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Research Article

Ahlfors Theorems for Differential Forms

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Some counterparts of theorems of Phragmén-Lindelöf and of Ahlfors are proved for differential forms of $\mathcal{WT}$-classes.

1. $\mathcal{WT}$-Forms

This paper is continuation of the earlier work [1], where the main topic was to examine the connection between quasiregular (qr) mappings and so-called $\mathcal{WT}$-classes of differential forms. We first recall some basic notation and terminology from [1].

Let $\mathcal{M}$ be a Riemannian manifold of class C^3, $\dim \mathcal{M} = n$, with or without boundary, and let

$$w \in L^p_{\mathrm{loc}}(\mathcal{M}), \quad \deg w = k, \quad 0 \le k \le n, \quad p > 1, \tag{1.1}$$

be a weakly closed differential form on $\mathcal{M}$, that is, for each form

$$\varphi \in W^1_{q,\mathrm{loc}}(\mathcal{M}), \quad \deg \varphi = k+1, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.2}$$

with a compact $\operatorname{supp} \varphi$ in $\mathcal{M}$ and such that $\operatorname{supp} \varphi \cap \partial \mathcal{M} = \emptyset$; we have

$$\int_{\mathcal{M}} \langle w, \delta\varphi \rangle \, dv = 0. \tag{1.3}$$

Here $\delta\varphi = (-1)^k \star^{-1} d \star \varphi$, $k = \deg\varphi$, and $\star\alpha$ is the orthogonal complement of a differential form α on a Riemannian manifold $\mathcal{M}$.

A weakly closed form w of the kind (1.1) is said to be of the class $\mathcal{WT}_1$ on $\mathcal{M}$ if there exists a weakly closed differential form

$$\theta \in L^q_{\mathrm{loc}}(\mathcal{M}), \quad \deg\theta = n - k, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.4}$$

such that almost everywhere on $\mathcal{M}$ we have

$$\nu_0 |\theta|^q \leq \langle w, *\theta \rangle \tag{1.5}$$

for some constant $\nu_0 > 0$.

The differential form (1.1) is said to be of the class $\mathcal{WT}_2$ on $\mathcal{M}$ if there exists a differential form (1.4) such that almost everywhere on $\mathcal{M}$

$$\nu_1 |w|^p \leq \langle w, *\theta \rangle, \tag{1.6}$$

$$|\theta| \leq \nu_2 |w|^{p-1}, \tag{1.7}$$

for some constants $\nu_1, \nu_2 > 0$.

Theorem 1.1. $\mathcal{WT}_2 \subset \mathcal{WT}_1$

For a proof see [1].

The following partial integration formula for differential forms is useful [1].

Lemma 1.2. *Let* $\alpha \in W^1_{p,\mathrm{loc}}(\mathcal{M})$ *and* $\beta \in W^1_q(\mathcal{M})$ *be differential forms,* $\deg\alpha + \deg\beta = n - 1$, $1/p + 1/q = 1$, $1 \leq p, q \leq \infty$, *and let* β *have a compact support* $\operatorname{supp}\beta \subset \mathcal{M}$. *Then*

$$\int_{\mathcal{M}} d\alpha \wedge \beta = (-1)^{\deg\alpha + 1} \int_{\mathcal{M}} \alpha \wedge d\beta. \tag{1.8}$$

In particular, the form α *is weakly closed if and only if* $d\alpha = 0$ *a.e. on* $\mathcal{M}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be Riemannian manifolds of dimensions $\dim\mathcal{A} = k$, $\dim\mathcal{B} = n - k$, $1 \leq k < n$, and with scalar products $\langle\cdot,\cdot\rangle_A$, $\langle\cdot,\cdot\rangle_B$, respectively. The Cartesian product $\mathcal{N} = \mathcal{A} \times \mathcal{B}$ has the natural structure of a Riemannian manifold with the scalar product

$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathcal{A}} + \langle\cdot,\cdot\rangle_{\mathcal{B}}. \tag{1.9}$$

We denote by $\pi : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ and $\eta : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ the natural projections of the manifold $\mathcal{N}$ onto submanifolds.

If $w_{\mathcal{A}}$ and $w_{\mathcal{B}}$ are volume forms on $\mathcal{A}$ and $\mathcal{B}$, respectively, then the differential form $w_{\mathcal{N}} = \pi^* w_{\mathcal{A}} \wedge \eta^* w_{\mathcal{B}}$ is a volume form on $\mathcal{N}$.

Let $y_1, \ldots, y_k$ be an orthonormal system of coordinates in $\mathbf{R}^k$, $1 \leq k \leq n$. Let $\mathcal{A}$ be a domain in $\mathbf{R}^k$, and let $\mathcal{B}$ be an $(n-k)$-dimensional Riemannian manifold. We consider the manifold $\mathcal{N} = \mathcal{A} \times \mathcal{B}$.

2. Boundary Sets

Below we introduce the notions of parabolic and hyperbolic type of boundary sets on noncompact Riemannian manifolds and study exhaustion functions of such sets. We also present some illuminating examples.

Let $\mathcal{M}$ be an n-dimensional noncompact Riemannian manifold without boundary. Boundary sets on $\mathcal{M}$ are analogies to prime ends due to Carathéodory (cf. e.g., [2]).

Let $\{\mathcal{U}_k\}$, $k = 1, 2, \ldots$ be a collection of open sets $\mathcal{U}_k \subset \mathcal{M}$ with the following properties:

(i) for all $k = 1, 2, \ldots$, $\overline{\mathcal{U}}_{k+1} \subset \mathcal{U}_k$,

(ii) $\bigcap_{k=1}^{\infty} \overline{\mathcal{U}}_k = \emptyset$.

A sequence with these properties will be called a chain on the manifold $\mathcal{M}$.

Let $\{\mathcal{U}'_k\}$, $\{\mathcal{U}''_k\}$ be two chains of open sets on M. We will say that the chain $\mathcal{U}'_k$ is *contained* in the chain $\{\mathcal{U}''_k\}$, if for each $m \geq 1$ there exists a number $k(m)$ such that for all $k > k(m)$ we have $\mathcal{U}'_k \subset \mathcal{U}''_m$. Two chains, each of which is contained in the other one, are called *equivalent*. Each equivalence class ξ of chains is called a *boundary set* of the manifold $\mathcal{M}$. To define ξ it is enough to determine at least one representative in the equivalence class. If the boundary set ξ is defined by the chain $\{\mathcal{U}_k\}$, then we will write $\xi \asymp \{\mathcal{U}_k\}$.

A sequence of points $m_k \in \mathcal{M}$ converges to ξ if for some (and, therefore, all) chain $\{\mathcal{U}_k\} \in \xi$ the following condition is satisfied: for every $k = 1, 2, \ldots$ there exists an integer $n(k)$ such that $m_n \in \mathcal{U}_k$ for all $n > n(k)$. A sequence (m_n) lies off a boundary set $\xi \asymp \{\mathcal{U}_k\}$, if for every $k = 1, 2, \ldots$ there exists a number $n(k)$ such that for all $n > n(k)$ $m_n \notin \mathcal{U}_k$.

A boundary set $\xi \asymp \{\mathcal{U}_k\}$ is called a set of ends of the manifold $\mathcal{M}$ if each of $\{\mathcal{U}_k\}$ has a compact boundary $\partial \mathcal{U}_k$. If in addition each of the sets U_k is connected, then $\xi \asymp \{U_k\}$ is called *an end of the manifold* $\mathcal{M}$.

2.1. Types of Boundary Set

Let D be an open set on $\mathcal{M}$ and let $A, B \subset D$ be closed subsets in D such that $\overline{A} \cap \overline{B} = \emptyset$. Each triple $(A, B; D)$ is called a condenser on $\mathcal{M}$.

We fix $p \geq 1$. The p-capacity of the condenser $(A, B; D)$ is defined by

$$\operatorname{cap}_p(A, B; D) = \inf \int_D |\nabla \varphi|^p * \mathbb{1}_{\mathcal{M}}, \tag{2.1}$$

where the infimum is taken over the set of all continuous functions of class $W^1_{p,\mathrm{loc}}(D)$ such that $\varphi(m)|_A = 0$, $\varphi(m)|_B = 1$. It is easy to see that for a pair $(A, B; D)$ and $(A_1, B_1; D)$ with $A_1 \subset A$, $B_1 \subset B$ we have

$$\operatorname{cap}_p(A_1, B_1; D) \leq \operatorname{cap}_p(A, B; D). \tag{2.2}$$

Let $\overline{A}, \overline{B}$ be compact in D. A standard approximation method shows that $\operatorname{cap}_p(A, B; D)$ does not change if one restricts the class of functions in the variational problem (2.1) to Lipschitz functions equal to 0 and 1 in the sets A and B, respectively.

Let $\{\mathcal{U}_k\}$ be an arbitrary chain on a manifold $\mathcal{M}$. We fix a subdomain $H \subset\subset \mathcal{M}$. If k is sufficiently large, the intersection $\overline{H} \cap \overline{\mathcal{U}_k} = \emptyset$ and we consider the condenser $(\overline{H}, \overline{\mathcal{U}_k}; \mathcal{M})$. Then it is clear that for $k = 1, 2, \ldots$

$$\operatorname{cap}_p\left(\overline{H}, \overline{\mathcal{U}_k}; \mathcal{M}\right) \geq \operatorname{cap}_p\left(\overline{H}, \overline{\mathcal{U}_{k+1}}; \mathcal{M}\right). \tag{2.3}$$

We will say that the chain $\{\mathcal{U}_k\}$ on $\mathcal{M}$ has *p-capacity zero*, if for every subdomain $H \subset\subset \mathcal{M}$ we have

$$\lim_{k \to \infty} \operatorname{cap}_p\left(\overline{H}, \overline{\mathcal{U}_k}; \mathcal{M}\right) = 0. \tag{2.4}$$

We will say that a boundary set ξ is of *p-parabolic type* if every chain $\{\mathcal{U}_k\} \asymp \xi$ is of p-capacity zero. A boundary set ξ is of *α-hyperbolic type* if at least one of the chains $\{\mathcal{U}_k\} \in \xi$ is not of p-parabolic type.

