Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 657935, 7 pages
doi:10.1155/2011/657935

Research Article

A Sharp Double Inequality between Harmonic and Identric Means

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Received 31 May 2011; Accepted 6 August 2011

Academic Editor: Ondřej Došlý

We find the greatest value p and the least value q in $(0, 1/2)$ such that the double inequality $H(pa+(1-p)b, pb+(1-p)a) < I(a,b) < H(qa+(1-q)b, qb+(1-q)a)$ holds for all $a, b > 0$ with $a \neq b$. Here, $H(a,b)$, and $I(a,b)$ denote the harmonic and identric means of two positive numbers a and b, respectively.

1. Introduction

The classical harmonic mean $H(a,b)$ and identric mean $I(a,b)$ of two positive numbers a and b are defined by

$$H(a,b) = \frac{2ab}{a+b}, \tag{1.1}$$

$$I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.2}$$

respectively. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for H and I can be found in the literature [1–17].

Let $M_p(a,b) = [(a^p+b^p)/2]^{1/p}$, $L(a,b) = (a-b)/(\log a - \log b)$, $G(a,b) = \sqrt{ab}$, $A(a,b) = (a+b)/2$, and $P(a,b) = (a-b)/[4\arctan(\sqrt{a/b}) - \pi]$ be the pth power, logarithmic, geometric, arithmetic, and Seiffert means of two positive numbers a and b with $a \neq b$, respectively. Then it is well-known that

$$\begin{aligned} \min\{a,b\} < H(a,b) &= M_{-1}(a,b) < G(a,b) \\ &= M_0(a,b) < L(a,b) \\ &< P(a,b) < I(a,b) < A(a,b) \\ &= M_1(a,b) < \max\{a,b\} \end{aligned} \tag{1.3}$$

for all $a,b > 0$ with $a \neq b$.

Long and Chu [18] answered the question: what are the greatest value p and the least value q such that $M_p(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) < M_q(a,b)$ for all $a,b > 0$ with $a \neq b$ and $\alpha,\beta > 0$ with $\alpha + \beta < 1$.

In [19], the authors proved that the double inequality

$$\alpha A(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta A(a,b) + (1-\beta)H(a,b) \tag{1.4}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/\pi$ and $\beta \geq 5/6$.

The following sharp bounds for I, $(LI)^{1/2}$, and $(L+I)/2$ in terms of power means are presented in [20]:

$$\begin{gathered} M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \qquad M_0(a,b) < \sqrt{L(a,b)I(a,b)} < M_{1/2}(a,b), \\ M_{\log 2/(1+\log 2)}(a,b) < \frac{L(a,b)+I(a,b)}{2} < M_{1/2}(a,b) \end{gathered} \tag{1.5}$$

for all $a,b > 0$ with $a \neq b$.

Alzer and Qiu [21] proved that the inequalities

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b) \tag{1.6}$$

hold for all positive real numbers a and b with $a \neq b$ if and only if $\alpha \leq 2/3$ and $\beta \geq 2/e = 0.73575$, and so forth.

For fixed $a,b > 0$ with $a \neq b$ and $x \in [0,1/2]$, let

$$f(x) = H(xa + (1-x)b, xb + (1-x)a). \tag{1.7}$$

Then it is not difficult to verify that $f(x)$ is continuous and strictly increasing in $[0,1/2]$. Note that $f(0) = H(a,b) < I(a,b)$ and $f(1/2) = A(a,b) > I(a,b)$. Therefore, it is natural to ask what are the greatest value p and the least value q in $(0,1/2)$ such that the double inequality $H(pa + (1-p)b, pb + (1-p)a) < I(a,b) < H(qa + (1-q)b, qb + (1-q)a)$ holds for all $a,b > 0$ with $a \neq b$. The main purpose of this paper is to answer these questions. Our main result is Theorem 1.1.

