Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 143079, 14 pages
doi:10.1155/2011/143079

Research Article

Almost Surely Asymptotic Stability of Exact and Numerical Solutions for Neutral Stochastic Pantograph Equations

Zhanhua Yu

Department of Mathematics, Harbin Institute of Technology at Weihai, Weihai 264209, China

Correspondence should be addressed to Zhanhua Yu, yuzhanhua@yeah.net

Received 26 March 2011; Revised 29 June 2011; Accepted 29 June 2011

Academic Editor: Nobuyuki Kenmochi

We study the almost surely asymptotic stability of exact solutions to neutral stochastic pantograph equations (NSPEs), and sufficient conditions are obtained. Based on these sufficient conditions, we show that the backward Euler method (BEM) with variable stepsize can preserve the almost surely asymptotic stability. Numerical examples are demonstrated for illustration.

1. Introduction

The neutral pantograph equation (NPE) plays important roles in mathematical and industrial problems (see [1]). It has been studied by many authors numerically and analytically. We refer to [1–7]. One kind of NPEs reads

$$[x(t) - N(x(qt))]' = f(t, x(t), x(qt)). \tag{1.1}$$

Taking the environmental disturbances into account, we are led to the following neutral stochastic pantograph equation (NSPE)

$$d[x(t) - N(x(qt))] = f(t, x(t), x(qt))dt + g(t, x(t), x(qt))dB(t), \tag{1.2}$$

which is a kind of neutral stochastic delay differential equations (NSDDEs).

Using the continuous semimartingale convergence theorem (cf. [8]), Mao et al. (see [9, 10]) studied the almost surely asymptotic stability of several kinds of NSDDEs. As most NSDDEs cannot be solved explicitly, numerical methods have become essential. Efficient numerical methods for NSDDEs can be found in [11–13]. The stability theory of numerical solutions is one of fundamental research topics in the numerical analysis. The almost surely asymptotic stability of numerical solutions for stochastic differential equations (SDEs) and stochastic functional differential equations (SFDEs) has received much more attention (see [14–19]). Corresponding to the continuous semimartingale convergence theorem (cf. [8]), the discrete semimartingale convergence theorem (cf. [17, 20]) also plays important roles in the almost surely asymptotic stability analysis of numerical solutions for SDEs and SFDEs (see [17–19]). To our best knowledge, no results on the almost surely asymptotic stability of exact and numerical solutions for the NSPE (1.2) can be found. We aim in this paper to study the almost surely asymptotic stability of exact and numerical solutions to NSPEs by using the continuous semimartingale convergence theorem and the discrete semimartingale convergence theorem. We prove that the backward Euler method (BEM) with variable stepsize can preserve the almost surely asymptotic stability under the conditions which guarantee the almost surely asymptotic stability of the exact solution.

In Section 2, we introduce some necessary notations and elementary theories of NSPEs (1.2). Moreover, we state the discrete semimartingale convergence theorem as a lemma. In Section 3, we study the almost surely asymptotic stability of exact solutions to NSPEs (1.2). Section 4 gives the almost surely asymptotic stability of the backward Euler method with variable stepsize. Numerical experiments are presented in the finial section.

2. Neutral Stochastic Pantograph Equation

Throughout this paper, unless otherwise specified, we use the following notations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous, and $\mathcal{F}_0$ contains all P-null sets). $B(t)$ is a scalar Brownian motion defined on the probability space. $|\cdot|$ denotes the Euclidean norm in R^n. The inner product of x, y in R^n is denoted by $\langle x, y\rangle$ or $x^T y$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\operatorname{trace}(A^T A)}$. Let $\mathcal{L}^1([0,T]; R^n)$ denote the family of all R^n-value measurable $\mathcal{F}_t$-adapted processes $f = \{f(t)\}_{0\leq t\leq T}$ such that $\int_0^T |f(t)|dt < \infty$ w.p.1. Let $\mathcal{L}^2([0,T]; R^n)$ denote the family of all R^n-value measurable $\mathcal{F}_t$-adapted processes $f = \{f(t)\}_{0\leq t\leq T}$ such that $\int_0^T |f(t)|^2 dt < \infty$ w.p.1.

Consider an n-dimensional neutral stochastic pantograph equation

$$d\left[x(t) - N(x(qt))\right] = f(t, x(t), x(qt))dt + g(t, x(t), x(qt))dB(t), \tag{2.1}$$

on $t \geq 0$ with $\mathcal{F}_0$-measurable bounded initial data $x(0) = x_0$. Here $0 < q < 1$, $f : R^+ \times R^n \times R^n \to R^n$, $g : R^+ \times R^n \times R^n \to R^n$, and $N : R^n \to R^n$.

