma + 1) A(t) B(t) \tag{2.21}$$

is satisfied. If

$$\int_0^\infty \frac{1}{B(t)} dt < \infty, \tag{2.22}$$

then (2.19) *is of the nonlinear limit circle type, that is, each solution $y(t)$ of* (2.19) *satisfies*

$$\int_0^\infty y^{\gamma+1}(t) dt < \infty. \tag{2.23}$$

Example 2.4. Consider the following second-order nonlinear differential equation

$$y'' + \left(1 + t^2\right)^{1/4} y' + \left(1 + t^2\right)^{5/2} y^{1/3} = 0, \tag{2.24}$$

here $b(t) = (1 + t^2)^{1/4}$, $r(t) = (1 + t^2)^{5/2}$, $\gamma = 1/3$. We can easily verify that all the conditions in Corollary 2.3 are fulfilled, so each solution of (2.24) is of the nonlinear limit circle type. We note further that the type of (2.24) cannot be determined since $b(t) \neq 0$.

Next, we give a necessary condition for the sublinear (1.5) to be of the nonlinear limit circle type.

Theorem 2.5. *Suppose condition* (2.4),

$$\begin{gathered} \int_0^\infty \frac{b^2(t)}{a(t)r(t)} dt < \infty, \\ \int_0^\infty \frac{a(t)[r'(t)]^2}{r^3(t)} dt < \infty \end{gathered} \tag{2.25}$$

hold. If $y(t)$ is a nonlinear limit circle type solution of (1.5), *then*

$$\int_0^\infty \frac{a(t)[y'(t)]^2}{r(t)} dt < \infty. \tag{2.26}$$

Proof. Let $y(t)$ be a nonlinear limit circle type solution of (1.5), then $y(t)$ is bounded by Theorem 2.1. Multiplying (1.5) by $y(t)/r(t)$, noting that

$$(a(t)y')' = (a(t)y'y)' - a(t)[y']^2, \tag{2.27}$$

and integrating by parts, we obtain

$$\begin{aligned}\frac{a(t)y'(t)y(t)}{r(t)} - \frac{a(t_1)y'(t_1)y(t_1)}{r(t_1)} + \int_{t_1}^{t}\frac{a(u)y'(u)y(u)r'(u)}{r^2(u)}du \\ - \int_{t_1}^{t}\frac{[a(u)y'(u)]'y(u)}{r(u)}du - \int_{t_1}^{t}\frac{a(u)[y'(u)]^2}{r(u)}du = 0.\end{aligned} \tag{2.28}$$

Using (1.5), we have

$$\begin{aligned}\frac{a(t)y'(t)y(t)}{r(t)} - \frac{a(t_1)y'(t_1)y(t_1)}{r(t_1)} + \int_{t_1}^{t}\frac{a(u)y'(u)y(u)r'(u)}{r^2(u)}du\int_{t_1}^{t}\frac{b(u)y'(u)y(u)}{r(u)}du \\ + \int_{t_1}^{t} y^{\gamma+1}(u) - \int_{t_1}^{t}\frac{a(u)[y'(u)]^2}{r(u)}du = 0.\end{aligned} \tag{2.29}$$

Denote

$$H(t) = \int_{t_1}^{t}\frac{a(u)[y'(u)]^2}{r(u)}du, \tag{2.30}$$

by Schwartz inequality, the boundedness of $y(t)$, and condition (2.14), we obtain

$$\int_{t_1}^{t}\frac{a(u)y'(u)y(u)r'(u)}{r^2(u)}du \le H^{1/2}(t)\left[\int_{t_1}^{t}\frac{a(u)[r'(u)]^2y^2(u)}{r^3(u)}du\right]^{1/2} < M_1H^{1/2}(t) \tag{2.31}$$

for some constant $M_1 > 0$. By Schwartz inequality, the boundedness of $y(t)$, and condition (2.25),

$$\int_{t_1}^{t}\frac{b(u)y'(u)y(u)}{r(u)}du \le \left[\int_{t_1}^{t}\frac{b^2(u)y^2(u)}{a(u)r(u)}du\right]^{1/2} H^{1/2}(t) \le M_2H^{1/2}(t), \tag{2.32}$$

for some constant $M_2 > 0$.

If $y(t)$ is not eventually monotonic, let $\{t_j\} \to \infty$ be an increasing sequence of zeros of $y'(t)$. Then, by (2.28), there exists some constant $M_3 > 0$, such that

$$(M_1 + M_2)H^{1/2}(t_j) + M_3 \ge H(t_j). \tag{2.33}$$

This implies $H(t_j) \le M_4 < \infty$ for all j and some $M_4 > 0$, so (2.26) holds.