Theorem 1.1. *If* $p, q \in (0, 1/2)$, *then the double inequality*

$$\begin{aligned} &H(pa + (1-p)b, pb + (1-p)a) \\ &\quad < I(a,b) \\ &\quad < H(qa + (1-q)b, qb + (1-q)a) \end{aligned} \tag{1.8}$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $p \leq (1 - \sqrt{1-2/e})/2$ *and* $q \geq (6-\sqrt{6})/12$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\lambda = (6-\sqrt{6})/12$ and $\mu = (1-\sqrt{1-2/e})/2$. Then from the monotonicity of the function $f(x) = H(xa + (1-x)b, xb + (1-x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.8) we only need to prove that inequalities

$$I(a,b) < H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a), \tag{2.1}$$

$$I(a,b) > H(\mu a + (1-\mu)b, \mu b + (1-\mu)a), \tag{2.2}$$

hold for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $r \in (0, 1/2)$, then from (1.1) and (1.2) one has

$$\begin{aligned} &\log H(ra + (1-r)b, rb + (1-r)a) - \log I(a,b) \\ &\quad = \log\left\{r(1-r)t^2 + \left[r^2 + (1-r)^2\right]t + r(1-r)\right\} \\ &\qquad - \log(t+1) - \frac{t\log t}{t-1} + 1 + \log 2. \end{aligned} \tag{2.3}$$

Let

$$\begin{aligned} g(t) = &\log\left\{r(1-r)t^2 + \left[r^2 + (1-r)^2\right]t + r(1-r)\right\} \\ &- \log(t+1) - \frac{t\log t}{t-1} + 1 + \log 2. \end{aligned} \tag{2.4}$$

Then simple computations lead to

$$g(1) = 0, \tag{2.5}$$

$$\lim_{t\to+\infty} g(t) = \log[r(1-r)] + 1 + \log 2, \tag{2.6}$$

$$g'(t) = \frac{g_1(t)}{(t-1)^2}, \tag{2.7}$$

where

$$g_1(t) = \log t - \frac{(t-1)\left[\left(2r^2-2r+1\right)t^2+4r(1-r)t+2r^2-2r+1\right]}{(t+1)\left[r(1-r)t^2+\left(2r^2-2r+1\right)t+r(1-r)\right]}, \tag{2.8}$$

$$g_1(1) = 0, \tag{2.9}$$

$$\lim_{t\to+\infty} g_1(t) = +\infty, \tag{2.10}$$

$$g_1'(t) = \frac{g_2(t)}{t(t+1)^2\left[r(1-r)t^2+\left(2r^2-2r+1\right)t+r(1-r)\right]^2}, \tag{2.11}$$

where

$$\begin{aligned} g_2(t) &= r^2(1-r)^2t^6 + \left(2r^4-4r^3-2r^2+4r-1\right)t^5 - \left(17r^4-34r^3+25r^2-8r+1\right)t^4 \\ &\quad +4\left(7r^4-14r^3+13r^2-6r+1\right)t^3 - \left(17r^4-34r^3+25r^2-8r+1\right)t^2 \\ &\quad +\left(2r^4-4r^3-2r^2+4r-1\right)t + r^2(1-r)^2, \end{aligned} \tag{2.12}$$

$$g_2(1) = 0, \tag{2.13}$$

$$\lim_{t\to+\infty} g_2(t) = +\infty, \tag{2.14}$$

$$\begin{aligned} g_2'(t) &= 6r^2(1-r)^2t^5 + 5\left(2r^4-4r^3-2r^2+4r-1\right)t^4 - 4\left(17r^4-34r^3+25r^2-8r+1\right)t^3 \\ &\quad +12\left(7r^4-14r^3+13r^2-6r+1\right)t^2 - 2\left(17r^4-34r^3+25r^2-8r+1\right)t \\ &\quad +2r^4-4r^3-2r^2+4r-1, \end{aligned} \tag{2.15}$$

$$g_2'(1) = 0, \tag{2.16}$$

$$\lim_{t\to+\infty} g_2'(t) = +\infty, \tag{2.17}$$

$$\begin{aligned} g_2''(t) &= 30r^2(1-r)^2t^4 + 20\left(2r^4-4r^3-2r^2+4r-1\right)t^3 - 12\left(17r^4-34r^3+25r^2-8r+1\right)t^2 \\ &\quad +24\left(7r^4-14r^3+13r^2-6r+1\right)t - 2\left(17r^4-34r^3+25r^2-8r+1\right), \end{aligned} \tag{2.18}$$