Let $C(R^n; R^+)$ denote the family of continuous functions from R^n to R^+. Let $C^{1,2}(R^+ \times R^n; R^+)$ denote the family of all nonnegative functions $V(t, x)$ on $R^+ \times R^n$ which are continuously once differentiable in t and twice differentiable in x. For each $V \in C^{1,2}(R^+ \times R^n; R^+)$, define an operator LV from $R^+ \times R^n \times R^n$ to R by

$$\begin{aligned} \mathrm{LV}(t, x, y) &= V_t(t, x - N(y)) + V_x(t, x - N(y)) f(t, x, y) \\ &\quad + \frac{1}{2}\operatorname{trace}\left[g^T(t, x, y) V_{xx}(t, x - N(y)) g(t, x, y)\right], \end{aligned} \tag{2.2}$$

where

$$V_t(t,x) = \frac{\partial V(t,x)}{\partial t}, \qquad V_x(t,x) = \left(\frac{\partial V(t,x)}{\partial x_1}, \dots, \frac{\partial V(t,x)}{\partial x_n}\right),$$
$$V_{xx}(t,x) = \left(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right)_{n\times n}. \tag{2.3}$$

To be precise, we first give the definition of the solution to (2.1) on $0 \le t \le T$.

Definition 2.1. A R^n-value stochastic process $x(t)$ on $0 \le t \le T$ is called a solution of (2.1) if it has the following properties:

(1) $\{x(t)\}_{0\le t\le T}$ is continuous and $\mathcal{F}_t$-adapted;

(2) $f(t,x(t),x(qt)) \in \mathcal{L}^1([0,T];R^n)$, $g(t,x(t),x(qt)) \in \mathcal{L}^2([0,T];R^n)$:

(3) $x(0) = x_0$, and (2.1) holds for every $t \in [0,T]$ with probability 1.

A solution $x(t)$ is said to be unique if any other solution $\overline{x}(t)$ is indistinguishable from it, that is,

$$P\{x(t) = \overline{x}(t),\ 0 \le t \le T\} = 1. \tag{2.4}$$

To ensure the existence and uniqueness of the solution to (2.1) on $t \in [0,T]$, we impose the following assumptions on the coefficients N, f, and g.

Assumption 2.2. Assume that both f and g satisfy the global Lipschitz condition and the linear growth condition. That is, there exist two positive constants L and K such that for all $x, y, \overline{x}, \overline{y} \in R^n$, and $t \in [0,T]$,

$$\left|f(t,x,y) - f(t,\overline{x},\overline{y})\right|^2 \vee \left|g(t,x,y) - g(t,\overline{x},\overline{y})\right|^2 \le L\left(|x-\overline{x}|^2 + \left|y-\overline{y}\right|^2\right), \tag{2.5}$$

and for all $x, y \in R^n$, and $t \in [0,T]$,

$$\left|f(t,x,y)\right|^2 \vee \left|g(t,x,y)\right|^2 \le K\left(1 + |x|^2 + \left|y\right|^2\right). \tag{2.6}$$

Assumption 2.3. Assume that there is a constant $\kappa \in (0,1)$ such that

$$\left|N(x) - N(y)\right| \le \kappa\left|x-y\right|, \quad \forall x, y \in R^n. \tag{2.7}$$

Under Assumptions 2.2 and 2.3, the following results can be derived.

Lemma 2.4. *Let Assumptions 2.2 and 2.3 hold. Let $x(t)$ be a solution to (2.1) with $\mathcal{F}_0$-measurable bounded initial data $x(0) = x_0$. Then*

$$E\left(\sup_{0\le t\le T}|x(t)|^2\right) \le \left(1 + \frac{(1-\kappa)\kappa + 3(1-\sqrt{\kappa})}{(1-\sqrt{\kappa})^2(1-\kappa)}E|x_0|^2\right)\exp\left\{\frac{6K(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\right\}. \tag{2.8}$$

The proof of Lemma 2.4 is similar to Lemma 6.2.4 in [21], so we omit the details.

Theorem 2.5. *Let Assumptions 2.2 and 2.3 hold, then for any $\mathcal{F}_0$-measurable bounded initial data $x(0) = x_0$, (2.1) has a unique solution $x(t)$ on $t \in [0, T]$.*

Based on Lemma 6.2.3 in [21] and Lemma 2.4, this theorem can be proved in the same way as Theorem 6.2.2 in [21], so the details are omitted.

The discrete semimartingale convergence theorem (cf. [17, 20]) will play an important role in this paper.

Lemma 2.6. *Let $\{A_i\}$ and $\{U_i\}$ be two sequences of nonnegative random variables such that both A_i and U_i are $\mathcal{F}_i$-measurable for $i = 1, 2, \ldots$, and $A_0 = U_0 = 0$ a.s. Let M_i be a real-valued local martingale with $M_0 = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable. Assume that $\{X_i\}$ is a nonnegative semimartingale with the Doob-Mayer decomposition*

$$X_i = \zeta + A_i - U_i + M_i. \tag{2.9}$$

If $\lim_{i\to\infty} A_i < \infty$ a.s., then for almost all $\omega \in \Omega$:

$$\lim_{i\to\infty} X_i < \infty, \qquad \lim_{i\to\infty} U_i < \infty, \tag{2.10}$$

that is, both X_i and U_i converge to finite random variables.

3. Almost Surely Asymptotic Stability of Neutral Stochastic Pantograph Equations

In this section, we investigate the almost surely asymptotic stability of (2.1). We assume (2.1) has a continuous unique global solution for given $\mathcal{F}_0$-measurable bounded initial data x_0. Moreover, we always assume that $f(t,0,0) = 0, g(t,0,0) = 0$, $N(0) = 0$ in the following sections. Therefore, (2.1) admits a trivial solution $x(t) = 0$.