Let

$$\{\mathcal{U}_k\}_{k=1}^{\infty}, \quad \overline{\mathcal{U}}_k \subset \mathcal{U}_{k+1}, \quad \bigcup_{k=1}^{\infty} \mathcal{U}_k = \mathcal{M} \tag{2.5}$$

be an arbitrary exhaustion of the manifold $\mathcal{M}$ by subdomains $\{\mathcal{U}_k\}$. The manifold $\mathcal{M}$ is of p-parabolic or p-hyperbolic type depending on the p-parabolicity or p-hyperbolicity of the boundary set $\{\mathcal{M} \setminus \overline{\mathcal{U}}_k\}$.

It is well-known, see [3], that a noncompact Riemannian manifold $\mathcal{M}$ without boundary is of p-parabolic type if and only if every solution of the inequality

$$\operatorname{div}_{\mathcal{M}}\left(|\nabla u|^{p-2} \nabla u\right) \geq 0 \tag{2.6}$$

which is bounded from above is a constant.

The classical parabolicity and hyperbolicity coincides with 2-parabolicity and 2-hyperbolicity, respectively. Therefore whenever we refer to parabolic or hyperbolic type (of a manifold or a boundary set) we mean 2-parabolicity or 2-hyperbolicity.

Example 2.1. The space $\mathbf{R}^n$ is of p-parabolic type for $p \geq n$ and p-hyperbolic type for $p < n$.

We next present a proposition that provides a convenient method of verifying the p-parabolicity and p-hyperbolicity of boundary sets.

Lemma 2.2 (see [4]). *Let ξ be a boundary set on $\mathcal{M}$. If for a chain $\{\mathcal{U}_k \asymp \xi\}$ and for a nonempty open set $H_0 \subset\subset \mathcal{M}$ the condition* (2.4) *holds, then the boundary set ξ is of p-parabolic type.*

2.2. A-Solutions

Let $\mathcal{M}$ be a Riemannian manifold and let

$$A : T(\mathcal{M}) \longrightarrow T(\mathcal{M}) \tag{2.7}$$

be a mapping defined a.e. on the tangent bundle $T(\mathcal{M})$. Suppose that for a.e. $m \in \mathcal{M}$ the mapping A is continuous on the fiber T_m, that is, for a.e. $m \in \mathcal{M}$ the function $A(m, \cdot) : \xi \in T_m \to T_m$ is defined and continuous; the mapping $m \to A_m(X)$ is measurable for all measurable vector fields X (see [5]).

Suppose that for a.e. $m \in \mathcal{M}$ and for all $\xi \in T_m$ the inequalities

$$\begin{gathered} \nu_1 |\xi|^p \le \langle \xi, A(m, \xi) \rangle, \\ |A(m, \xi)| \le \nu_2 |\xi|^{p-1} \end{gathered} \tag{2.8}$$

hold with $p > 1$ and for some constants $\nu_1, \nu_2 > 0$. It is clear that we have $\nu_1 \le \nu_2$.

We consider the equation

$$\operatorname{div} A(m, \nabla f) = 0. \tag{2.9}$$

Solutions to (2.9) are understood in the weak sense, that is, A-solutions are $W^1_{p,\mathrm{loc}}$-functions satisfying the integral identity

$$\int_{\mathcal{M}} \langle \nabla \theta, A(m, \nabla f) \rangle * \mathbb{1}_{\mathcal{M}} = 0 \tag{2.10}$$

for all $\theta \in W^1_p(\mathcal{M})$ with a compact support $\operatorname{supp} \theta \subset \mathcal{M}$.

A function f in $W^1_{p,\mathrm{loc}}(\mathcal{M})$ is an A-*subsolution* of (2.9) in $\mathcal{M}$ if

$$\operatorname{div} A(m, \nabla f) \ge 0 \tag{2.11}$$

weakly in $\mathcal{M}$, that is,

$$\int_{\mathcal{M}} \langle \nabla \theta, A(m, \nabla f) \rangle * \mathbb{1}_{\mathcal{M}} \le 0 \tag{2.12}$$

whenever $\theta \in W^1_p(\mathcal{M})$, is nonnegative with a compact support in $\mathcal{M}$.

A basic example of such an equation is the p-Laplace equation

$$\operatorname{div} \left(|\nabla f|^{p-2} \nabla f \right) = 0. \tag{2.13}$$

3. Exhaustion Functions

Below we introduce exhaustion and special exhaustion functions on Riemannian manifolds and give illustrating examples.

3.1. Exhaustion Functions of Boundary Sets

Let $h : \mathcal{M} \to (0, h_0)$, $0 < h_0 \leq \infty$, be a locally Lipschitz function. For arbitrary $t \in (0, h_0)$ we denote by

$$B_h(t) = \{m \in \mathcal{M} : h(m) < t\}, \qquad \Sigma_h(t) = \{m \in \mathcal{M} : h(m) = t\} \tag{3.1}$$

the h-balls and h-spheres, respectively.

Let $h : \mathcal{M} \to \mathbf{R}$ be a locally Lipschitz function such that: there exists a compact $K \subset \mathcal{M}$ such that $|\nabla h(x)| > 0$ for a.e. $m \in \mathcal{M} \setminus K$. We say that the function h is an exhaustion function for a boundary set Ξ of $\mathcal{M}$ if for an arbitrary sequence of points $m_k \in \mathcal{M}$, $k = 1, 2, \ldots$ the function $h(m_k) \to h_0$ if and only if $m_k \to \xi$.

It is easy to see that this requirement is satisfied if and only if for an arbitrary increasing sequence $t_1 < t_2 < \cdots < h_0$ the sequence of the open sets $V_k = \{m \in \mathcal{M} : h(m) > t_k\}$ is a chain, defining a boundary set ξ. Thus the function h exhausts the boundary set ξ in the traditional sense of the word.

The function $h : \mathcal{M} \to (0, h_0)$ is called the exhaustion function of the manifold $\mathcal{M}$ if the following two conditions are satisfied

(i) for all $t \in (0, h_0)$ the h-ball $\overline{B_h(t)}$ is compact;

(ii) for every sequence $t_1 < t_2 < \cdots < h_0$ with $\lim_{k\to\infty} t_k = h_0$, the sequence of h-balls $\{B_h(t_k)\}$ generates an exhaustion of $\mathcal{M}$, that is,

$$B_h(t_1) \subset B_h(t_2) \subset \cdots \subset B_h(t_k) \subset \cdots, \quad \bigcup_k B_h(t_k) = \mathcal{M}. \tag{3.2}$$

Example 3.1. Let $\mathcal{M}$ be a Riemannian manifold. We set $h(m) = \operatorname{dist}(m, m_0)$ where $m_0 \in \mathcal{M}$ is a fixed point. Because $|\nabla h(m)| = 1$ almost everywhere on $\mathcal{M}$, the function h defines an exhaustion function of the manifold $\mathcal{M}$.

3.2. Special Exhaustion Functions

Let $\mathcal{M}$ be a noncompact Riemannian manifold with the boundary $\partial\mathcal{M}$ (possibly empty). Let A satisfy (2.8) and let $h : \mathcal{M} \to (0, h_0)$ be an exhaustion function, satisfying the following conditions:

(a_1) there is $h' > 0$ such that $h^{-1}([0, h'])$ is compact and h is a solution of (2.9) in the open set $h^{-1}((h', h_0))$;

(a_2) for a.e. t_1, $t_2 \in (h', h_0)$, $t_1 < t_2$,

$$\int_{\Sigma_h(t_2)} \left\langle \frac{\nabla h}{|\nabla h|}, A(x, \nabla h) \right\rangle d\mathcal{H}^{n-1} = \int_{\Sigma_h(t_1)} \left\langle \frac{\nabla h}{|\nabla h|}, A(x, \nabla h) \right\rangle d\mathcal{H}^{n-1}. \tag{3.3}$$

Here $d\mathcal{H}^{n-1}$ is the element of the $(n-1)$-dimensional Hausdorff measure on Σ_h. Exhaustion functions with these properties will be called *the special exhaustion functions of* $\mathcal{M}$ *with respect to* A. In most cases the mapping A will be the p-Laplace operator (2.13).

Since the unit vector $\nu = \nabla h/|\nabla h|$ is orthogonal to the h-sphere Σ_h, the condition (a_2) means that the flux of the vector field $A(m, \nabla h)$ through h-spheres $\Sigma_h(t)$ is constant.

Suppose that the function $A(m, \xi)$ *is continuously differentiable. If*

(b_1) $h \in C^2(\mathcal{M} \setminus K)$ *and satisfies* (2.9), *and*

(b_2) *at every point* $m \in \mathcal{M}$ *where* $\partial\mathcal{M}$ *has a tangent plane* $T_m(\partial\mathcal{M})$ *the condition*

$$\langle A(m, \nabla h(m)), \nu\rangle = 0 \tag{3.4}$$

is satisfied where ν *is a unit vector of the inner normal to the boundary* $\partial\mathcal{M}$, *then* h *is a special exhaustion function of the manifold* $\mathcal{M}$.