$$g_2''(1) = -2\left(24r^2-24r+5\right), \tag{2.19}$$

$$\lim_{t\to+\infty} g_2''(t) = +\infty, \tag{2.20}$$

$$\begin{aligned} g_2'''(t) &= 120r^2(1-r)^2t^3 + 60\left(2r^4-4r^3-2r^2+4r-1\right)t^2 \\ &\quad -24\left(17r^4-34r^3+25r^2-8r+1\right)t + 24\left(7r^4-14r^3+13r^2-6r+1\right), \end{aligned} \tag{2.21}$$

$$g_2'''(1) = -12\left(24r^2-24r+5\right), \tag{2.22}$$

$$\lim_{t\to\infty} g_2'''(t) = \infty, \tag{2.23}$$

$$\begin{aligned} g_2^{(4)}(t) &= 360r^2(1-r)^2t^2 + 120\left(2r^4-4r^3-2r^2+4r-1\right)t \\ &\quad -24\left(17r^4-34r^3+25r^2-8r+1\right), \end{aligned} \tag{2.24}$$

$$g_2^{(4)}(1) = 48\left(4r^4 - 8r^3 - 10r^2 + 14r - 3\right), \tag{2.25}$$

$$\lim_{t\to+\infty} g_2^{(4)}(t) = +\infty, \tag{2.26}$$

$$g_2^{(5)}(t) = 720r^2(1-r)^2t + 120\left(2r^4 - 4r^3 - 2r^2 + 4r - 1\right), \tag{2.27}$$

$$g_2^{(5)}(1) = 120\left(8r^4 - 16r^3 + 4r^2 + 4r - 1\right). \tag{2.28}$$

We divide the proof into two cases.

Case 1 ($r = \lambda = (6-\sqrt{6})/12$). Then (2.19), (2.22), (2.25), and (2.28) lead to

$$g_2''(1) = 0, \tag{2.29}$$

$$g_2'''(1) = 0, \tag{2.30}$$

$$g_2^{(4)}(1) = \frac{13}{3} > 0, \tag{2.31}$$

$$g_2^{(5)}(1) = \frac{65}{3} > 0. \tag{2.32}$$

From (2.27) we clearly see that $g_2^{(5)}(t)$ is strictly increasing in $[1,+\infty)$, then inequality (2.32) leads to the conclusion that $g_2^{(5)}(t) > 0$ for $t \in [1,+\infty)$, hence $g_2^{(4)}(t)$ is strictly increasing in $[1,+\infty)$.

It follows from inequality (2.31) and the monotonicity of $g_2^{(4)}(t)$ that $g_2'''(t)$ is strictly increasing in $[1,+\infty)$. Then (2.30) implies that $g_2'''(t) > 0$ for $t \in [1,+\infty)$, so $g_2''(t)$ is strictly increasing in $[1,+\infty)$.

From (2.29) and the monotonicity of $g_2''(t)$ we clearly see that $g_2'(t)$ is strictly increasing in $[1,+\infty)$.

From (2.5), (2.7), (2.9), (2.11), (2.13), (2.16), and the monotonicity of $g_2'(t)$ we conclude that

$$g(t) > 0 \tag{2.33}$$

for $t \in (1,+\infty)$.

Therefore, inequality (2.1) follows from (2.3) and (2.4) together with inequality (2.33).