To be precise, let us give the definition on the almost surely asymptotic stability of (2.1).

Definition 3.1. The solution $x(t)$ to (2.1) is said to be almost surely asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \quad \text{a.s.} \tag{3.1}$$

for any bounded $\mathcal{F}_0$-measurable bounded initial data $x(0)$.

Lemma 3.2. *Let $\rho : R^+ \to (0,\infty)$ and $z : [0,\infty) \to R^n$ be a continuous functions. Assume that*

$$\sigma_1 := \limsup_{t\to\infty} \frac{\rho(t)}{\rho(qt)} < \frac{1}{\kappa},$$
$$\sigma_2 := \limsup_{t\to\infty} \left[\rho(t)\left|z(t) - N(z(qt))\right|\right] < \infty. \tag{3.2}$$

Then,

$$\limsup_{t\to\infty} \left[\rho(t)|z(t)|\right] \le \frac{\sigma_2}{1-\kappa\sigma_1}. \tag{3.3}$$

Proof. Using the idea of Lemma 3.1 in [9], we can obtain the desired result. □

Lemma 3.3. *Suppose that* (2.1) *has a continuous unique global solution $x(t)$ for given $\mathcal{F}_0$-measurable bounded initial data x_0. Let Assumption 2.3 hold. Assume that there are functions $U \in C^{1,2}(R^+ \times R^n; R^+)$, $w \in C(R^n; R^+)$, and four positive constants $\lambda_1 > \lambda_2, \lambda_3, \lambda_4$ such that*

$$LU(t,x,y) \le -\lambda_1 w(x) + q\lambda_2 w(y), \quad (t,x,y) \in R^+ \times R^n \times R^n,$$
$$U(t, x - N(y)) \le \lambda_3 w(x) + \lambda_4 w(y), \quad (t,x) \in R^+ \times R^n. \tag{3.4}$$

Then, for any $\varepsilon \in (0,\gamma^*)$

$$\limsup_{t\to\infty} t^{(\gamma^*-\varepsilon)} U(t, x(t) - N(x(qt))) < \infty \quad a.s., \tag{3.5}$$

where γ^* is positive and satisfies

$$\lambda_1 = \lambda_2 q^{-\gamma^*}. \tag{3.6}$$

That is,

$$\lim_{t\to\infty} U(t, x(t) - N(x(qt))) = 0 \quad a.s. \tag{3.7}$$

Proof. Choose $V(t,x(t)) = t^{\gamma} U(t, x(t) - N(x(qt)))$ for $(t,x) \in R^+ \times R^n$ and $\gamma > 0$. Similar to the proof of Lemma 2.2 in [9], the desired conclusion can be obtained by using the continuous semimartingale convergence theorem (cf. [8]). □

Theorem 3.4. *Suppose that* (2.1) *has a continuous unique global solution $x(t)$ for given $\mathcal{F}_0$-measurable bounded initial data x_0. Let Assumption 2.3 hold. Assume that there are four positive constants $\lambda_1 - \lambda_4$ such that*

$$2(x - N(y))^T f(t,x,y) \le -\lambda_1 |x|^2 + \lambda_2 |y|^2,$$
$$|g(t,x,y)|^2 \le \lambda_3 |x|^2 + \lambda_4 |y|^2 \tag{3.8}$$

for $t \geq 0$ *and* $x, y \in R^n$. *If*

$$\lambda_1 - \lambda_3 > \frac{\lambda_2 + \lambda_4}{q}, \tag{3.9}$$

then, the global solution $x(t)$ *to* (2.1) *is almost surely asymptotically stable.*

Proof. Let $U(t, x) = w(x) = |x|^2$. Applying Lemma 3.3 and Lemma 3.2 with $\rho = 1$, we can obtain the desired conclusion. □

Theorem 3.4 gives sufficient conditions of the almost surely asymptotic stability of NSPEs (2.1). Based on this result, we will investigate the almost surely asymptotic stability of the BEM with variable stepsize for (2.1) in the following section.

4. Almost Surely Asymptotic Stability of the Backward Euler Method

To define the BEM for (2.1), we introduce a mesh $H = \{m; t_{-m}, t_{-m+1}, \ldots, t_0, t_1, \ldots, t_n, \ldots\}$ as follows. Let $h_n = t_{n+1} - t_n$, $h_{-m-1} = t_{-m}$. Set $t_0 = \gamma_0 > 0$ and $t_m = q^{-1}\gamma_0$. We define $m - 1$ grid points $t_1 < t_2 < \cdots < t_{m-1}$ in (t_0, t_m) by

$$t_i = t_0 + i\Delta_0, \quad \text{for } i = 1, 2, \ldots, m-1, \tag{4.1}$$

where $\Delta_0 = (t_m - t_0)/m$ and define the other grid points by

$$t_{km+i} = q^{-k} t_i, \quad \text{for } k = -1, 0, 1, \ldots, \quad i = 0, 1, 2, \ldots, m-1. \tag{4.2}$$

It is easy to see that the grid point t_n satisfies $qt_n = t_{n-m}$ for $n \geq 0$, and the step size h_n satisfies

$$qh_n = h_{n-m}, \quad \text{for } n \geq 0, \quad \lim_{n \to \infty} h_n = \infty. \tag{4.3}$$

