

Research Article

A Sharp Double Inequality between Harmonic and Identric Means

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We find the greatest value p and the least value q in $(0, 1/2)$ such that the double inequality $H(pa + (1-p)b, pb + (1-p)a) < I(a, b) < H(qa + (1-q)b, qb + (1-q)a)$ holds for all $a, b > 0$ with $a \neq b$. Here, $H(a, b)$, and $I(a, b)$ denote the harmonic and identric means of two positive numbers a and b , respectively.

1. Introduction

The classical harmonic mean $H(a, b)$ and identric mean $I(a, b)$ of two positive numbers a and b are defined by

$$H(a, b) = \frac{2ab}{a+b}, \quad (1.1)$$

$$I(a, b) = \begin{cases} \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}, & a \neq b, \\ a, & a = b, \end{cases} \quad (1.2)$$

respectively. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for H and I can be found in the literature [1–17].

Let $M_p(a, b) = [(a^p + b^p)/2]^{1/p}$, $L(a, b) = (a - b)/(\log a - \log b)$, $G(a, b) = \sqrt{ab}$, $A(a, b) = (a + b)/2$, and $P(a, b) = (a - b)/[4 \arctan(\sqrt{a/b}) - \pi]$ be the p th power, logarithmic,

geometric, arithmetic, and Seiffert means of two positive numbers a and b with $a \neq b$, respectively. Then it is well-known that

$$\begin{aligned} \min\{a, b\} < H(a, b) = M_{-1}(a, b) < G(a, b) \\ &= M_0(a, b) < L(a, b) \\ &< P(a, b) < I(a, b) < A(a, b) \\ &= M_1(a, b) < \max\{a, b\} \end{aligned} \quad (1.3)$$

for all $a, b > 0$ with $a \neq b$.

Long and Chu [18] answered the question: what are the greatest value p and the least value q such that $M_p(a, b) < A^\alpha(a, b)G^\beta(a, b)H^{1-\alpha-\beta}(a, b) < M_q(a, b)$ for all $a, b > 0$ with $a \neq b$ and $\alpha, \beta > 0$ with $\alpha + \beta < 1$.

In [19], the authors proved that the double inequality

$$\alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b) \quad (1.4)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/\pi$ and $\beta \geq 5/6$.

The following sharp bounds for I , $(LI)^{1/2}$, and $(L+I)/2$ in terms of power means are presented in [20]:

$$\begin{aligned} M_{2/3}(a, b) < I(a, b) < M_{\log 2}(a, b), \quad M_0(a, b) < \sqrt{L(a, b)I(a, b)} < M_{1/2}(a, b), \\ M_{\log 2/(1+\log 2)}(a, b) < \frac{L(a, b) + I(a, b)}{2} < M_{1/2}(a, b) \end{aligned} \quad (1.5)$$

for all $a, b > 0$ with $a \neq b$.

Alzer and Qiu [21] proved that the inequalities

$$\alpha A(a, b) + (1 - \alpha)G(a, b) < I(a, b) < \beta A(a, b) + (1 - \beta)G(a, b) \quad (1.6)$$

hold for all positive real numbers a and b with $a \neq b$ if and only if $\alpha \leq 2/3$ and $\beta \geq 2/e = 0.73575$, and so forth.

For fixed $a, b > 0$ with $a \neq b$ and $x \in [0, 1/2]$, let

$$f(x) = H(xa + (1 - x)b, xb + (1 - x)a). \quad (1.7)$$

Then it is not difficult to verify that $f(x)$ is continuous and strictly increasing in $[0, 1/2]$. Note that $f(0) = H(a, b) < I(a, b)$ and $f(1/2) = A(a, b) > I(a, b)$. Therefore, it is natural to ask what are the greatest value p and the least value q in $(0, 1/2)$ such that the double inequality $H(pa + (1 - p)b, pb + (1 - p)a) < I(a, b) < H(qa + (1 - q)b, qb + (1 - q)a)$ holds for all $a, b > 0$ with $a \neq b$. The main purpose of this paper is to answer these questions. Our main result is Theorem 1.1.