Case 2 ($r = \mu = (1-\sqrt{1-2/e})/2$). Then (2.19), (2.22), (2.25), and (2.28) lead to

$$g_2''(1) = -\frac{2}{e}(5e-12) < 0, \tag{2.34}$$

$$g_2'''(1) = -\frac{12}{e}(5e-12) < 0, \tag{2.35}$$

$$g_2^{(4)}(1) = -\frac{48}{e^2}\left(3e^2 - 7e - 1\right) < 0, \tag{2.36}$$

$$g_2^{(5)}(1) = \frac{120}{e^2}\left(2 + 2e - e^2\right) > 0. \tag{2.37}$$

From (2.27) and (2.37) we know that ${g_2}^{(4)}(t)$ is strictly increasing in $[1,+\infty)$. Then (2.26) and (2.36) lead to the conclusion that there exists $t_1 > 1$ such that ${g_2}^{(4)}(t) < 0$ for $t \in [1,t_1)$ and ${g_2}^{(4)}(t) > 0$ for $t \in (t_1,+\infty)$, hence $g_2'''(t)$ is strictly decreasing in $[1,t_1]$ and strictly increasing in $[t_1,+\infty)$.

It follows from (2.23) and (2.35) together with the piecewise monotonicity of $g_2'''(t)$ that there exists $t_2 > t_1 > 1$ such that $g_2''(t)$ is strictly decreasing in $[1,t_2]$ and strictly increasing in $[t_2,+\infty)$. Then (2.20) and (2.34) lead to the conclusion that there exists $t_3 > t_2 > 1$ such that $g_2'(t)$ is strictly decreasing in $[1,t_3]$ and strictly increasing in $[t_3,+\infty)$.

From (2.16) and (2.17) together with the piecewise monotonicity of $g_2'(t)$ we clearly see that there exists $t_4 > t_3 > 1$ such that $g_2'(t) < 0$ for $t \in (1,t_4)$ and $g_2'(t) > 0$ for $t \in (t_4,+\infty)$. Therefore, $g_2(t)$ is strictly decreasing in $[1,t_4]$ and strictly increasing in $[t_4,+\infty)$. Then (2.11)–(2.14) lead to the conclusion that there exists $t_5 > t_4 > 1$ such that $g_1(t)$ is strictly decreasing in $[1,t_5]$ and strictly increasing in $[t_5,+\infty)$.

It follows from (2.7)–(2.10) and the piecewise monotonicity of $g_1(t)$ that there exists $t_6 > t_5 > 1$ such that $g(t)$ is strictly decreasing in $[1,t_6]$ and strictly increasingin $[t_6,+\infty)$.

Note that (2.6) becomes

$$\lim_{t\to+\infty} g(t) = \log[r(1-r)] + 1 + \log 2 = 0 \tag{2.38}$$

for $r = \mu = (1-\sqrt{1-2/e})/2$.

From (2.5) and (2.38) together with the piecewise monotonicity of $g(t)$ we clearly see that

$$g(t) < 0 \tag{2.39}$$

for $t \in (1,+\infty)$.

Therefore, inequality (2.2) follows from (2.3) and (2.4) together with inequality (2.39).

Next, we prove that the parameter $\lambda = (6-\sqrt{6})/12$ is the best possible parameter in $(0,1/2)$ such that inequality (2.1) holds for all $a,b > 0$ with $a \neq b$. In fact, if $r < \lambda = (6-\sqrt{6})/12$, then (2.19) leads to $g_2''(1) = -2(24r^2 - 24r + 5) < 0$. From the continuity of $g_2''(t)$ we know that there exists $\delta > 0$ such that

$$g_2''(t) < 0 \tag{2.40}$$

for $t \in (1, 1+\delta)$.

It follows from (2.3)–(2.5), (2.7), (2.9), (2.11), (2.13), and (2.16) that $I(a,b) > H(ra + (1-r)b, rb + (1-r)a)$ for $a/b \in (1,1+\delta)$.

Finally, we prove that the parameter $\mu = (1-\sqrt{1-2/e})/2$ is the best possible parameter in $(0,1/2)$ such that inequality (2.2) holds for all $a,b > 0$ with $a \neq b$. In fact, if $(1-\sqrt{1-2/e})/2 = \mu < r < 1/2$, then (2.6) leads to $\lim_{t\to+\infty} g(t) > 0$. Hence, there exists $T > 1$ such that

$$g(t) > 0 \tag{2.41}$$

for $t \in (T,+\infty)$.