If $y(t)$ is eventually monotonic, then $y(t)y'(t) \leq 0$ for all $t \geq t_1$ ($t_1 \geq 0$ large enough). Using (2.28), we can repeat the type of argument used above to obtain that (2.26) holds. This completes the proof of Theorem 2.5. □

The following theorem gives sufficient conditions to ensure that (1.5) being of the nonlinear limit point type.

Theorem 2.6. *Suppose condition* (2.4), (2.9), (2.10), (2.25) *hold. If*

$$\int_0^\infty \frac{\left[(a(t)r(t))'\right]^2}{a(t)r^3(t)}dt < \infty, \tag{2.34}$$

$$\begin{gathered} \int_0^\infty \frac{b^2(t)r^{3\alpha-\beta}(t)}{a^{3\beta-\alpha}(t)}dt < \infty, \\ \int_0^\infty \int_0^t \frac{\left[A(\xi)B(\xi) - \left(\dot{A}(\xi)/\gamma+1\right)\right]\left(r^\alpha(\xi)/a^\beta(\xi)\right)}{B(t)}d\xi dt < \infty, \end{gathered} \tag{2.35}$$

$$\int_0^\infty \frac{1}{B(u)}du = \infty, \tag{2.36}$$

then (1.5) *is of the nonlinear limit point type.*

Proof. As in the proof of Theorem 2.2, we define

$$V(x,z,s) = \frac{z^2}{2} + B(t)\frac{x^{\gamma+1}}{\gamma+1} + \int_0^t \left[A(\xi)B(\xi) - \frac{\dot{A}(\xi)}{\gamma+1}\right]\frac{r^\alpha(\xi)}{a^\beta(\xi)}y^{\gamma+1}(\xi)d\xi, \tag{2.37}$$

we differentiate it to obtain

$$\begin{aligned} \dot{V} &= z\dot{z} + \dot{B}(t)\frac{x^{\gamma+1}}{\gamma+1} + B(t)x^\gamma\dot{x} + \left[A(t)B(t) - \frac{\dot{B}(t)}{\gamma+1}\right]y^{\gamma+1}(t) \\ &= z\left(\dot{A}(t)x - B(t)x^\gamma\right) + \dot{B}(t)\frac{x^{\gamma+1}}{\gamma+1} + B(t)x^\gamma(z - A(t)x) \\ &\quad + \left[A(t)B(t) - \frac{\dot{B}(t)}{\gamma+1}\right]y^{\gamma+1}(t) = \dot{A}(t)xz. \end{aligned} \tag{2.38}$$

Let $y(t) = x(s)$ be any nontrivial solution of (1.5) with $y(t_1) = x(s(t_1)) = x(s_1) \neq 0$. Theorem 2.1 implies that $y(t) = x(s)$ is bounded, so $|x(s)|^{(1-\gamma)/2} \leq K_1$ for some constant $K_1 > 0$. Hence

$$\begin{aligned}
\dot{V}(s) &\geq -\left|\dot{A}(t)\right||x(s)|^{(1-\gamma)/2}|x(s)|^{(1+\gamma)/2}|z(s)| \\
&\geq -K_1 \frac{\left|\dot{A}(t)\right|}{[a(t)r(t)]^{(\beta-\alpha)/2}} \times \left[\frac{z^2}{2} + \frac{B(t)x^{\gamma+1}(t)}{2}\right] \\
&\geq -K_1 \frac{\left|\dot{A}(t)\right|}{[a(t)r(t)]^{(\beta-\alpha)/2}} \times \left[\frac{z^2}{2} + \frac{B(t)x^{\gamma+1}(t)}{\gamma+1}\right] \\
&\geq -K_1 \frac{\left|\dot{A}(t)\right|}{[a(t)r(t)]^{(\beta-\alpha)/2}} \times V(t).
\end{aligned} \tag{2.39}$$

Let $H(t) = K_1|\dot{A}(t)|/B^{1/2}(t)$, then we can write $\dot{V} + H(t)V \geq 0$. So

$$\frac{d}{ds}\left(V(s)\exp\int_{s_1}^{s} H(\tau(\xi))d\xi\right) \geq 0. \tag{2.40}$$