Theorem 1.1. *If $p, q \in (0, 1/2)$, then the double inequality*

$$\begin{aligned} &H(pa + (1 - p)b, pb + (1 - p)a) \\ &< I(a, b) \\ &< H(qa + (1 - q)b, qb + (1 - q)a) \end{aligned} \tag{1.8}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq (1 - \sqrt{1 - 2/e})/2$ and $q \geq (6 - \sqrt{6})/12$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\lambda = (6 - \sqrt{6})/12$ and $\mu = (1 - \sqrt{1 - 2/e})/2$. Then from the monotonicity of the function $f(x) = H(xa + (1 - x)b, xb + (1 - x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.8) we only need to prove that inequalities

$$I(a, b) < H(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a), \tag{2.1}$$

$$I(a, b) > H(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a), \tag{2.2}$$

hold for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $r \in (0, 1/2)$, then from (1.1) and (1.2) one has

$$\begin{aligned} &\log H(ra + (1 - r)b, rb + (1 - r)a) - \log I(a, b) \\ &= \log \left\{ r(1 - r)t^2 + [r^2 + (1 - r)^2]t + r(1 - r) \right\} \\ &\quad - \log(t + 1) - \frac{t \log t}{t - 1} + 1 + \log 2. \end{aligned} \tag{2.3}$$

Let

$$\begin{aligned} g(t) &= \log \left\{ r(1 - r)t^2 + [r^2 + (1 - r)^2]t + r(1 - r) \right\} \\ &\quad - \log(t + 1) - \frac{t \log t}{t - 1} + 1 + \log 2. \end{aligned} \tag{2.4}$$

Then simple computations lead to

$$g(1) = 0, \tag{2.5}$$

$$\lim_{t \rightarrow +\infty} g(t) = \log[r(1 - r)] + 1 + \log 2, \tag{2.6}$$

$$g'(t) = \frac{g_1(t)}{(t - 1)^2}, \tag{2.7}$$

where

$$g_1(t) = \log t - \frac{(t-1)[(2r^2-2r+1)t^2+4r(1-r)t+2r^2-2r+1]}{(t+1)[r(1-r)t^2+(2r^2-2r+1)t+r(1-r)]}, \quad (2.8)$$

$$g_1(1) = 0, \quad (2.9)$$

$$\lim_{t \rightarrow +\infty} g_1(t) = +\infty, \quad (2.10)$$

$$g_1'(t) = \frac{g_2(t)}{t(t+1)^2[r(1-r)t^2+(2r^2-2r+1)t+r(1-r)]^2}, \quad (2.11)$$

where

$$\begin{aligned} g_2(t) = & r^2(1-r)^2t^6 + (2r^4-4r^3-2r^2+4r-1)t^5 - (17r^4-34r^3+25r^2-8r+1)t^4 \\ & + 4(7r^4-14r^3+13r^2-6r+1)t^3 - (17r^4-34r^3+25r^2-8r+1)t^2 \\ & + (2r^4-4r^3-2r^2+4r-1)t + r^2(1-r)^2, \end{aligned} \quad (2.12)$$

$$g_2(1) = 0, \quad (2.13)$$

$$\lim_{t \rightarrow +\infty} g_2(t) = +\infty, \quad (2.14)$$

$$\begin{aligned} g_2'(t) = & 6r^2(1-r)^2t^5 + 5(2r^4-4r^3-2r^2+4r-1)t^4 - 4(17r^4-34r^3+25r^2-8r+1)t^3 \\ & + 12(7r^4-14r^3+13r^2-6r+1)t^2 - 2(17r^4-34r^3+25r^2-8r+1)t \\ & + 2r^4-4r^3-2r^2+4r-1, \end{aligned} \quad (2.15)$$

$$g_2'(1) = 0, \quad (2.16)$$

$$\lim_{t \rightarrow +\infty} g_2'(t) = +\infty, \quad (2.17)$$

$$\begin{aligned} g_2''(t) = & 30r^2(1-r)^2t^4 + 20(2r^4-4r^3-2r^2+4r-1)t^3 - 12(17r^4-34r^3+25r^2-8r+1)t^2 \\ & + 24(7r^4-14r^3+13r^2-6r+1)t - 2(17r^4-34r^3+25r^2-8r+1), \end{aligned} \quad (2.18)$$

$$g_2''(1) = -2(24r^2-24r+5), \quad (2.19)$$

$$\lim_{t \rightarrow +\infty} g_2''(t) = +\infty, \quad (2.20)$$

$$\begin{aligned} g_2'''(t) = & 120r^2(1-r)^2t^3 + 60(2r^4-4r^3-2r^2+4r-1)t^2 \\ & - 24(17r^4-34r^3+25r^2-8r+1)t + 24(7r^4-14r^3+13r^2-6r+1), \end{aligned} \quad (2.21)$$