For the given mesh H, we define the BEM for (2.1) as follows:

$$\begin{aligned} Y_{n+1} - N(Y_{n+1-m}) &= Y_n - N(Y_{n-m}) + h_n f(t_{n+1}, Y_{n+1}, Y_{n+1-m}) \\ &\quad + g(t_n, Y_n, Y_{n-m})\Delta B_n, \quad n \geq -m, \\ Y_{-m} - N(Y_{-m-m}) &= x_0 - N(x_0) + h_{-m-1} f(t_{-m}, Y_{-m}, Y_{-m-m}) \\ &\quad + g(0, x_0, x_0)B(t_{-m}). \end{aligned} \tag{4.4}$$

Here, $Y_n (n \geq -m)$ is an approximation value of $x(t_n)$ and $\mathcal{F}_{t_n}$-measurable. $\Delta B_n = B(t_{n+1}) - B(t_n)$ is the Brownian increment. The approximations $Y_{n-m} (n = -m, -m+1, \ldots, -1)$ are calculated by the following formulae:

$$Y_{n-m} = (1 - \theta_n)x_0 + \theta_n Y_{-m}, \quad n = -m, -m+1, \ldots, -1, \tag{4.5}$$

where $\theta_n = qt_n/t_{-m}$. As a standard hypothesis, we assume that the BEM (4.4) is well defined.

To be precise, let us introduce the definition on the almost surely asymptotic stability of the BEM (4.4).

Definition 4.1. The approximate solution Y_n to the BEM (4.4) is said to be almost surely asymptotically stable if

$$\lim_{n\to\infty} Y_n = 0 \quad \text{a.s.} \tag{4.6}$$

for any bounded $\mathcal{F}_0$-measurable bounded initial data x_0.

Theorem 4.2. *Assume that the BEM* (4.4) *is well defined. Let Assumption 2.3 hold. Let conditions* (3.8) *and* (3.9) *hold. Then the BEM approximate solution* (4.4) *obeys*

$$\lim_{n\to\infty} Y_n = 0 \quad a.s. \tag{4.7}$$

That is, the approximate solution Y_n to the BEM (4.4) *is almost surely asymptotically stable.*

Proof. Set $\overline{Y}_n = Y_n - N(Y_{n-m})$. For $n \geq 0$, from (4.4), we have

$$\left|\overline{Y}_{n+1} - h_n f(t_{n+1}, Y_{n+1}, Y_{n+1-m})\right|^2 = \left|\overline{Y}_n + g(t_n, Y_n, Y_{n-m})\Delta B_n\right|^2. \tag{4.8}$$

Then, we can obtain that

$$\begin{aligned}\left|\overline{Y}_{n+1}\right|^2 \leq{}& \left|\overline{Y}_n\right|^2 + 2h_n\left\langle \overline{Y}_{n+1}, f(t_{n+1}, Y_{n+1}, Y_{n+1-m})\right\rangle + \left|g(t_n, Y_n, Y_{n-m})\Delta B_n\right|^2 \\ &+ 2\left\langle \overline{Y}_n, g(t_n, Y_n, Y_{n-m})\right\rangle \Delta B_n,\end{aligned} \tag{4.9}$$

which subsequently leads to

$$\begin{aligned}\left|\overline{Y}_{n+1}\right|^2 \leq{}& \left|\overline{Y}_n\right|^2 + 2h_n\left\langle \overline{Y}_{n+1}, f(t_{n+1}, Y_{n+1}, Y_{n+1-m})\right\rangle \\ &+ \left|g(t_n, Y_n, Y_{n-m})\right|^2 h_n + \overline{m}_n,\end{aligned} \tag{4.10}$$

where

$$\overline{m}_n = 2\left\langle \overline{Y}_n, g(t_n, Y_n, Y_{n-m})\right\rangle \Delta B_n + \left|g(t_n, Y_n, Y_{n-m})\right|^2 \left(\Delta B_n^2 - h_n\right). \tag{4.11}$$

By conditions (3.8) and (3.9), we have

$$\begin{aligned}\left|\overline{Y}_{n+1}\right|^2 \leq{}& \left|\overline{Y}_n\right|^2 - \lambda_1 h_n |Y_{n+1}|^2 + \lambda_2 h_n |Y_{n+1-m}|^2 \\ &+ \left(\lambda_3 |Y_n|^2 + \lambda_4 |Y_{n-m}|^2\right) h_n + \overline{m}_n.\end{aligned} \tag{4.12}$$

Using the equality $|a+b|^2 \le 2|a|^2 + 2|b|^2$, we obtain that

$$\left|\overline{Y}_{n+1}\right|^2 \ge \frac{1}{2}|Y_{n+1}|^2 - |N(Y_{n+1-m})|^2,$$
$$\left|\overline{Y}_{n}\right|^2 \le 2|Y_n|^2 + 2|N(Y_{n-m})|^2. \tag{4.13}$$

Inserting these inequalities to (4.12) and using Assumption 2.3 yield

$$\left(\frac{1}{2}+\lambda_1 h_n\right)|Y_{n+1}|^2 \le (2+\lambda_3 h_n)|Y_n|^2 + \left(\kappa^2 + \lambda_2 h_n\right)|Y_{n+1-m}|^2 + \left(2\kappa^2 + \lambda_4 h_n\right)|Y_{n-m}|^2 + \overline{m}_n. \tag{4.14}$$