Therefore, $H(ra + (1-r)b, rb + (1-r)a) > I(a,b)$ for $a/b \in (T, +\infty)$, follows from (2.3) and (2.4) together with inequality (2.41). □

Acknowledgments

This research was supported by the Natural Science Foundation of China under Grant 11071069 and Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

References

[1] Y. M. Chu and B. Y. Long, "Sharp inequalities between means," *Mathematical Inequalities and Applications*, vol. 14, no. 3, pp. 647–655, 2011.
[2] Y.-M. Chu, S.-S. Wang, and C. Zong, "Optimal lower power mean bound for the convex combination of harmonic and logarithmic means," *Abstract and Applied Analysis*, vol. 2011, Article ID 520648, 9 pages, 2011.
[3] M.-K. Wang, Y.-M. Chu, and Y.-F. Qiu, "Some comparison inequalities for generalized Muirhead and identric means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 295620, 10 pages, 2010.
[4] Y.-M. Chu and B.-Y. Long, "Best possible inequalities between generalized logarithmic mean and classical means," *Abstract and Applied Analysis*, vol. 2010, Article ID 303286, 13 pages, 2010.
[5] Y.-M. Chu and W.-F. Xia, "Inequalities for generalized logarithmic means," *Journal of Inequalities and Applications*, vol. 2009, Article ID 763252, 7 pages, 2009.
[6] H. Alzer, "A harmonic mean inequality for the gamma function," *Journal of Computational and Applied Mathematics*, vol. 87, no. 2, pp. 195–198, 1997.
[7] H. Alzer, "An inequality for arithmetic and harmonic means," *Aequationes Mathematicae*, vol. 46, no. 3, pp. 257–263, 1993.
[8] H. Alzer, "Inequalities for arithmetic, geometric and harmonic means," *The Bulletin of the London Mathematical Society*, vol. 22, no. 4, pp. 362–366, 1990.
[9] H. Alzer, "Ungleichungen für Mittelwerte," *Archiv der Mathematik*, vol. 47, no. 5, pp. 422–426, 1986.
[10] J. Sándor, "Two inequalities for means," *International Journal of Mathematics and Mathematical Sciences*, vol. 18, no. 3, pp. 621–623, 1995.
[11] J. Sándor, "On refinements of certain inequalities for means," *Archivum Mathematicum*, vol. 31, no. 4, pp. 279–282, 1995.
[12] J. Sándor, "On certain identities for means," *Studia Universitatis Babeş-Bolyai, Mathematica*, vol. 38, no. 4, pp. 7–14, 1993.
[13] J. Sándor, "A note on some inequalities for means," *Archiv der Mathematik*, vol. 56, no. 5, pp. 471–473, 1991.
[14] J. Sándor, "On the identric and logarithmic means," *Aequationes Mathematicae*, vol. 40, no. 2-3, pp. 261–270, 1990.
[15] P. R. Mercer, "Refined arithmetic, geometric and harmonic mean inequalities," *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 4, pp. 1459–1464, 2003.
[16] M. K. Vamanamurthy and M. Vuorinen, "Inequalities for means," *Journal of Mathematical Analysis and Applications*, vol. 183, no. 1, pp. 155–166, 1994.
[17] W. Gautschi, "A harmonic mean inequality for the gamma function," *SIAM Journal on Mathematical Analysis*, vol. 5, pp. 278–281, 1974.
[18] B.-Y. Long and Y.-M. Chu, "Optimal power mean bounds for the weighted geometric mean of classical means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 905679, 6 pages, 2010.
[19] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang, and G.-D. Wang, "The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean," *Journal of Inequalities and Applications*, vol. 2010, Article ID 436457, 7 pages, 2010.
[20] H. Alzer, "Bestmögliche Abschätzungen für spezielle Mittelwerte," *Zbornik Radova Prirodno-Matematichkog Fakulteta, Serija za Matematiku*, vol. 23, no. 1, pp. 331–346, 1993.
[21] H. Alzer and S.-L. Qiu, "Inequalities for means in two variables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.