$$g_2'''(1) = -12(24r^2-24r+5), \quad (2.22)$$

$$\lim_{t \rightarrow \infty} g_2'''(t) = \infty, \quad (2.23)$$

$$\begin{aligned} g_2^{(4)}(t) = & 360r^2(1-r)^2t^2 + 120(2r^4-4r^3-2r^2+4r-1)t \\ & - 24(17r^4-34r^3+25r^2-8r+1), \end{aligned} \quad (2.24)$$

$$g_2^{(4)}(1) = 48(4r^4 - 8r^3 - 10r^2 + 14r - 3), \tag{2.25}$$

$$\lim_{t \rightarrow +\infty} g_2^{(4)}(t) = +\infty, \tag{2.26}$$

$$g_2^{(5)}(t) = 720r^2(1-r)^2t + 120(2r^4 - 4r^3 - 2r^2 + 4r - 1), \tag{2.27}$$

$$g_2^{(5)}(1) = 120(8r^4 - 16r^3 + 4r^2 + 4r - 1). \tag{2.28}$$

We divide the proof into two cases.

Case 1 ($r = \lambda = (6 - \sqrt{6})/12$). Then (2.19), (2.22), (2.25), and (2.28) lead to

$$g_2''(1) = 0, \tag{2.29}$$

$$g_2'''(1) = 0, \tag{2.30}$$

$$g_2^{(4)}(1) = \frac{13}{3} > 0, \tag{2.31}$$

$$g_2^{(5)}(1) = \frac{65}{3} > 0. \tag{2.32}$$

From (2.27) we clearly see that $g_2^{(5)}(t)$ is strictly increasing in $[1, +\infty)$, then inequality (2.32) leads to the conclusion that $g_2^{(5)}(t) > 0$ for $t \in [1, +\infty)$, hence $g_2^{(4)}(t)$ is strictly increasing in $[1, +\infty)$.

It follows from inequality (2.31) and the monotonicity of $g_2^{(4)}(t)$ that $g_2'''(t)$ is strictly increasing in $[1, +\infty)$. Then (2.30) implies that $g_2'''(t) > 0$ for $t \in [1, +\infty)$, so $g_2''(t)$ is strictly increasing in $[1, +\infty)$.

From (2.29) and the monotonicity of $g_2''(t)$ we clearly see that $g_2'(t)$ is strictly increasing in $[1, +\infty)$.

From (2.5), (2.7), (2.9), (2.11), (2.13), (2.16), and the monotonicity of $g_2'(t)$ we conclude that

$$g(t) > 0 \tag{2.33}$$

for $t \in (1, +\infty)$.

Therefore, inequality (2.1) follows from (2.3) and (2.4) together with inequality (2.33).

Case 2 ($r = \mu = (1 - \sqrt{1 - 2/e})/2$). Then (2.19), (2.22), (2.25), and (2.28) lead to

$$g_2''(1) = -\frac{2}{e}(5e - 12) < 0, \tag{2.34}$$

$$g_2'''(1) = -\frac{12}{e}(5e - 12) < 0, \tag{2.35}$$

$$g_2^{(4)}(1) = -\frac{48}{e^2}(3e^2 - 7e - 1) < 0, \tag{2.36}$$

$$g_2^{(5)}(1) = \frac{120}{e^2}(2 + 2e - e^2) > 0. \tag{2.37}$$

From (2.27) and (2.37) we know that $g_2^{(4)}(t)$ is strictly increasing in $[1, +\infty)$. Then (2.26) and (2.36) lead to the conclusion that there exists $t_1 > 1$ such that $g_2^{(4)}(t) < 0$ for $t \in [1, t_1)$ and $g_2^{(4)}(t) > 0$ for $t \in (t_1, +\infty)$, hence $g_2'''(t)$ is strictly decreasing in $[1, t_1]$ and strictly increasing in $[t_1, +\infty)$.

It follows from (2.23) and (2.35) together with the piecewise monotonicity of $g_2'''(t)$ that there exists $t_2 > t_1 > 1$ such that $g_2''(t)$ is strictly decreasing in $[1, t_2]$ and strictly increasing in $[t_2, +\infty)$. Then (2.20) and (2.34) lead to the conclusion that there exists $t_3 > t_2 > 1$ such that $g_2'(t)$ is strictly decreasing in $[1, t_3]$ and strictly increasing in $[t_3, +\infty)$.