Let $A_n = 1 + 2\lambda_1 h_n$, $B_n = 3 - 2\lambda_1 h_n + 2\lambda_3 h_n$, $C_n = 2\kappa^2 + 2\lambda_2 h_n$, and $D_n = 4\kappa^2 + 2\lambda_4 h_n$. Using these notations, (4.14) implies that

$$|Y_{n+1}|^2 - |Y_n|^2 \le \frac{B_n}{A_n}|Y_n|^2 + \frac{C_n}{A_n}|Y_{n+1-m}|^2 + \frac{D_n}{A_n}|Y_{n-m}|^2 + \frac{2}{A_n}\overline{m}_n. \tag{4.15}$$

Then, we can conclude that

$$|Y_n|^2 \le |Y_0|^2 + \sum_{i=0}^{n-1}\frac{B_i}{A_i}|Y_i|^2 + \sum_{i=0}^{n-1}\frac{C_i}{A_i}|Y_{i+1-m}|^2 + \sum_{i=0}^{n-1}\frac{D_i}{A_i}|Y_{i-m}|^2 + \sum_{i=0}^{n-1}\frac{2}{A_i}\overline{m}_i. \tag{4.16}$$

Note that

$$\begin{aligned}
\sum_{i=0}^{n-1}\frac{C_i}{A_i}|Y_{i+1-m}|^2 &= \sum_{i=-m+1}^{n-m}\frac{C_{i+m-1}}{A_{i+m-1}}|Y_i|^2 \\
&= \sum_{i=-m+1}^{-1}\frac{C_{i+m-1}}{A_{i+m-1}}|Y_i|^2 + \sum_{i=0}^{n-1}\frac{C_{i+m-1}}{A_{i+m-1}}|Y_i|^2 - \sum_{i=n-m+1}^{n-1}\frac{C_{i+m-1}}{A_{i+m-1}}|Y_i|^2, \\
\sum_{i=0}^{n-1}\frac{D_i}{A_i}|Y_{i-m}|^2 &= \sum_{i=-m}^{n-m-1}\frac{D_{i+m}}{A_{i+m}}|Y_i|^2 \\
&= \sum_{i=-m}^{-1}\frac{D_{i+m}}{A_{i+m}}|Y_i|^2 + \sum_{i=0}^{n-1}\frac{D_{i+m}}{A_{i+m}}|Y_i|^2 - \sum_{i=n-m}^{n-1}\frac{D_{i+m}}{A_{i+m}}|Y_i|^2.
\end{aligned} \tag{4.17}$$

We, therefore, have

$$
\begin{aligned}
&|Y_n|^2+\sum_{i=n-m+1}^{n-1}\frac{C_{i+m-1}}{A_{i+m-1}}|Y_i|^2+\sum_{i=n-m}^{n-1}\frac{D_{i+m}}{A_{i+m}}|Y_i|^2\\
&\quad\le |Y_0|^2+\sum_{i=-m+1}^{-1}\frac{C_{i+m-1}}{A_{i+m-1}}|Y_i|^2+\sum_{i=-m}^{-1}\frac{D_{i+m}}{A_{i+m}}|Y_i|^2\\
&\qquad+\sum_{i=0}^{n-1}\left(\frac{B_i}{A_i}+\frac{C_{i+m-1}}{A_{i+m-1}}+\frac{D_{i+m}}{A_{i+m}}\right)|Y_i|^2+\sum_{i=0}^{n-1}\frac{2}{A_i}\overline{m}_i.
\end{aligned}
\tag{4.18}
$$

Similar to (4.15), from (4.4), we can obtain that

$$
\begin{aligned}
|Y_0|^2-|Y_{-1}|^2 &\le \frac{B_{-1}}{A_{-1}}|Y_{-1}|^2+\frac{C_{-1}}{A_{-1}}|Y_{-m}|^2\\
&\quad+\frac{D_{-1}}{A_{-1}}\left[2(1-\theta_{-1})^2|x_0|^2+2\theta_{-1}^2|Y_{-m}|^2\right]+\frac{2}{A_{-1}}\overline{m}_{-1},\\
|Y_n|^2-|Y_{n-1}|^2 &\le \frac{B_{n-1}}{A_{n-1}}|Y_{n-1}|^2+\frac{C_{n-1}}{A_{n-1}}\left[2(1-\theta_n)^2|x_0|^2+2\theta_n^2|Y_{-m}|^2\right]\\
&\quad+\frac{D_{n-1}}{A_{n-1}}\left[2(1-\theta_{n-1})^2|x_0|^2+2\theta_{n-1}^2|Y_{-m}|^2\right]\\
&\quad+\frac{2}{A_{n-1}}\overline{m}_{n-1},\quad -m+1\le n\le -1,\\
|Y_{-m}|^2-|x_0|^2 &\le \frac{B_{-m-1}+D_{-m-1}}{A_{-m-1}}|x_0|^2+\frac{C_{-m-1}}{A_{-m-1}}\left[2(1-\theta_{-m})^2|x_0|^2+2\theta_{-m}^2|Y_{-m}|^2\right]\\
&\quad+\frac{2}{A_{-m-1}}\overline{m}_{-m-1},
\end{aligned}
\tag{4.19}
$$