The proof of this statement is simple. Consider the open set

$$\mathcal{M}(t_1, t_2) = \{m \in \mathcal{M} : t_1 < h(m) < t_2\}, \quad 0 < t_1 < t_2 < \infty, \tag{3.5}$$

with the boundary $\partial\mathcal{M}(t_1, t_2)$. Using the Stokes formula, we have for noncritical values $t_1 < t_2$ (for the definition of critical values of C^k-functions see, e.g., [6, Part II, Chapter 2, Section 10])

$$\begin{aligned}
&\int_{\Sigma_h(t_2)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m, \nabla h) \right\rangle d\mathcal{H}^{n-1} - \int_{\Sigma_h(t_1)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m, \nabla h) \right\rangle d\mathcal{H}^{n-1} \\
&\quad = \int_{\partial\mathcal{M}(t_1,t_2) \cup \cup_{i=1,2} \Sigma_h(t_i)} \langle \nu, A(m, \nabla h)\rangle d\mathcal{H}^{n-1} = \int_{\partial\mathcal{M}(t_1,t_2)} \langle \nu, A(m, \nabla h)\rangle d\mathcal{H}^{n-1} \\
&\quad = \int_{\mathcal{M}(t_1,t_2)} \operatorname{div} A(m, \nabla h) * \mathbb{1} = 0,
\end{aligned} \tag{3.6}$$

and (a_2) follows.

Example 3.2. We fix an integer k, $1 \le k \le n$, and set

$$d_k(x) = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}. \tag{3.7}$$

It is easy to see that $|\nabla d_k(x)| = 1$ everywhere in $\mathbf{R}^n \setminus \Sigma_0$ where $\Sigma_0 = \{x \in \mathbf{R}^n : d_k(x) = 0\}$. We will call the set

$$B_k(t) = \{x \in \mathbf{R}^n : d_k(x) < t\} \tag{3.8}$$

a k-ball and the set

$$\Sigma_k(t) = \{x \in \mathbf{R}^n : d_k(x) = t\} \tag{3.9}$$

a k-sphere in $\mathbf{R}^n$.

We will say that an unbounded domain $D \subset \mathbf{R}^n$ is k-admissible if for each $t > \inf_{x\in D} d_k(x)$ the set $D \cap B_k(t)$ has compact closure.

It is clear that every unbounded domain $D \subset \mathbf{R}^n$ is n-admissible. In the general case the domain D is k-admissible if and only if the function $d_k(x)$ is an exhaustion function of D. It is not difficult to see that if a domain $D \subset \mathbf{R}^n$ is k-admissible, then it is l-admissible for all $k < l < n$.

Fix $1 \le k < n$. Let Δ be a bounded domain in the $(n-k)$-plane $x_1 = \cdots = x_k = 0$ and let

$$D = \{x = (x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in \mathbf{R}^n : (x_{k+1}, \dots, x_n) \in \Delta\} \tag{3.10}$$

be a domain in $\mathbf{R}^n$.

The domain D is k-admissible. The k-spheres $\Sigma_k(t)$ are orthogonal to the boundary ∂D and therefore $\langle \nabla d_k, \nu\rangle = 0$ everywhere on the boundary. The function

$$h(x) = \begin{cases} \log d_k(x), & p = k, \\ d_k^{(p-n)/(p-1)}(x), & p \neq k, \end{cases} \tag{3.11}$$

is a special exhaustion function of the domain D. Therefore for $p \ge k$ the domain D is of p-parabolic type and for $p < k$ p-hyperbolic type.

Example 3.3. Fix $1 \le k < n$. Let Δ be a bounded domain in the plane $x_1 = \cdots = x_k = 0$ with a piecewise smooth boundary and let

$$D = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : (x_{k+1}, \dots, x_n) \in \Delta\} = \mathbf{R}^{n-k} \times \Delta \tag{3.12}$$

be the cylinder domain with base Δ.

The domain D is k-admissible. The k-spheres $\Sigma_k(t)$ are orthogonal to the boundary ∂D and therefore $\langle \nabla d_k, \nu\rangle = 0$ everywhere on the boundary, where d_k is as in Example 3.2.

Let $h = \phi(d_k)$ where ϕ is a C^2-function. We have $\nabla h = \phi' \nabla d_k$ and

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla h|^{n-2} \frac{\partial h}{\partial x_i}\right) &= \sum_{i=1}^{k} \frac{\partial}{\partial x_i}\left((\phi')^{n-1} \frac{\partial d_k}{\partial x_i}\right) \\ &= (n-1)(\phi')^{n-2}\phi'' + \frac{k-1}{d_k}(\phi')^{n-1}. \end{aligned} \tag{3.13}$$

From the equation

$$(n-1)\phi'' + \frac{k-1}{d_k}\phi' = 0 \tag{3.14}$$

we conclude that *the function*

$$h(x) = (d_k(x))^{(n-k)/(n-1)} \tag{3.15}$$

satisfies (2.13) *in $D \setminus K$ and thus it is a special exhaustion function of the domain D.*

Example 3.4. Let (r,θ), where $r \geq 0$, $\theta \in S^{n-1}(1)$, be the spherical coordinates in R^n. Let $U \subset S^{n-1}(1)$, $\partial U \neq \emptyset$, be an arbitrary domain on the unit sphere $S^{n-1}(1)$. We fix $0 \leq r_1 < \infty$ and consider the domain

$$D = \{(r,\theta) \in R^n : r_1 < r < \infty,\ \theta \in U\}. \tag{3.16}$$

As mentioned above it is easy to verify that the given domain is n-admissible and *the function*

$$h(|x|) = \log \frac{|x|}{r_1} \tag{3.17}$$

is a special exhaustion function of the domain D for $p = n$.

Example 3.5. Fix $1 \leq n \leq p$. Let $x_1, x_2, \ldots, x_n$ be an orthonormal system of coordinates in $\mathbf{R}^n$,$1 \leq n < p$. Let $D \subset \mathbf{R}^n$ be an unbounded domain with piecewise smooth boundary and let $\mathcal{B}$ be an $(p-n)$-dimensional compact Riemannian manifold with or without boundary. We consider the manifold $\mathcal{M} = D \times \mathcal{B}$.

We denote by $x \in D$, $b \in \mathcal{B}$, and $(x,b) \in \mathcal{M}$ the points of the corresponding manifolds. Let $\pi : D \times \mathcal{B} \to D$ and $\eta : D \times \mathcal{B} \to \mathcal{B}$ be the natural projections of the manifold $\mathcal{M}$.

Assume now that the function h is a function on the domain D satisfying the conditions (b_1), (b_2) and (2.13). We consider the function $h^* = h \circ \pi : \mathcal{M} \to (0,\infty)$.

We have

$$\begin{aligned}
\nabla h^* &= \nabla(h \circ \pi) = (\nabla_x h) \circ \pi, \\
\operatorname{div}\left(|\nabla h^*|^{p-2}\nabla h^*\right) &= \operatorname{div}\left(|\nabla(h \circ \pi)|^{p-2}\nabla(h \circ \pi)\right) \\
&= \operatorname{div}\left(|\nabla_x h|^{p-2} \circ \pi(\nabla_x h) \circ \pi\right) = \left(\sum_{i=1}^{n} \partial\partial x_i \left(|\nabla_x h|^{p-2}\partial h \partial x_i\right)\right) \circ \pi.
\end{aligned} \tag{3.18}$$

Because h is a special exhaustion function of D we have

$$\operatorname{div}\left(|\nabla h^*|^{p-2}\nabla h^*\right) = 0. \tag{3.19}$$

Let $(x,b) \in \partial\mathcal{M}$ be an arbitrary point where the boundary $\partial\mathcal{M}$ has a tangent hyperplane and let ν be a unit normal vector to $\partial\mathcal{M}$.

If $x \in \partial D$, then $\nu = \nu_1 + \nu_2$ where the vector $\nu_1 \in \mathbf{R}^k$ is orthogonal to ∂D and ν_2 is a vector from $T_b(\mathcal{B})$. Thus

$$\langle \nabla h^*, \nu \rangle = \langle (\nabla_x h) \circ \pi, \nu_1 \rangle = 0, \tag{3.20}$$

because h is a special exhaustion function on D and satisfies the property (b_2) on ∂D. If $b \in \partial\mathcal{B}$, then the vector ν is orthogonal to $\partial\mathcal{B} \times \mathbf{R}^n$ and

$$\langle \nabla h^*, \nu \rangle = \langle (\nabla_x h) \circ \pi, \nu \rangle = 0, \tag{3.21}$$

because the vector $(\nabla_x h) \circ \pi$ is parallel to $\mathbf{R}^n$.

The other requirements for a special exhaustion function for the manifold $\mathcal{M}$ are easy to verify.

Therefore, *the function*

$$h^* = h^*(x,b) = h \circ \pi : \mathcal{M} \longrightarrow (0,\infty) \tag{3.22}$$

is a special exhaustion function on the manifold $\mathcal{M} = D \times \mathcal{B}$.

Example 3.6. Let $\mathcal{A}$ be a compact Riemannian manifold, $\dim \mathcal{A} = k$, with piecewise smooth boundary or without boundary. We consider the Cartesian product $\mathcal{M} = \mathcal{A} \times \mathbf{R}^n$, $n \geq 1$. We denote by $a \in \mathcal{A}$, $x \in \mathbf{R}^n$ and $(a,x) \in \mathcal{M}$ the points of the corresponding spaces. It is easy to see that the function

$$h(a,x) = \begin{cases} \log|x|, & p = n, \\ |x|^{(p-n)/(p-1)}, & p \neq n, \end{cases} \tag{3.23}$$

is a special exhaustion function for the manifold $\mathcal{M}$. Therefore, for $p \geq n$ the given manifold is of p-parabolic type and for $p < n$ p-hyperbolic type.

Example 3.7. Let (r,θ), where $r \geq 0$, $\theta \in S^{n-1}(1)$, be the spherical coordinates in $\mathbf{R}^n$. Let $U \subset S^{n-1}(1)$ be an arbitrary domain on the unit sphere $S^{n-1}(1)$. We fix $0 \leq r_1 < r_2 < \infty$ and consider the domain

$$D = \{(r,\theta) \in \mathbf{R}^n : r_1 < r < r_2,\ \theta \in U\} \tag{3.24}$$

with the metric

$$ds^2_{\mathcal{M}} = \alpha^2(r)dr^2 + \beta^2(r)dl^2_\theta, \tag{3.25}$$

where $\alpha(r),\ \beta(r) > 0$ are C^0-functions on $[r_1, r_2)$ and dl_θ is an element of length on $S^{n-1}(1)$.