From (2.16) and (2.17) together with the piecewise monotonicity of $g_2'(t)$ we clearly see that there exists $t_4 > t_3 > 1$ such that $g_2'(t) < 0$ for $t \in (1, t_4)$ and $g_2'(t) > 0$ for $t \in (t_4, +\infty)$. Therefore, $g_2(t)$ is strictly decreasing in $[1, t_4]$ and strictly increasing in $[t_4, +\infty)$. Then (2.11)–(2.14) lead to the conclusion that there exists $t_5 > t_4 > 1$ such that $g_1(t)$ is strictly decreasing in $[1, t_5]$ and strictly increasing in $[t_5, +\infty)$.

It follows from (2.7)–(2.10) and the piecewise monotonicity of $g_1(t)$ that there exists $t_6 > t_5 > 1$ such that $g(t)$ is strictly decreasing in $[1, t_6]$ and strictly increasing in $[t_6, +\infty)$.

Note that (2.6) becomes

$$\lim_{t \rightarrow +\infty} g(t) = \log[r(1-r)] + 1 + \log 2 = 0 \quad (2.38)$$

for $r = \mu = (1 - \sqrt{1 - 2/e})/2$.

From (2.5) and (2.38) together with the piecewise monotonicity of $g(t)$ we clearly see that

$$g(t) < 0 \quad (2.39)$$

for $t \in (1, +\infty)$.

Therefore, inequality (2.2) follows from (2.3) and (2.4) together with inequality (2.39).

Next, we prove that the parameter $\lambda = (6 - \sqrt{6})/12$ is the best possible parameter in $(0, 1/2)$ such that inequality (2.1) holds for all $a, b > 0$ with $a \neq b$. In fact, if $r < \lambda = (6 - \sqrt{6})/12$, then (2.19) leads to $g_2''(1) = -2(24r^2 - 24r + 5) < 0$. From the continuity of $g_2''(t)$ we know that there exists $\delta > 0$ such that

$$g_2''(t) < 0 \quad (2.40)$$

for $t \in (1, 1 + \delta)$.

It follows from (2.3)–(2.5), (2.7), (2.9), (2.11), (2.13), and (2.16) that $I(a, b) > H(ra + (1-r)b, rb + (1-r)a)$ for $a/b \in (1, 1 + \delta)$.

Finally, we prove that the parameter $\mu = (1 - \sqrt{1 - 2/e})/2$ is the best possible parameter in $(0, 1/2)$ such that inequality (2.2) holds for all $a, b > 0$ with $a \neq b$. In fact, if $(1 - \sqrt{1 - 2/e})/2 = \mu < r < 1/2$, then (2.6) leads to $\lim_{t \rightarrow +\infty} g(t) > 0$. Hence, there exists $T > 1$ such that

$$g(t) > 0 \quad (2.41)$$

for $t \in (T, +\infty)$.

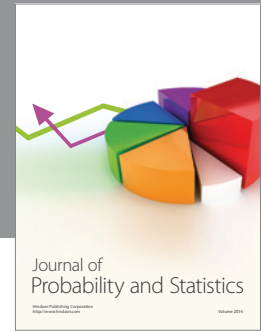
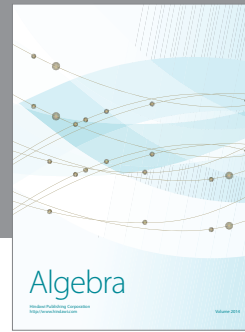
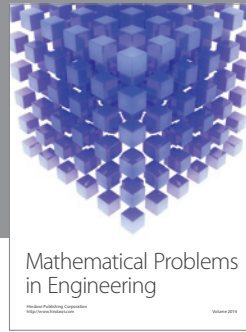
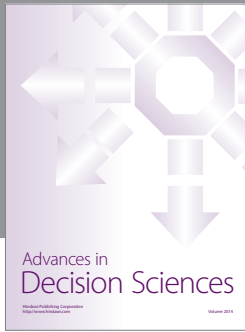
Therefore, $H(ra + (1 - r)b, rb + (1 - r)a) > I(a, b)$ for $a/b \in (T, +\infty)$, follows from (2.3) and (2.4) together with inequality (2.41). \square

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