Integrating the above inequality, we obtain

$$V(s)\exp\int_{s_1}^{s} H(\tau(\xi) \geq V(s_1)). \tag{2.41}$$

Condition (2.9) implies

$$\int_{s_1}^{s} H(\tau(\xi))d\xi < \infty, \tag{2.42}$$

and since $V(s_1) > 0$, we have

$$V(s) \geq K_2 > 0 \quad \text{for } s \geq s_1. \tag{2.43}$$

Rewriting $V(x,z,s)$ in terms of y and t and dividing (2.42) by $B(t)$, we obtain

$$\begin{aligned}
&\frac{\left[y'(t)\right]^2 a(t)}{2r(t)} + \frac{y^{\gamma+1}(t)}{\gamma+1} + \left[\frac{b(t)}{[a(t)r(t)]^{\beta-\alpha}} + \alpha\frac{[a(t)r(t)]'}{r^2(t)}\right]y(t)y'(t) \\
&\quad + \left[\frac{b^2(t)r^{3\alpha-\beta}(t)}{2a^{3\beta-\alpha}(t)} + \alpha^2\frac{\left[(a(t)r(t))'\right]^2}{2a(t)r^3(t)} + \alpha\frac{b(t)[a(t)r(t)]'}{[a(t)r(t)]^{2\beta}}\right]y^2(t) \\
&\quad + \int_0^t \frac{\left[A(\xi)B(\xi) - \left(\dot{B}(\xi)/(\gamma+1)\right)\right]\left(r^{\alpha}(\xi)/a^{\beta}(\xi)\right)y^{\gamma+1}(\xi)}{B(t)}d\xi \geq \frac{K_2}{B(t)}.
\end{aligned} \tag{2.44}$$

If $y(t)$ is a limit circle type solution, then $y(t)$ is bounded, $\int_{t_1}^{\infty} y^{\gamma+1}(t)dt < \infty$, and by condition (2.4) and (2.25), we get

$$\int_{t_1}^{\infty} \frac{\left[y'(t)\right]^2 a(t)}{2r(t)} dt < \infty, \tag{2.45}$$

according to Theorem 2.5. By condition (2.34), we get

$$\int_{t_1}^{\infty} \alpha^2 \frac{\left[(a(t)r(t))'\right]^2}{2a(t)r^3(t)} y^2(t)dt < \infty. \tag{2.46}$$

By the Schwartz inequality,

$$\begin{aligned}\int_{t_1}^{\infty} \alpha \frac{[a(t)r(t)]'}{r^2(t)} y(t)y'(t)dt &\le \left(\int_{t_1}^{\infty} \alpha^2 \frac{\left[(a(t)r(t))'\right]^2 y^2(t)}{a(t)r^3(t)} dt\right)^{1/2} \\ &\quad\times \left(\int_{t_1}^{\infty} \frac{\left[y'(t)\right]^2 a(t)}{r(t)} dt\right)^{1/2} < \infty.\end{aligned} \tag{2.47}$$

Noticing condition (2.35) and $y(t)$ being bounded, we get

$$\int_{t_1}^{\infty} \frac{b^2(t)r^{3\alpha-\beta}(t)}{a^{3\beta-\alpha}(t)} y^2(t)dt < \infty, \tag{2.48}$$

so, by the Schwartz inequality, we have

$$\begin{aligned}\int_{t_1}^{\infty} \frac{b(t)[a(t)r(t)]'}{[a(t)r(t)]^{2\beta}} y^2(t)dt &\le \left(\int_{t_1}^{\infty} \frac{b^2(t)r^{3\alpha-\beta}(t)}{a^{3\beta-\alpha}(t)} y^2(t)dt\right)^{1/2} \\ &\quad\times \left(\int_{t_1}^{\infty} \frac{\left[(a(t)r(t))'\right]^2}{a(t)r^3(t)} y^2(t)dt\right)^{1/2} < \infty, \\ \int_{t_1}^{\infty} \frac{b(t)}{[a(t)r(t)]^{\beta-\alpha}} y(t)y'(t)dt &\le \left(\int_{t_1}^{\infty} \frac{b^2(t)r^{3\alpha-\beta}(t)y^2(t)}{a^{3\beta-\alpha}(t)} dt\right)^{1/2} \\ &\quad\times \left(\int_{t_1}^{\infty} \frac{\left[y'(t)\right]^2 a(t)}{r(t)} dt\right)^{1/2} < \infty,\end{aligned} \tag{2.49}$$

Consequently, integrating both sides of (2.44), we see that the integrand of the left side of (2.44) is bounded, but the integrand of the right side of (2.44) tends to infinity according to condition (2.36). This leads to a contradiction, so $y(t)$ is a limit point case solution of (1.5), and (1.5) is of the nonlinear limit point type. $\square$

Similarly, when $a(t) \equiv 1$, (1.5) becomes (2.19) and we get the following corollary.