The manifold $\mathcal{M} = (D, ds^2_{\mathcal{M}})$ is a warped Riemannian product. In the case $\alpha(r) \equiv 1$, $\beta(r) = 1$, and $U = S^{n-1}$ the manifold $\mathcal{M}$ is isometric to a cylinder in $\mathbf{R}^{n+1}$. In the case $\alpha(r) \equiv 1$, $\beta(r) = r$, and $U = S^{n-1}$ the manifold $\mathcal{M}$ is a spherical annulus in $\mathbf{R}^n$.

The volume element in the metric (3.25) is given by the expression

$$d\sigma_{\mathcal{M}} = \alpha(r)\beta^{n-1}(r)dr\ dS^{n-1}(1). \tag{3.26}$$

If $\phi(r,\theta) \in C^1(D)$, then the length of the gradient $\nabla\phi$ in $\mathcal{M}$ takes the form

$$|\nabla\phi|^2 = \frac{1}{\alpha^2}(\phi'_r)^2 + \frac{1}{\beta^2}|\nabla_\theta\phi|^2, \tag{3.27}$$

where $\nabla_\theta\phi$ is the gradient in the metric of the unit sphere $S^{n-1}(1)$.

For the special exhaustion function $h(r,\theta) \equiv h(r)$ (2.13) reduces to the following form

$$\frac{d}{dr}\left(\left(\frac{1}{\alpha(r)}\right)^{p-1}(h'_r(r))^{p-1}\beta^{n-1}(r)\right) = 0. \tag{3.28}$$

Solutions of this equation are the functions

$$h(r) = C_1\int_{r_1}^{r}\frac{\alpha(t)}{\beta^{(n-1)/(p-1)}(t)}\,dt + C_2 \tag{3.29}$$

where C_1 and C_2 are constants.

Because the function h satisfies obviously the boundary condition (a_2) as well as the other conditions of Section 3.2, we see that under the assumption

$$\int^{r_2}\frac{\alpha(t)}{\beta^{(n-1)/(p-1)}(t)}dt = \infty \tag{3.30}$$

the function

$$h(r) = \int_{r_1}^{r}\frac{\alpha(t)}{\beta^{(n-1)/(p-1)}(t)}dt \tag{3.31}$$

is a special exhaustion function on the manifold $\mathcal{M}$.

Theorem 3.8. *Let $h : \mathcal{M} \to (0, h_0)$ be a special exhaustion function of a boundary set ξ of the manifold $\mathcal{M}$. Then*

(i) *if $h_0 = \infty$, the set ξ is of p-parabolic type,*

(ii) *if $h_0 < \infty$, the set ξ is of p-hyperbolic type.*

Proof. Choose $0 < t_1 < t_2 < h_0$ such that $K \subset B_h(t_1)$. We need to estimate the p-capacity of the condenser $(B_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M})$. We have

$$\operatorname{cap}_p\left(\overline{B}_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M}\right) = \frac{J}{(t_2 - t_1)^{p-1}}, \tag{3.32}$$

where

$$J = \int_{\Sigma_h(t)} |\nabla h|^{p-1} d\mathcal{H}^{n-1}_{\mathcal{M}} \tag{3.33}$$

is a quantity independent of $t > h(K) = \sup\{h(m) : m \in K\}$. Indeed, for the variational problem (2.1) we choose the function φ_0, $\varphi_0(m) = 0$ for $m \in B_h(t_1)$,

$$\varphi_0(m) = \frac{h(m) - t_1}{t_2 - t_1}, \quad m \in B_h(t_2) \setminus B_h(t_1) \tag{3.34}$$

and $\varphi_0(m) = 1$ for $m \in \mathcal{M} \setminus B_h(t_2)$. Using the Kronrod-Federer formula [7, Theorem 3.2.22], we get

$$\begin{aligned}
\operatorname{cap}_p(B_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M}) &\leq \int_{\mathcal{M}} |\nabla \varphi_0|^p * \mathbb{1}_{\mathcal{M}} \\
&\leq \frac{1}{(t_2 - t_1)^p} \int_{t_1 < h(m) < t_2} |\nabla h(m)|^p * \mathbb{1}_{\mathcal{M}} \\
&= \int_{t_1}^{t_2} dt \int_{\Sigma_h(t)} |\nabla h(m)|^{p-1} d\mathcal{H}^{n-1}_{\mathcal{M}}.
\end{aligned} \tag{3.35}$$

Because the special exhaustion function satisfies (2.13) and the boundary condition (a_2), one obtains for arbitrary τ_1, τ_2, $h(K) < \tau_1 < \tau_2 < h_0$

$$\begin{aligned}
&\int_{\Sigma_h(t_2)} |\nabla h|^{p-1} d\mathcal{H}^{n-1}_{\mathcal{M}} - \int_{\Sigma_h(t_1)} |\nabla h|^{p-1} d\mathcal{H}^{n-1}_{\mathcal{M}} \\
&\quad = \int_{\Sigma_h(t_2)} |\nabla h|^{p-2} \langle \nabla h, \nu \rangle d\mathcal{H}^{n-1}_{\mathcal{M}} - \int_{\Sigma_h(t_1)} |\nabla h|^{p-2} \langle \nabla h, \nu \rangle d\mathcal{H}^{n-1}_{\mathcal{M}} \\
&\quad = \int_{t_1 < h(m) < t_2} \operatorname{div}_{\mathcal{M}}\left(|\nabla h|^{p-2} \nabla h\right) * \mathbb{1}_{\mathcal{M}} = 0.
\end{aligned} \tag{3.36}$$

Thus we have established the inequality

$$\operatorname{cap}_p(B_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M}) \leq \frac{J}{(t_2 - t_1)^{p-1}}. \tag{3.37}$$

By the conditions, imposed on the special exhaustion function, the function φ_0 is an extremal in the variational problem (2.1). Such an extremal is unique and therefore the preceding inequality holds in fact with equality. This conclusion proves (3.32).

If $h_0 = \infty$, then letting $t_2 \to \infty$ in (3.32) we conclude the parabolicity of the type of ξ. Let $h_0 < \infty$. Consider an exhaustion $\{\mathcal{U}_k\}$ and choose $t_0 > 0$ such that the h-ball $B_h(t_0)$ contains the compact set K.

Set $t_k = \sup_{m \in \partial \mathcal{U}_k} h(m)$. Then for $t_k > t_0$ we have

$$\operatorname{cap}_p\left(\overline{\mathcal{U}}_{k_0}, \mathcal{U}_k; \mathcal{M}\right) \geq \operatorname{cap}_p(B_h(t_0), B_h(t_k); \mathcal{M}) = \frac{J}{(t_k - t_0)^{p-1}}, \tag{3.38}$$

hence

$$\liminf_{k \to \infty} \operatorname{cap}_p\left(\overline{\mathcal{U}}_{k_0}, \mathcal{U}_k; \mathcal{M}\right) \geq \frac{J}{(h_0 - t_0)^{p-1}} > 0, \tag{3.39}$$

and the boundary set ξ is of p-hyperbolic type. □

4. Energy Integral

The fundamental result of this section is an estimate for the rate of growth of the energy integral of forms of the class $\mathcal{WT}_2$ on noncompact manifolds under various boundary conditions for the forms. As an application we get Phragmén-Lindelöf type theorems for the forms of this class and we prove some generalizations of the classical theorem of Ahlfors concerning the number of distinct asymptotic tracts of an entire function of finite order.

4.1. Boundary Conditions

Let $\mathcal{M}$ be an n-dimensional Riemannian manifold with nonempty boundary $\partial\mathcal{M}$. We will fix a closed differential form w, $\deg w = k$, $1 \leq k \leq n$, $w \in L^p_{\text{loc}}(\mathcal{M})$ of class $\mathcal{WT}_1$ and the complementary closed form θ, $\deg \theta = n - k$, $\theta \in L^q_{\text{loc}}(\mathcal{M})$, satisfying the condition (1.4). We assume that there exists a differential form $Z \in W^1_{p,\text{loc}}$ with continuous coefficients for which $dZ = w$.

Let $h : \mathcal{M} \to (0, h_0)$ be an exhaustion function of $\mathcal{M}$. As mentioned before we let $B_h(\tau)$ be an h-ball and $\Sigma_h(\tau)$ an h-sphere.