where A_n, B_n, C_n, D_n $(n=-m-1,\dots,-1)$ are defined as before,

$$
\begin{aligned}
\overline{m}_n &= 2\langle Y_n-N(Y_{n-m}), g(t_n,Y_n,Y_{n-m})\rangle\Delta B_n\\
&\quad+\left|g(t_n,Y_n,Y_{n-m})\right|^2\left(\Delta B_n^2-h_n\right),\quad -m\le n\le -1,\\
\overline{m}_{-m-1} &= 2\langle x_0-N(x_0), g(0,x_0,x_0)\rangle B(t_{-m})+\left|g(0,x_0,x_0)\right|^2\left(B^2(t_{-m})-t_{-m}\right).
\end{aligned}
\tag{4.20}
$$

From (4.19), we have

$$
|Y_0|^2\le A|x_0|^2+B|Y_{-m}|^2+\sum_{i=-m+1}^{-1}\frac{B_i}{A_i}|Y_i|^2+\sum_{i=-m-1}^{-1}\frac{2}{A_i}\overline{m}_i,
\tag{4.21}
$$

where

$$A = 1 + \frac{B_{-m-1} + D_{-m-1}}{A_{-m-1}} + \sum_{i=-m}^{-1} \frac{2D_i}{A_i}\left((1-\theta_i)^2\right) + \sum_{i=-m-1}^{-2} \frac{2C_i}{A_i}\left((1-\theta_{i+1})^2\right),$$
$$B = \frac{B_{-m}}{A_{-m}} + \frac{B_{-1}}{A_{-1}} + \sum_{i=-m}^{-1} \frac{2D_i}{A_i}\theta_i^2 + \sum_{i=-m-1}^{-2} \frac{2C_i}{A_i}\theta_{i+1}^2. \tag{4.22}$$

Obviously $A > 0$. By (4.18) and (4.21), we can obtain that

$$|Y_n|^2 + \sum_{i=n-m+1}^{n-1} \frac{C_{i+m-1}}{A_{i+m-1}}|Y_i|^2 + \sum_{i=n-m}^{n-1} \frac{D_{i+m}}{A_{i+m}}|Y_i|^2$$
$$\leq A|x_0|^2 + \left(B + \frac{D_0}{A_0}\right)|Y_{-m}|^2 + \sum_{i=-m+1}^{n-1}\left(\frac{B_i}{A_i} + \frac{C_{i+m-1}}{A_{i+m-1}} + \frac{D_{i+m}}{A_{i+m}}\right)|Y_i|^2 + M_n, \tag{4.23}$$

where $M_n = \sum_{i=-m-1}^{n-1}(2/A_i)\overline{m}_i$. Similar to the proof in [18], we can obtain that M_n is a martingale with $M_{-m-1} = 0$. Note that $h_{i+m-1} \leq h_{i+m}$ and $h_{i+m} = h_i/q$ for $i \geq -m$. Then, we have

$$\frac{B_i}{A_i} + \frac{C_{i+m-1}}{A_{i+m-1}} + \frac{D_{i+m}}{A_{i+m}} \leq \frac{(3+6\kappa^2) - 2(\lambda_1 h_i - \lambda_3 h_i - \lambda_2 h_{i+m-1} - \lambda_4 h_{i+m})}{1 + 2\lambda_1 h_i}$$
$$\leq \frac{1}{1+2\lambda_1 h_i}\left\{\left(3 + 6\kappa^2\right) - 2\left(\lambda_1 - \lambda_3 - \frac{\lambda_2}{q} - \frac{\lambda_4}{q}\right)h_i\right\}. \tag{4.24}$$

Using the condition (3.9) and $\lim_{i\to\infty} h_i = \infty$, we obtain that there exists an integer i^* such that

$$\frac{B_i}{A_i} + \frac{C_{i+m-1}}{A_{i+m-1}} + \frac{D_{i+m}}{A_{i+m}} \geq 0, \quad i \leq i^*,$$
$$\frac{B_i}{A_i} + \frac{C_{i+m-1}}{A_{i+m-1}} + \frac{D_{i+m}}{A_{i+m}} < 0, \quad i > i^*. \tag{4.25}$$

Set $U_{-m-1} = 0$,

$$U_{-m} = \begin{cases} \left(B + \dfrac{D_0}{A_0}\right)|Y_{-m}|^2 & \text{if } \left(B + \dfrac{D_0}{A_0}\right) > 0, \\ 0 & \text{if } \left(B + \dfrac{D_0}{A_0}\right) \leq 0, \end{cases}$$

$$U_n = \begin{cases} U_{-m} + \sum_{i=-m+1}^{n-1} \left(\frac{B_i}{A_i} + \frac{C_{i+m-1}}{A_{i+m-1}} + \frac{D_{i+m}}{A_{i+m}} \right) |Y_i|^2 & \text{if } -m+1 \le n \le i^*+1, \\ U_{-m} + \sum_{i=-m+1}^{i^*} \left(\frac{B_i}{A_i} + \frac{C_{i+m-1}}{A_{i+m-1}} + \frac{D_{i+m}}{A_{i+m}} \right) |Y_i|^2 & \text{if } n > i^*+1, \end{cases}$$

$$V_n = \begin{cases} 0 & \text{if } -m-1 \le n \le i^*+1, \\ -\sum_{i=i^*+1}^{n-1} \left(\frac{B_i}{A_i} + \frac{C_{i+m-1}}{A_{i+m-1}} + \frac{D_{i+m}}{A_{i+m}} \right) |Y_i|^2 & \text{if } n > i^*+1. \end{cases} \tag{4.26}$$