Corollary 2.7. *Suppose condition* (2.4), (2.10), (2.20) *hold. If*

$$\int_0^\infty \frac{b^2(t)}{r(t)}dt < \infty, \qquad \int_0^\infty \frac{\left[(r(t))'\right]^2}{r^3(t)}dt < \infty, \qquad \int_0^\infty b^2(t)r^{3\alpha-\beta}(t)dt < \infty,$$
$$\int_0^\infty \int_0^t \frac{\left[A(\xi)B(\xi) - \left(\dot{A}(\xi)/(\gamma+1)\right)\right]r^{\alpha(\xi)}}{B(t)}d\xi dt < \infty, \qquad \int_0^\infty \frac{1}{r^{\beta-\alpha}(u)}du = \infty, \tag{2.50}$$

then (2.19) *is of the nonlinear limit point type.*

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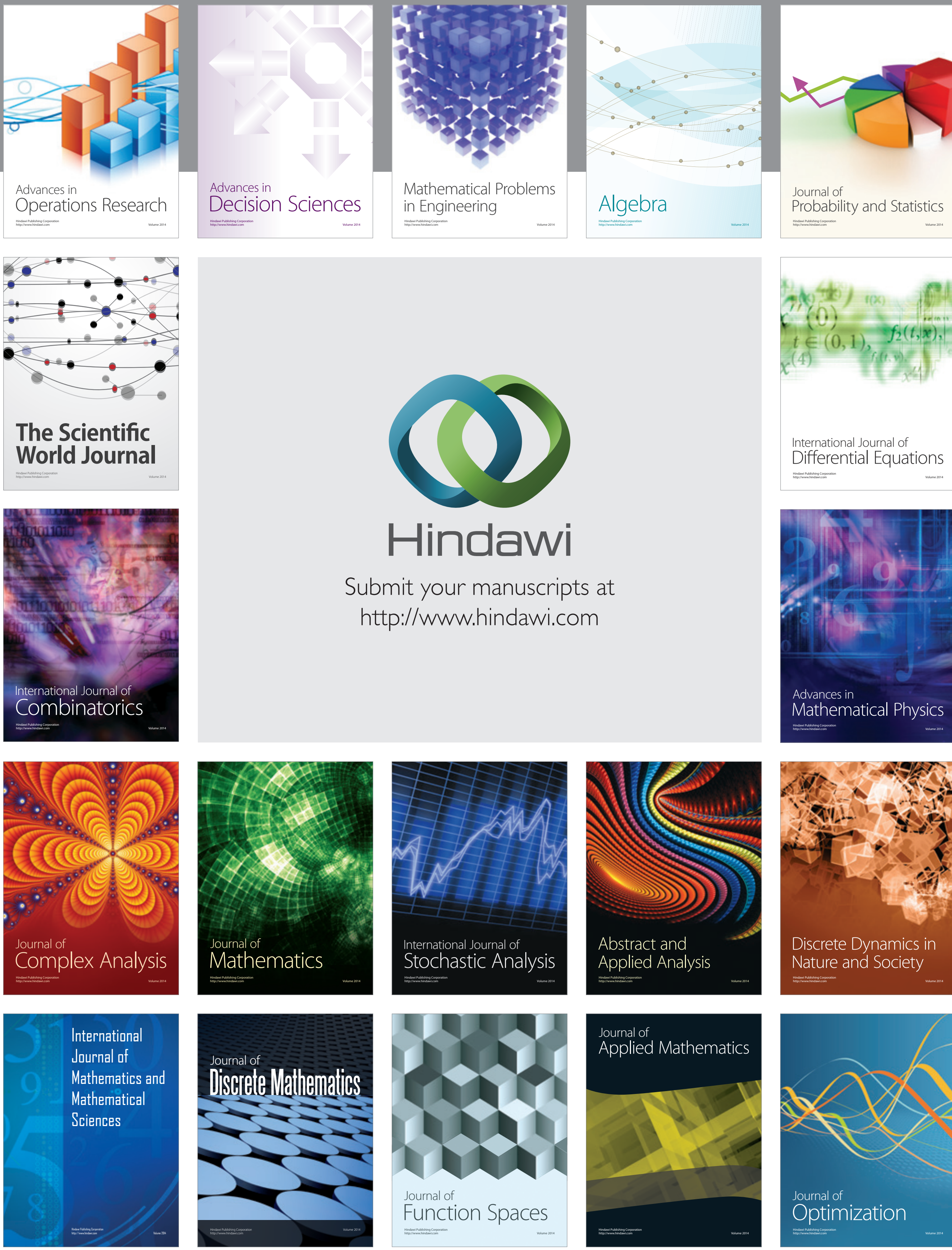
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