4.2. Dirichlet Condition with Zero Boundary Values

We will say that the form $Z \in W^1_{p,\text{loc}}$(with continuous coefficients and such that $dZ = w$) satisfies Dirichlet's condition with zero boundary values on $\partial\mathcal{M}$ if for every differential form $v \in L^q_{\text{loc}}(\mathcal{M})$, $\deg v = n - k$, and for almost every $\tau \in (0, h_0)$

$$\int_{B_h(\tau)} w \wedge v + (-1)^{k-1} \int_{B_h(\tau)} Z \wedge dv = \int_{\Sigma_h(\tau)} Z \wedge v. \tag{4.1}$$

In particular, the form Z satisfies the boundary condition (4.1), if its coefficients are continuous and if its support does not intersect with $\partial\mathcal{M}$, that is

$$\operatorname{supp} Z \cap \partial\mathcal{M} = \emptyset, \quad \text{where } \operatorname{supp} Z = \overline{\{m \in \mathcal{M} : Z(m) \neq 0\}}. \tag{4.2}$$

If $\mathcal{M}$ is compact then (4.1) takes the form

$$\int_{\mathcal{M}} w \wedge v + (-1)^{k-1} \int_{\mathcal{M}} Z \wedge dv = 0. \tag{4.3}$$

4.3. Neumann Condition with Zero Boundary Values

We will say that a form Z satisfies Neumann's condition with zero boundary values, if for every differential form $v \in W^1_{p,\mathrm{loc}}(\mathcal{M})$, $\deg v = k - 1$, and for almost every $\tau \in (0, h_0)$

$$\int_{B_h(\tau)} dv \wedge \theta = \int_{\Sigma_h(\tau)} v \wedge \theta. \tag{4.4}$$

If $\mathcal{M}$ is compact then (4.4) takes the form

$$\int_{\mathcal{M}} dv \wedge \theta = 0. \tag{4.5}$$

4.4. Mixed Zero Boundary Condition

We will say that a form Z satisfies mixed zero boundary condition if for an arbitrary function $\phi \in C^1(\mathcal{M})$ and for almost every $\tau \in (0, h_0)$ we have

$$\int_{B_h(\tau)} \phi w \wedge \theta + (-1)^{n-1} \int_{B_h(\tau)} Z \wedge \theta \wedge d\phi = \int_{\Sigma_h(\tau)} \phi Z \wedge \theta. \tag{4.6}$$

If $\mathcal{M}$ is compact then (4.6) takes the form

$$\int_{\mathcal{M}} \phi w \wedge \theta + (-1)^{n-1} \int_{\mathcal{M}} Z \wedge \theta \wedge d\phi = 0. \tag{4.7}$$

We assume that the form

$$Z \in C^2(\operatorname{int} \mathcal{M}) \cap C^1(\partial\mathcal{M}) \tag{4.8}$$

has the property (4.1). On the basis of Stokes' formula (the standard Stokes formula with generalized derivatives) we conclude that for almost every $\tau \in (0, h_0)$

$$\int_{B_h(\tau)} dZ \wedge v + (-1)^{k-1} \int_{B_h(\tau)} Z \wedge dv = \int_{\partial B_h(\tau)} Z \wedge v \tag{4.9}$$

holds. Therefore we get

$$\int_{\partial B_h(\tau)\setminus\Sigma_h(\tau)} Z \wedge v = 0, \quad \forall v \in W^1_{q,\mathrm{loc}}(\mathcal{M}). \tag{4.10}$$

This implies that the restriction of Z onto the boundary $\partial\mathcal{M}$ is the zero form, that is,

$$Z|_{\partial\mathcal{M}}(m) = 0 \quad \text{at every point } m \in \partial\mathcal{M}. \tag{4.11}$$

We next clarify the geometric meaning of the condition (4.11). We assume that $m \in \partial\mathcal{M}$ is a point where the boundary $\partial\mathcal{M}$ has a tangent plane $T_m(\partial\mathcal{M})$ and that the form Z satisfies the regularity condition (4.8) in some neighborhood of the point m.

Proposition 4.1. *If a form Z is simple at a point $m \in \mathcal{M}$, then the condition* (4.11) *is fulfilled if and only if the form Z is of the form*

$$Z = \omega \wedge dx_n, \tag{4.12}$$

where ω is a form, $\deg\omega = \deg Z - 1$.

Proof. We give an orthonormal system of coordinates $x_1, \ldots, x_n$ at the point m such that the hyperplane $T_m(\partial\mathcal{M})$ is given by the equation $x_n = 0$. Let $\deg Z = l$. Because the form Z is simple, we can represent it as follows

$$Z = \left(\sum_{i=1}^{l} a_{1,i}dx_i + a_{1,n}dx_n\right) \wedge \cdots \wedge \left(\sum_{i=1}^{l} a_{l,i}dx_i + a_{l,n}dx_n\right), \tag{4.13}$$

where $a_{i,j} = a_{i,j}(m)$ are some constants. The condition (4.11) can now be rewritten as follows

$$\left(\sum_{i=1}^{l} a_{1,i}dx_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^{l} a_{l,i}dx_i\right) = 0, \tag{4.14}$$

and we easily obtain (4.12).

The proof of the converse implication is obvious. □

We next clarify the geometric meaning of the Neumann condition (4.4). We fix the forms

$$Z, v \in C^2(\mathrm{int}\,\mathcal{M}) \cap C^1(\partial\mathcal{M}). \tag{4.15}$$

By Stokes' formula we have for almost every $\tau \in (0, h_0)$

$$\int_{\partial B_h(\tau)} v \wedge \theta = \int_{B_h(\tau)} dv \wedge \theta + (-1)^{k-1}\int_{B_h(\tau)} v \wedge d\theta. \tag{4.16}$$

Because the form θ is closed, condition (4.4) gives

$$\int_{\Sigma_h(\tau)} v \wedge \theta = 0 \quad \forall v \in W^1_{p,\mathrm{loc}}. \tag{4.17}$$

Therefore

$$\theta|_{\partial\mathcal{M}}(m) = 0 \tag{4.18}$$

at every point $m \in \partial\mathcal{M}$.

Exactly in the same way we verify that the mixed zero boundary condition (4.6) is equivalent to the condition

$$Z \wedge \theta|_{\partial\mathcal{M}}(m) = 0 \tag{4.19}$$

at every point $m \in \partial\mathcal{M}$.

Consider the case of quasilinear equations (2.9). Let $m \in \partial\mathcal{M}$ be a regular point and let $x_1, \ldots, x_n$ be local coordinates in a neighborhood of this point. We have

$$\begin{aligned} \theta &= *\sum_{i=1}^{n} A_i(m, \nabla f(m))\,dx_i \\ &= \sum_{i=1}^{n} (-1)^{i-1} A_i(m, \nabla f(m))\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n. \end{aligned} \tag{4.20}$$

We set $Z = f$. In the case (4.1) we choose $v = \phi\theta$ where θ is an arbitrary locally Lipschitz function. We obtain

$$\int_{B_h(\tau)} \phi\, df \wedge \theta + \int_{B_h(\tau)} f\, d(\phi\theta) = \int_{\Sigma_h(\tau)} \phi f \theta \tag{4.21}$$

and further

$$\int_{B_h(\tau)} \sum_{i=1}^{n} (\phi f)_{x_i} A_i(m, \nabla f) * \mathbb{1} = \int_{\Sigma_h(\tau)} \phi f \theta, \quad \forall \phi. \tag{4.22}$$

This condition characterizes generalized solutions of (2.9) with zero Dirichlet boundary condition on $\partial\mathcal{M}$.

On the other side, choosing in the case of the Neumann condition (4.4) for v an arbitrary locally Lipschitz function ϕ we get for almost every $\tau \in (0, h)$

$$\int_{B_h(\tau)} \langle \nabla\phi, A(m, \nabla f) \rangle * \mathbb{1} = \int_{\Sigma_h(\tau)} \phi \langle A(m, \nabla f), \nu \rangle\, d\mathcal{H}^{n-1}_{\mathcal{M}} \tag{4.23}$$

which characterizes generalized solutions of (2.9) with zero Neumann boundary conditions on $\partial\mathcal{M}$.

It is easy to see that at every point of the boundary we have

$$(-1)^{i-1}dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n\Big|_{\partial\mathcal{M}}=\cos(\nu,x_i)\,d\mathcal{H}^{n-1}_{\mathcal{M}},\tag{4.24}$$

where (ν,x_i) is the angle between the inner normal vector ν to $\partial\mathcal{M}$ and the direction $0x_i$; $d\mathcal{H}^{n-1}_{\mathcal{M}}$ is the element of surface area on $\mathcal{M}$.

Thus, at a regular boundary point, the condition (4.18) is equivalent to the requirement

$$\langle A(m,\nabla f(m)),\nu\rangle=0.\tag{4.25}$$

Using (4.11) we see that the condition (4.6) is equivalent to the traditional mixed boundary condition at regular boundary points.

4.5. Maximum Principle for $\mathcal{WT}$-Forms

Let $\mathcal{M}$ be a compact Riemannian manifold with nonempty boundary, $\dim\mathcal{M}=n$, and let $v\in L^p_{\mathrm{loc}}$, $\deg w=k$, $1\le k\le n$, be a differential form of class $\mathcal{WT}_1$ on $\mathcal{M}$. Let θ, $\deg\theta=n-k$, be a form complementary to the form w.

Theorem 4.2. *Suppose that there exists a differential form $Z\in W^1_{p,\mathrm{loc}}(\mathcal{M})$, $dZ=w$ on $\mathcal{M}$. If either* (4.3) *or* (4.5) *holds, then $\theta\equiv 0$ on $\mathcal{M}$.*

Proof. We assume that (4.3) holds and set $v=\theta$. Then (4.3) yields

$$\int_{\mathcal{M}} w\wedge\theta=0.\tag{4.26}$$

Because

$$\begin{aligned}(-1)^{k(n-k)}(-1)^{k(n-k)}*(w\wedge\theta)&=(-1)^{k(n-k)}*^{-1}(w\wedge\theta)\\&=*^{-1}\Big(w\wedge(-1)^{k(n-k)}\theta\Big)\\&=*^{-1}(w\wedge*(*\theta))=\langle w,*\theta\rangle,\end{aligned}\tag{4.27}$$

we get

$$\int_{\mathcal{M}} w\wedge\theta=\int_{\mathcal{M}}*(w\wedge\theta)*\mathbb{1}=\int_{\mathcal{M}}\langle w,*\theta\rangle*\mathbb{1}.\tag{4.28}$$

Using (1.5) we deduce

$$0=\int_{\mathcal{M}} w\wedge\theta\ge\nu_0\int_{\mathcal{M}}|\theta|^q*\mathbb{1}.\tag{4.29}$$

We assume that the boundary condition (4.5) holds. Choose $v = Z$. Then (4.5) gives

$$\int_{\mathcal{M}} w \wedge \theta = 0. \tag{4.30}$$

As above, we arrive at the inequality (4.29). This inequality implies that $\theta \equiv 0$ on $\mathcal{M}$. □

In order to illustrate Theorem 4.2 we consider the example of generalized solutions $f \in W^1_{p,\mathrm{loc}}(\mathcal{M})$ of (2.9) under the condition

$$\nu_0 |A(m,\xi)|^p \leq \langle \xi, A(m,\xi) \rangle \tag{4.31}$$

for all $\xi \in T_m(\mathcal{M})$ with the constants $p > 1$ and $\nu_0 > 0$.