Obviously,

$$\lim_{n\to\infty} U_n = U_{-m} + \sum_{i=-m+1}^{i^*} \left(\frac{B_i}{A_i} + \frac{C_{i+m-1}}{A_{i+m-1}} + \frac{D_{i+m}}{A_{i+m}} \right) |Y_i|^2 < \infty, \quad \text{a.s.} \tag{4.27}$$

Moreover, (4.23) implies that

$$|Y_n|^2 \le C|x_0|^2 + U_n - V_n + M_n. \tag{4.28}$$

Here $C = \max\{A, 1\}$. According to (4.27), using Lemma 2.6 yields

$$\limsup_{n\to\infty} |Y_n|^2 < \infty \quad \text{a.s.}, \qquad \lim_{n\to\infty} V_n < \infty \quad \text{a.s.} \tag{4.29}$$

Then, we have

$$\lim_{i\to\infty} -\left(\frac{B_i}{A_i} + \frac{C_{i+m-1}}{A_{i+m-1}} + \frac{D_{i+m}}{A_{i+m}} \right) |Y_i|^2 = 0 \quad \text{a.s.} \tag{4.30}$$

Note that

$$\lim_{i\to\infty} -\left(\frac{B_i}{A_i} + \frac{C_{i+m-1}}{A_{i+m-1}} + \frac{D_{i+m}}{A_{i+m}} \right) = \frac{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4}{\lambda_1} > 0. \tag{4.31}$$

We therefore obtain that

$$\lim_{n\to\infty} |Y_n|^2 = 0 \quad \text{a.s.} \tag{4.32}$$

Then, the desired conclusion is obtained. This completes the proof. □

5. Numerical Experiments

In this section, we present numerical experiments to illustrate theoretical results of stability presented in the previous sections.

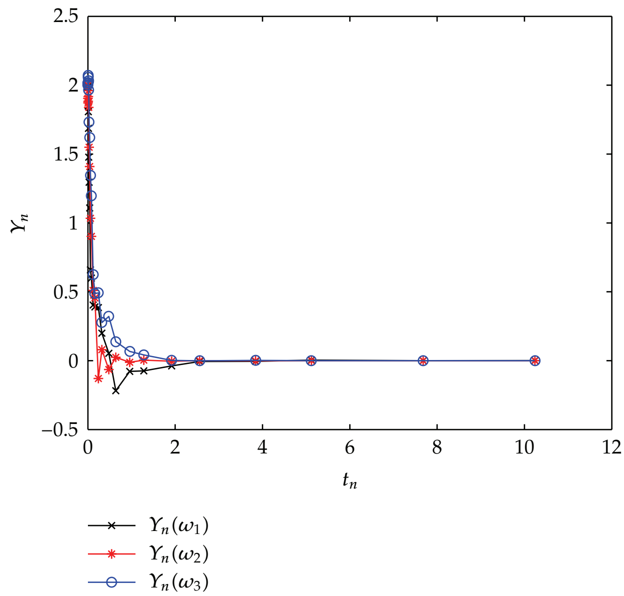

Figure 1: Almost surely asymptotic stability with $x_0 = 2, t_0 = 0.01,\ m = 2$.

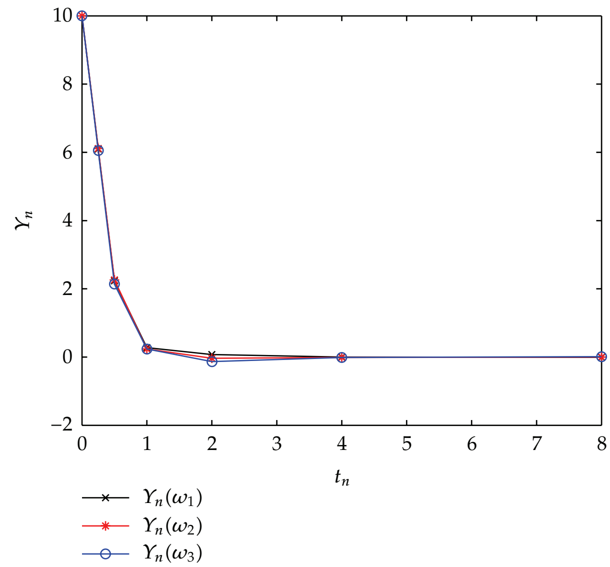

Figure 2: Almost surely asymptotic stability with $x_0 = 10,\ t_0 = 1,\ m = 1$.