Setting $Z = f$ we get

Corollary 4.3. *Suppose that the manifold $\mathcal{M}$ is compact and the boundary $\partial\mathcal{M}$ is not empty. If the function f satisfies the condition* (4.22) *or* (4.23), *then* $f \equiv \mathrm{const}$ *on $\mathcal{M}$.*

5. Estimates for Energy Integral: Applications

This chapter is devoted to Phragmén-Lindelöf and Ahlfors theorems for differential forms.

5.1. Basic Theorem

Let $\mathcal{M}$ be a noncompact Riemannian manifold, $\dim \mathcal{M} = n$. We consider a class $\mathcal{F}$ of differential forms $Z \in W^1_{p,\mathrm{loc}}(\mathcal{M})$, $\deg Z = k-1$, such that the form $dZ = w$ satisfies the conditions (1.1) and is in the class $\mathcal{WT}_2$. Let $\theta \in L^q_{\mathrm{loc}}$ be a form satisfying the condition (1.4), complementary to w.

If the boundary $\partial\mathcal{M}$ is nonempty then we will assume that the form Z satisfies on $\partial\mathcal{M}$ some boundary condition B. In the case considered below such a boundary condition can be any of the conditions (4.1), (4.2), (4.4), and (4.6). We will denote by $\mathcal{F}_B(\mathcal{M})$ the set of forms Z, $dZ \in \mathcal{WT}_2$, satisfying the boundary condition B on $\mathcal{M}$. In particular, below we will operate with the classes of the $\mathcal{F}_D$, $\mathcal{F}_0$, $\mathcal{F}_N$, and $\mathcal{F}_{DN}$ forms corresponding to the boundary conditions (4.1), (4.2), (4.4), and (4.6), respectively.

We fix a locally Lipschitz exhaustion function $h : \mathcal{M} \to (0, h_0)$, $0 < h_0 \leq \infty$. Let $\tau \in (0, h_0)$ and let $B_h(\tau)$ be an h-ball, and $\Sigma_h(\tau)$ its boundary sphere as before.

We introduce a characteristic $\varepsilon(\tau)$ setting

$$\varepsilon(\tau; \mathcal{F}_B) = \inf \frac{\int_{\Sigma_h(\tau)} |w|^p |\nabla h|^{-1} d\mathcal{H}^{n-1}_{\mathcal{M}}}{\left| \int_{\Sigma_h(\tau)} \langle Z, *\theta \rangle d\mathcal{H}^{n-1}_{\mathcal{M}} \right|}, \tag{5.1}$$

where the infimum is taken over all $Z \in \mathcal{F}_B(\mathcal{M})$, $Z \neq 0$.

Some estimates of (5.1) are given in [8, 9].

Under these circumstances we have the following theorem.

Theorem 5.1. *Suppose that the form $Z \in \mathcal{F}_B(\mathcal{M})$ satisfies one of the boundary condition* (4.1), (4.4), *or* (4.6). *Then for almost all $\tau \in (0, h_0)$ and for an arbitrary τ_0 the following relation holds*

$$\frac{d}{d\tau}\left(I(\tau)\exp\left\{-\nu_1\int_{\tau_0}^{\tau}\varepsilon(t;\mathcal{F}_B)dt\right\}\right) \geq 0, \tag{5.2}$$

where

$$I(\tau) = \int_{B_h(\tau)} |w|^p * \mathbb{1}. \tag{5.3}$$

In particular, for all $\tau_1 < \tau_2$ we have

$$I(\tau_1) \leq I(\tau_2)\exp\left\{-\nu_1\int_{\tau_1}^{\tau_2}\varepsilon(t)dt\right\}. \tag{5.4}$$

Proof. The Kronrod-Federer formula yields

$$I(\tau) = \int_0^{\tau} dt \int_{\Sigma_h(\tau)} |w|^p \frac{d\mathcal{H}^{n-1}_{\mathcal{M}}}{|\nabla h|} \tag{5.5}$$

and, in particular, the function $I(\tau)$ is absolutely continuous on closed intervals of $(0, h_0)$. Now it is enough to prove the inequality

$$\frac{d}{d\tau}I(\tau) \geq \nu_1 I(\tau)\ \varepsilon(\tau). \tag{5.6}$$

From (5.5) we have for almost every $\tau \in (0, h_0)$

$$\frac{d}{d\tau}I(\tau) = \int_{\Sigma_h(\tau)} |w|^p \frac{d\mathcal{H}^{n-1}_{\mathcal{M}}}{|\nabla h|}. \tag{5.7}$$

By (1.6) we obtain

$$\begin{aligned} I(\tau) &= \int_{B_h(\tau)} |w|^p * \mathbb{1} \leq \nu_1^{-1}\int_{B_h(\tau)} \langle w, *\theta\rangle * \mathbb{1} \\ &= \nu_1^{-1}\int_{B_h(\tau)} w \wedge \theta = \nu_1^{-1}\int_{B_h(\tau)} dZ \wedge \theta. \end{aligned} \tag{5.8}$$

However, the form Z is weakly closed and satisfies one of the conditions (4.1), (4.2), or (4.4). Therefore for a.e. $\tau \in (0, h_0)$,

$$\int_{B_h(\tau)} dZ \wedge \theta = \int_{\Sigma_h(\tau)} Z \wedge \theta. \tag{5.9}$$

Thus we get

$$I(\tau) \le \nu_1^{-1} \int_{\Sigma_h(\tau)} \langle Z, *\theta \rangle \, d\mathcal{H}_{\mathcal{M}}^{n-1}. \tag{5.10}$$

Further from (5.1) it follows that

$$\int_{\Sigma_h(\tau)} |w|^p \frac{d\mathcal{H}_{\mathcal{M}}^{n-1}}{|\nabla h|} \ge \varepsilon(\tau; \mathcal{F}_B) \left| \int_{\Sigma_h(\tau)} \langle Z, *\theta \rangle \, d\mathcal{H}_{\mathcal{M}}^{n-1} \right|. \tag{5.11}$$

Combining the above inequalities we obtain

$$I(\tau) \le \frac{\nu_1^{-1}}{\varepsilon(\tau; \mathcal{F}_B)} \int_{\Sigma_h(\tau)} |w|^p \frac{d\mathcal{H}_{\mathcal{M}}^{n-1}}{|\nabla h|}. \tag{5.12}$$

This inequality together with the equality (5.7) yields

$$I(\tau) \le \frac{\nu_1^{-1}}{\varepsilon(\tau; \mathcal{F}_B)} \frac{d}{d\tau} I\ (\tau). \tag{5.13}$$

We thus obtain the desired conclusion (5.6). □

We will need also some other estimates of the energy integral. We now prove the first of these inequalities. Denote by $\mathcal{F}(B_h(\tau))$ the set of all differential forms

$$Z_0 \in C^1(B_h(\tau)), \quad \deg Z_0 = k-1, \quad dZ_0 = 0, \tag{5.14}$$

such that for almost every $\tau \in (0, h_0)$ and for an arbitrary Lipschitz function ϕ the following formula holds

$$\int_{\Sigma_h(\tau)} \phi \ Z_0 \wedge \theta = \int_{B_h(\tau)} d\phi \wedge Z_0 \wedge \theta. \tag{5.15}$$

Theorem 5.2. *If the differential form $Z \in \mathcal{F}_B(\mathcal{M})$, $dZ \in \mathcal{WT}_2$, satisfies the boundary condition* (4.1), (4.4), *or* (4.6), *then for all $\tau_1 < \tau_2 < h_0$ and for an arbitrary form $Z_0 \in \mathcal{F}(B_h(\tau_2))$ the following relation holds*

$$\nu_1 \int_{B_h(\tau_1)} |dZ|^p * \mathbb{1} \le \frac{p}{\tau_2 - \tau_1} \int_{B_h(\tau_2) \setminus B_h(\tau_1)} |\nabla h| |(Z - Z_0) \wedge \theta| * \mathbb{1}. \tag{5.16}$$

Proof. We consider the function

$$\phi(m) = \begin{cases} 1 & \text{for } m \in B_h(\tau_1), \\ \dfrac{\tau_2 - h(m)}{\tau_2 - \tau_1} & \text{for } m \in B_h(\tau_2) \setminus B_h(\tau_1), \\ 0 & \text{for } m \in \mathcal{M} \setminus B_h(\tau_2). \end{cases} \tag{5.17}$$

Suppose that the form Z satisfies the condition (4.1). Setting in (4.1) $v = (\phi)^p Z \wedge \theta$ we get

$$\int_{B_h(\tau_2)} (\phi)^p w \wedge \theta + (-1)^{k-1} \int_{B_h(\tau_2)} Z \wedge d(\phi)^p \wedge \theta = 0, \tag{5.18}$$

or

$$\int_{B_h(\tau_2)} (\phi)^p \langle w, *\theta \rangle * \mathbb{1} = (-1)^k \, p \int_{B_h(\tau_2)} (\phi)^{p-1} \, Z \wedge d\phi \wedge \theta. \tag{5.19}$$

The function $(\phi)^p$ is locally Lipschitz on $\overline{B}_h(\tau_2)$ and $\phi|_{\Sigma_h(\tau_2)} = 0$. Thus by (5.15) we get

$$\begin{aligned} \int_{B_h(\tau_2)} d\phi^p \wedge Z_0 \wedge \theta = \int_{B_h(\tau_2)} & d(\phi)^p \wedge Z_0 \wedge \theta \\ & + \int_{B_h(\tau_2)} (\phi)^p \, dZ_0 \wedge \theta + \int_{B_h(\tau_2)} (\phi)^p \, Z_0 \wedge d\theta \\ & = \int_{\Sigma_h(\tau_2)} (\phi)^p \, Z_0 \wedge \theta = 0. \end{aligned} \tag{5.20}$$

Hence we arrive at the relation

$$\int_{B_h(\tau_2)} (\phi)^p \langle w, *\theta \rangle \; * \mathbb{1} = (-1)^k \, p \int_{B_h(\tau_2)} (\phi)^{p-1} (Z - Z_0) \wedge d\phi \wedge \theta, \tag{5.21}$$

which by (1.6) yields

$$\nu_1 \int_{B_h(\tau_2)} (\phi)^p |w|^p * \mathbb{1} \le \frac{p}{\tau_2 - \tau_1} \int_{B_h(\tau_2)} (\phi)^{p-1} \, |(Z - Z_0) \wedge \theta| |\nabla h| * \mathbb{1}. \tag{5.22}$$

Observing that $\phi(m) = 1$ for $m \in B_h(\tau_1)$ and $\phi(m) = 0$ for $m \in \mathcal{M} \setminus B_h(\tau_2)$ we obtain

$$\nu_1 \int_{B_h(\tau_1)} |w|^p * \mathbb{1} \le \frac{p}{\tau_2 - \tau_1} \int_{B_h(\tau_2) \setminus B_h(\tau_1)} (\phi)^{p-1} |\nabla h| |(Z - Z_0) \wedge \theta| * \mathbb{1}. \tag{5.23}$$

Because $|\phi| \le 1$, the inequality (5.16) follows.

Let the form Z satisfy the condition (4.4). We choose $v = (\phi)^p Z$ and observe that

$$v|_{\Sigma_h(\tau_2)} = 0. \tag{5.24}$$

Then we get

$$\int_{B_h(\tau_2)} (\phi)^p \; w \wedge \theta = -\int_{B_h(\tau_2)} d(\phi)^p \wedge Z \wedge \theta = (-1)^k \; p \int_{B_h(\tau_2)} (\phi)^{p-1} Z \wedge d\phi \wedge \theta. \tag{5.25}$$

Further details of the proof in this case are similar to those carried out above.

We assume that the form Z satisfies the mixed boundary condition (4.6). Observing that

$$(\phi)^p|_{\Sigma_h(\tau_2)} = 0, \tag{5.26}$$

we get

$$\int_{B_h(\tau_2)} (\phi)^p \; w \wedge \theta = (-1)^n \int_{B_h(\tau_2)} Z \wedge \theta \wedge d(\phi)^p = (-1)^{n-k} \; p \int_{B_h(\tau_2)} (\phi)^{p-1} \; Z \wedge d\phi \wedge \theta. \tag{5.27}$$

Arguing as above we complete the proof of the theorem. □

There is also an estimate for the energy integral which does not use the complementary form θ of $dZ = w$. Such an estimate is given in the next theorem.

Theorem 5.3. *If the form Z, $dZ \in \mathcal{WT}_2(\mathcal{M})$, satisfies on $\partial\mathcal{M}$ one of the boundary conditions* (4.1), (4.4), *or* (4.6), *then for all $0 < \tau_1 < \tau_2 < h_0$ and for an arbitrary form $Z_0 \in \mathcal{F}(B_h(\tau_2))$ we have*

$$\int_{B_h(\tau_1)} |dZ|^p * \mathbb{1} \le \left(\frac{p\nu_2}{(\tau_2 - \tau_1)\nu_1} \right)^p \int_{B_h(\tau_2)\setminus B_h(\tau_1)} |\nabla h|^p |Z - Z_0|^p * \mathbb{1}. \tag{5.28}$$

Proof. We use the earlier established relation (5.22). We estimate the integral on the right hand side of (5.22). By (1.7) we get

$$\begin{aligned} \int_{\mathcal{M}} (\phi)^{p-1} |\nabla h| |(Z - Z_0)| \wedge \theta * \mathbb{1} &\le \int_{\mathcal{M}} (\phi)^{p-1} |\nabla h| |Z - Z_0| |\theta| * \mathbb{1} \\ &\le \nu_2 \int_{\mathcal{M}} (\phi)^{p-1} |\nabla h| |Z - Z_0| |w|^{p-1} * \mathbb{1} \\ &\le \nu_2 \left(\int_{\mathcal{M}} |\nabla h|^p |Z - Z_0|^p * \mathbb{1} \right)^{1/p} \left(\int_{\mathcal{M}} \phi^p |w|^p * \mathbb{1} \right)^{(p-1)/p}. \end{aligned} \tag{5.29}$$

From (5.22) we get

$$\left(\frac{\nu_1}{\nu_2}\right)^p \int_{\mathcal{M}} |w|^p * \mathbb{1} \le \left(\frac{p}{\tau_2 - \tau_1}\right)^p \int_{\mathcal{M}} |\nabla h|^p |Z - Z_0|^p * \mathbb{1}. \tag{5.30}$$

Using the facts that $\phi = 1$ on $B_h(\tau_1)$ and $\phi = 0$ on $\mathcal{M} \setminus B_h(\tau_2)$ we easily obtain (5.28). □

5.2. Phragmén-Lindelöf Theorem

Let $\mathcal{M}$ be an n-dimensional noncompact Riemannian manifold with or without boundary and let $w \in \mathcal{W}\mathcal{T}_2$ be a differential form as in (1.1), $\deg w = k$, and θ its complementary form as in (1.4).

We assume that there exists a differential form $Z \in W^1_{p,\mathrm{loc}}$ with $dZ = w$. If the boundary $\partial\mathcal{M}$ is nonempty, then we will assume that Z satisfies the boundary condition of Dirichlet (4.1), Neumann's condition (4.4), or the condition (4.6).

We fix a locally Lipschitz exhaustion function $h : \mathcal{M} \to (0, h_0)$, $0 < h_0 \le \infty$. Let, as above

$$I(\tau; Z) = \int_{B_h(\tau)} |dZ|^p \ * \mathbb{1}, \tag{5.31}$$

and let

$$\begin{aligned} \mu(\tau; Z) &= \inf \int_{\tau < h(m) < \tau+1} |\nabla h| |(Z - Z_0) \wedge \theta| * \mathbb{1}, \\ m(\tau; Z) &= \inf \int_{\tau < h(m) < \tau+1} |\nabla h|^p \ |Z - Z_0|^p * \mathbb{1}, \end{aligned} \tag{5.32}$$

where the infimum is taken over all closed forms Z_0, satisfying conditions (5.14), (5.15) on $B_h(\tau)$.

The following theorem exhibits a generalization of the classical Phragmén-Lindelöf principle for holomorphic functions.

Theorem 5.4. *Suppose that the form Z, $dZ \in \mathcal{W}\mathcal{T}_2(\mathcal{M})$, satisfies one of the boundary conditions (4.1), (4.4) or (4.6). The following alternatives hold: either the form $dZ = 0$ a.e. on the manifold $\mathcal{M}$, or for all $\tau_0 \in (0, h_0)$ we have*

$$\liminf_{\tau \to h_0} I(\tau; Z) \exp\left\{-\nu_1 \int_{\tau_0}^{\tau} \varepsilon(t; \mathcal{F}_B) dt\right\} > 0; \tag{5.33}$$

$$\liminf_{\tau \to h_0} \mu(\tau; Z) \exp\left\{-\nu_1 \int_{\tau_0}^{\tau} \varepsilon(t; \mathcal{F}_B) dt\right\} > 0, \tag{5.34}$$

$$\liminf_{\tau \to h_0} m(\tau; Z) \exp\left\{-\nu_1 \int_{\tau_0}^{\tau} \varepsilon(t; \mathcal{F}_B) dt\right\} > 0. \tag{5.35}$$

Proof. The property (5.33) follows readily from (5.4) and is presented here only for the sake of completeness.

By (5.4) and (5.16), for a.e. $\tau \in (\tau_0, h_0)$ we have

$$I(\tau_0) \le I(\tau) \exp\left\{-\nu_1 \int_{\tau_0}^{\tau} \varepsilon(t)dt\right\}$$
$$\le p\nu_1^{-1} \int_{B_h(\tau+1)\setminus B_h(\tau)} |\nabla h||(Z - Z_0) \wedge \theta| * \mathbb{1} \exp\left\{-\nu_1 \int_{\tau_0}^{\tau} \varepsilon(t)dt\right\}. \tag{5.36}$$

Therefore we get

$$I(\tau_0) \le p\nu_1^{-1} \mu(\tau; Z) \exp\left\{-\nu_1 \int_{\tau_0}^{\tau} \varepsilon(t)dt\right\}. \tag{5.37}$$

Analogously, using (5.28) we get

$$I(\tau_0) \le \left(\frac{\nu_2}{\nu_1}\right)^p p^p m(\tau; Z) \exp\left\{-\nu_1 \int_{\tau_0}^{\tau} \varepsilon(t)dt\right\}. \tag{5.38}$$

If we now assume that the form $w \not\equiv 0$, then $I(\tau_0) > 0$ for some $\tau_0 \in (0, h_0)$. From this there easily follow (5.34) and (5.35). □

5.3. Integral of Energy and Allocation of Finite Forms

There is another application of the above estimates of energy integrals connected with a generalization of the classical Denjoy-Carleman-Ahlfors theorem about the number of different asymptotic tracts of an entire function of a given order. In the present case this theorem can be interpreted as a statement concerning the connection between the number of finite forms in the class $\mathcal{WT}_2$ defined on the manifold $\mathcal{M}$ and the rate of growth of their energy integrals.

Let $\mathcal{M}$ be an n-dimensional noncompact Riemannian manifold with or without boundary. We fix a locally Lipschitz exhaustion function $h : \mathcal{M} \to (0, h_0)$, $0 < h_0 \le \infty$, of the manifold $\mathcal{M}$.