Consider the following scalar problem:

$$d\left[x(t) - \frac{1}{2}x(0.5t)\right] = (-8x(t) + x(0.5t))dt + \sin(x(0.5t))dB(t), \quad t \geq 0,$$
$$x(0) = x_0. \tag{5.1}$$

For the test (5.1), we have $\lambda_1 = 11$, $\lambda_2 = 4$, $\lambda_3 = 0$, and $\lambda_4 = 1$ corresponding to Theorem 3.4. By Theorem 3.4, the solution to (5.1) is almost surely asymptotically stable.

Theorem 4.2 shows that the BEM approximation to (5.1) is almost surely asymptotically stable. In Figure 1, We compute three different paths $(Y_n(\omega_1), Y_n(\omega_2), Y_n(\omega_3))$ using the BEM (4.4) with $x_0 = 2$, $t_0 = 0.01$, $m = 2$. In Figure 2, three different paths $(Y_n(\omega_1), Y_n(\omega_2), Y_n(\omega_3))$ of BEM approximations are computed with $x_0 = 10$, $t_0 = 1$, $m = 1$. The results demonstrate that these paths are asymptotically stable.

Acknowledgments

The author would like to thank the referees for their helpful comments and suggestions.

References

[1] A. Iserles, "On the generalized pantograph functional-differential equation," *The European Journal of Applied Mathematics*, vol. 4, no. 1, pp. 1–38, 1993.

[2] M. Huang and S. Vandewalle, "Discretized stability and error growth of the nonautonomous pantograph equation," *SIAM Journal on Numerical Analysis*, vol. 42, no. 5, pp. 2020–2042, 2005.

[3] W. Wang and S. Li, "On the one-leg h-methods for solving nonlinear neutral functional differential equations," *Applied Mathematics and Computation*, vol. 193, pp. 285–301, 2007.

[4] Y. Liu, "Asymptotic behaviour of functional-differential equations with proportional time delays," *The European Journal of Applied Mathematics*, vol. 7, no. 1, pp. 11–30, 1996.

[5] T. Koto, "Stability of Runge-Kutta methods for the generalized pantograph equation," *Numerische Mathematik*, vol. 84, no. 2, pp. 233–247, 1999.

[6] A. Bellen, S. Maset, and L. Torelli, "Contractive initializing methods for the pantograph equation of neutral type," in *Recent Trends in Numerical Analysis*, vol. 3, pp. 35–41, 2000.

[7] S. F. Ma, Z. W. Yang, and M. Z. Liu, "H_α-stability of modified Runge-Kutta methods for nonlinear neutral pantograph equations," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 1128–1142, 2007.

[8] R. S. Lipster and A. N. Shiryayev, *Theory of Martingale*, Horwood, Chichester, UK, 1989.

[9] X. Mao, "Asymptotic properties of neutral stochastic differential delay equations," *Stochastics and Stochastics Reports*, vol. 68, no. 3-4, pp. 273–295, 2000.

[10] X. Mao, Y. Shen, and C. Yuan, "Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching," *Stochastic Processes and Their Applications*, vol. 118, no. 8, pp. 1385–1406, 2008.

[11] S. Gan, H. Schurz, and H. Zhang, "Mean square convergence of stochastic θ-methods for nonlinear neutral stochastic differential delay equations," *International Journal of Numerical Analysis and Modeling*, vol. 8, no. 2, pp. 201–213, 2011.

[12] S. Zhou and F. Wu, "Convergence of numerical solutions to neutral stochastic delay differential equations with Markovian switching," *Journal of Computational and Applied Mathematics*, vol. 229, no. 1, pp. 85–96, 2009.

[13] B. Yin and Z. Ma, "Convergence of the semi-implicit Euler method for neutral stochastic delay differential equations with phase semi-Markovian switching," *Applied Mathematical Modelling*, vol. 35, no. 5, pp. 2094–2109, 2011.

[14] S. Pang, F. Deng, and X. Mao, "Almost sure and moment exponential stability of Euler-Maruyama discretizations for hybrid stochastic differential equations," *Journal of Computational and Applied Mathematics*, vol. 213, no. 1, pp. 127–141, 2008.

[15] D. J. Higham, X. Mao, and C. Yuan, "Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations," *SIAM Journal on Numerical Analysis*, vol. 45, no. 2, pp. 592–607, 2007.

[16] Y. Saito and T. Mitsui, "T-stability of numerical scheme for stochastic differential equaions," in *Contributions in Numerical Mathematics*, vol. 2 of *World Scientific Series in Applicable Analysis*, pp. 333–344, 1993.

[17] A. Rodkina and H. Schurz, "Almost sure asymptotic stability of drift-implicit θ-methods for bilinear ordinary stochastic differential equations in $\mathbb{R}^1$," *Journal of Computational and Applied Mathematics*, vol. 180, no. 1, pp. 13–31, 2005.

[18] F. Wu, X. Mao, and L. Szpruch, "Almost sure exponential stability of numerical solutions for stochastic delay differential equations," *Numerische Mathematik*, vol. 115, no. 4, pp. 681–697, 2010.

[19] Q. Li and S. Gan, "Almost sure exponential stability of numerical solutions for stochastic delay differential equations with jumps," *Journal of Applied Mathematics and Computing*. In press.

[20] A. N. Shiryayev, *Probablity*, Springer, Berlin, Germany, 1996.

[21] X. Mao, *Stochastic Differential Equations and Their Applications*, Horwood, New York, NY, USA, 1997.