We assume that there are $L \ge 1$ mutually disjoint domains $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_L$ on $\mathcal{M}$ such that $\mathcal{O}_i \cap \partial\mathcal{M} = \emptyset$ if the boundary $\partial\mathcal{M}$ is nonempty. We also assume that on each domain $\mathcal{O}_i$ there is given a differential form Z_i with continuous coefficients and the properties:

(i) $\deg Z_i = k - 1$, $dZ \not\equiv 0$,

(ii) $dZ_i = w \in \mathcal{WT}_2$ with structure constants p, ν_1, ν_2, independent of $i = 1, 2, \ldots, L$,

(iii) Z_i satisfies on $\partial\mathcal{O}_i$ the zero boundary condition (4.2).

We define a form Z on $\mathcal{M}$ by setting $Z|_{\mathcal{O}_i} = Z_i$ and $Z = 0$ on $\mathcal{M} \setminus \bigcup_{i=1}^{L} \mathcal{O}_i$.

According to Theorem 4.2 each of the domains $\mathcal{O}_i$ has a noncompact closure. Then by Theorem 5.4 the "narrower" the intersection of the domains $\mathcal{O}_i$ with h-spheres $\Sigma_h(t)$ for $t \to \infty$, the higher is the rate of growth of the form Z. Below we will consider the Denjoy-Carleman-Ahlfors theorem as a statement on the connection between the number L of mutually disjoint domains $\mathcal{O}_i$ on $\mathcal{M}_0$ and the rate of growth of the energy of the form Z (or of the form Z itself) with respect to an exhaustion function $h(m)$ of the manifold $\mathcal{M}$. We will prove that such a formulation of the problem contains, in particular, the classical Denjoy-Carleman-Ahlfors problem for holomorphic functions of the complex plane. In the case of harmonic functions of $\mathbf{R}^n$ see [10, 11] for the history of the problem.

We next introduce some necessary notation. We consider an open subset $D \subset \mathcal{M}$ with a noncompact closure and we assume that the restriction of the form Z to D satisfies condition (4.2).

The function $h|_D : D \to (0, \infty)$ is an exhaustion function of D. We fix an h-ball $B_h(\tau)$. Considering the variational problem (5.1) for the class of forms Z, satisfying the boundary condition (4.2) on D_1 we define the characteristic

$$\varepsilon(t; D) = \varepsilon(t; \mathcal{F}_0) \tag{5.39}$$

where $\mathcal{F}_0$ is defined in Section 5.1 and in (4.1).

Following [12] we introduce the N-mean

$$E(t; N) = \inf \frac{1}{N} \sum_{k=1}^{N} \varepsilon(t; D_k) \tag{5.40}$$

where the infimum is taken over all decompositions of D into N nonintersecting open sets $D_1, D_2, \ldots, D_N$ with noncompact closures.

We record the following simple result.

Lemma 5.5. *Let $D_1 \subset D_2$ be arbitrary open subsets of $\mathcal{M}$ with noncompact closures. Then*

$$\varepsilon(t; D_2) \le \varepsilon(t; D_1), \tag{5.41}$$

$$\varepsilon(t; \mathcal{M}) \le E(t; N), \quad N \ge 1. \tag{5.42}$$

Proof. It is enough to observe that each pair of forms Z, Z_0 admissible for the variational problem (5.1) for the set D_1 is also admissible for this problem for the set D_2.

From (5.41) we get (5.42). □

We next derive a more general assertion about the monotonicity of N-means.

Lemma 5.6. *For arbitrary $N > 1$ we have*

$$E(t; N + 1) \ge E(t; N). \tag{5.43}$$

Proof. We consider an arbitrary family of open subsets $\{D_k\}$, $k = 1, 2, \ldots, N+1$, admissible for the infimum in (5.40). It is not difficult to see that

$$\frac{1}{N+1}\sum_{k=1}^{N+1}\varepsilon(t; D_k) = \frac{1}{N+1}\sum_{k=1}^{N+1}\left(\frac{1}{N}\sum_{j=1, j\neq k}^{N+1}\varepsilon(t; D_j)\right). \tag{5.44}$$

Because

$$\frac{1}{N}\sum_{j=1, j\neq k}^{N+1}\varepsilon(t; D_k) \geq E(t; N), \tag{5.45}$$

we see that

$$\frac{1}{N+1}\sum_{k=1}^{N+1}\varepsilon(t; D_k) \geq E(t; N) \tag{5.46}$$

and the lemma is proved. □

The next theorem provides a solution to the aforementioned problem concerning the connection between the number L of finite forms on $\mathcal{M}$, and the rate of growth of the total energy of these forms or the sum of their L^p-norms on an h-ball $B_h(\tau)$.

Theorem 5.7. *Suppose that the manifold $\mathcal{M}$ satisfies the properties listed in the beginning of this subsection and that for some $N = 1, 2, \ldots$*

$$\int^{h_0} h_0 E(t, N)\,dt = \infty. \tag{5.47}$$

If the differential form Z, $dZ \in \mathcal{WT}_2(\mathcal{M})$, is such that

$$\liminf_{\tau\to h_0}\int_{h(m)<\tau}|dZ|^p \ast \mathbb{1}\exp\left\{-\nu_1\int_{\tau_0}^{\tau}E(t; N)\,dt\right\} = 0, \tag{5.48}$$

or

$$\liminf_{\substack{\tau',\tau''\to h_0\\ 0<\tau'<\tau''<h_0}}\int_{\tau'<h(m)<\tau''}|\nabla h||Z\wedge\theta| \ast \mathbb{1}\exp\left\{-\nu_1\int_{\tau_0}^{\tau'}E(t; N)\,dt\right\} = 0, \tag{5.49}$$

or

$$\liminf_{\substack{\tau',\tau''\to h_0\\ 0<\tau'<\tau''<h_0}}\int_{\tau'<h(m)<\tau''}|\nabla h|^p|Z|^p \ast \mathbb{1}\exp\left\{-\nu_1\int_{\tau_0}^{\tau'}E(t; N)\,dt\right\} = 0, \tag{5.50}$$

then $L < N$.

Proof. We assume that there exists N mutually nonintersecting domains $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_N$ on the set $\mathcal{M}_0$ and the forms Z_i defined on $\mathcal{O}_i$ with above properties. We denote by

$$d(\mathcal{O}_k) = \inf_{m\in\mathcal{O}_k} h(m), \quad d = \max_{1\le k\le N} d(\mathcal{O}_k). \tag{5.51}$$

Fix $\tau_0 > d$. Using the inequality (5.4) from Theorem 5.1 for an arbitrary $k = 1, 2, \dots, N$ and a.e. $0 < \tau_0 < \tau' < h_0$ we have

$$I_k(\tau_0) \exp\left\{ \nu_1 \int_{\tau_0}^{\tau'} \varepsilon_k(t)dt \right\} \le I_k(\tau'), \tag{5.52}$$

where

$$I_k(\tau') = \int_{\mathcal{O}_k\cap B_h(\tau')} |dZ_k|^p * \mathbb{1}, \qquad \varepsilon_k(\tau') = \varepsilon(\tau'; \mathcal{O}_k). \tag{5.53}$$

Adding these inequalities, we get

$$\min_{1\le k\le N} I_k(\tau_0) \sum_{k=1}^{N} \exp\left\{ \nu_1 \int_{\tau_0}^{\tau'} \varepsilon_k(t)dt \right\} \le I(\tau'), \tag{5.54}$$

where

$$I(\tau') = \int_{B_h(\tau')} |dZ|^p * \mathbb{1}. \tag{5.55}$$

Applying the arithmetic-geometric mean inequality

$$\frac{1}{N} \sum_{k=1}^{N} \exp\left\{ \nu_1 \int_{\tau_0}^{\tau'} \varepsilon_k(t)dt \right\} \ge \prod_{k=1}^{N} \exp\left\{ \frac{\nu_1}{N} \int_{\tau_0}^{\tau'} \varepsilon_k(t)dt \right\} \tag{5.56}$$

we get

$$\min_{1\le k\le N} I_k(\tau_0) N \exp\left\{ \frac{\nu_1}{N} \int_{\tau_0}^{\tau'} \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k(t)dt \right\} \le I(\tau'). \tag{5.57}$$

The domains $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_N$ are nonintersecting. Therefore for all $\tau_0 < t < h_0$ we have

$$\frac{1}{N}\sum_{k=1}^{N}\varepsilon_k(t) \geq E(t;N). \tag{5.58}$$

The preceding inequality gives now

$$\min_{1\leq k\leq N} I_k(\tau_0) N \exp\left\{\nu_1 \int_{\tau_0}^{\tau'} E(t;N)dt\right\} \leq I(\tau'). \tag{5.59}$$

For the estimation of the integral $I(\tau')$ we use the inequalities (5.16) and (5.28) and obtain

$$\min_{1\leq k\leq N} I_k(\tau_0) \leq C_1 \int_{\tau'<h(m)<\tau''} |\nabla h||Z\wedge\theta| * \mathbb{1} \exp\left\{-\nu_1 \int_{\tau_0}^{\tau'} E(t;N)dt\right\} \tag{5.60}$$

or

$$\min_{1\leq k\leq N} I_k(\tau_0) \leq C_2 \int_{\tau'<h(m)<\tau''} |\nabla h|^p |Z|^p * \mathbb{1} \ \exp\left\{-\nu_1 \int_{\tau_0}^{\tau'} E(t;N)dt\right\}, \tag{5.61}$$

where

$$C_1 = \frac{p}{\tau'' - \tau'}, \qquad C_2 = \left(\frac{p\nu_2}{(\tau''-\tau')\nu_1}\right). \tag{5.62}$$

On the basis of the conditions (5.47)–(5.50) imposed on the form Z, for some k, $1 \leq k \leq N$, we have $I_k(\tau_0) = 0$. Thus $dZ_k(m) \equiv 0$ on $\mathcal{O}_k \cap B_h(\tau_0)$. Because we have chosen $\tau_0 > d$ arbitrarily, we can conclude that at least on one of the components $\mathcal{O}_k$, $dZ_k \equiv 0$. A contradiction. □

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