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Research Article

On Some k-Dimensional Cyclic Systems of Difference Equations

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Motivated by Iričanin and Stević's paper (2006) in which for the first time were considered some cyclic systems of difference equations, here we study the global attractivity of some nonlinear k-dimensional cyclic systems of higher-order difference equations. To do this, we use the transformation method from Berenhaut et al. (2007) and Berenhaut and Stević (2007). The main results in this paper also extend our recent results in the work of (Liu and Yang 2010, in press).

1. Introduction

Motivated by papers [1, 2], in [3], we proved that the unique positive equilibrium points of the following difference equations:

$$y_n = \frac{y_{n-k}^r + y_{n-l}^r}{1 + y_{n-k}^r y_{n-l}^r}, \quad y_n = \frac{y_{n-k}^r y_{n-l}^r y_{n-m}^r + y_{n-k}^r + y_{n-l}^r + y_{n-m}^r}{y_{n-k}^r y_{n-l}^r + y_{n-k}^r y_{n-m}^r + y_{n-l}^r y_{n-m}^r + 1}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $k, l, m \in \mathbb{N}$, $1 \le k < l < m$, and $r \in (0, 1]$ are globally asymptotically stable, respectively.

Motivated by paper [4] by Iričanin and Stević, in which for the first time were considered some cyclic systems of difference equations, here we mainly investigate the global attractivity of the k-dimensional system of higher-order difference equations

$$z_n^{(i)} = \frac{1 + \left(z_{n-l}^{(i)}\right)^r \left(z_{n-m}^{(\theta(i+1))}\right)^r}{\left(z_{n-l}^{(i)}\right)^r + \left(z_{n-m}^{(\theta(i+1))}\right)^r}, \quad i = 1, 2, \ldots, k, \ n \in \mathbb{N}_0, \tag{1.2}$$

where $r \in (0,1]$, $k,l,m \in \mathbb{N}$, $k \geq 2$, $\theta(s) = s(\bmod k)$, and $\theta(k) = k$, as well as the following counterpart difference equation system:

$$z_n^{(i)} = \frac{\left(z_{n-p}^{(i)}\right)^r + \left(z_{n-q}^{(\theta(i+1))}\right)^r}{1 + \left(z_{n-p}^{(i)}\right)^r \left(z_{n-q}^{(\theta(i+1))}\right)^r}, \quad i = 1,2,\ldots,k, \ n \in \mathbb{N}_0, \tag{1.3}$$

where $r \in (0,1]$, $k,p,q \in \mathbb{N}$, $k \geq 2$.

Furthermore, we also present some similar results regarding the k-dimensional cyclic difference equation system:

$$z_n^{(i)} = \frac{\left(z_{n-u}^{(i)}\right)^r \left(z_{n-l}^{(\theta(i+1))}\right)^r \left(z_{n-m}^{(\theta(i+2))}\right)^r + \left(z_{n-u}^{(i)}\right)^r + \left(z_{n-l}^{(\theta(i+1))}\right)^r + \left(z_{n-m}^{(\theta(i+2))}\right)^r}{\left(z_{n-u}^{(i)}\right)^r \left(z_{n-l}^{(\theta(i+1))}\right)^r + \left(z_{n-u}^{(i)}\right)^r \left(z_{n-m}^{(\theta(i+2))}\right)^r + \left(z_{n-l}^{(\theta(i+1))}\right)^r \left(z_{n-m}^{(\theta(i+2))}\right)^r + 1}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where $i = 1,2,\ldots,k$ and $r \in (0,1]$, $k,u,l,m \in \mathbb{N}$, $k \geq 3$, as well as the following counterpart difference equation system

$$z_n^{(i)} = \frac{\left(z_{n-v}^{(i)}\right)^r \left(z_{n-p}^{(\theta(i+1))}\right)^r + \left(z_{n-v}^{(i)}\right)^r \left(z_{n-q}^{(\theta(i+2))}\right)^r + \left(z_{n-p}^{(\theta(i+1))}\right)^r \left(z_{n-q}^{(\theta(i+2))}\right)^r + 1}{\left(z_{n-v}^{(i)}\right)^r \left(z_{n-p}^{(\theta(i+1))}\right)^r \left(z_{n-q}^{(\theta(i+2))}\right)^r + \left(z_{n-v}^{(i)}\right)^r + \left(z_{n-p}^{(\theta(i+1))}\right)^r + \left(z_{n-q}^{(\theta(i+2))}\right)^r}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where $i = 1,2,\ldots,k$ and $r \in (0,1]$, $k,v,p,q \in \mathbb{N}$, $k \geq 3$.

For some recent papers on systems of difference equations, see, for example, [4–10] and the related references therein. Some related scalar equations have been studied mainly by the semicycle structure analysis which was unnecessarily complicated and only useful for lower-order equations, as it was shown by Berg and Stević in [11] (see also papers [12–22]). Thus in this paper we will investigate systems (1.2)–(1.5) by the transformation method from [1, 2] which makes the proofs more concise and elegant, and moreover, it is also effective for higher-order ones.

2. Preliminary Lemmas

Before proving the main results in Section 3, in this section we will first present some useful lemmas which are extensions of those ones in [1].

Lemma 2.1. *Define a mapping* $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\mathcal{F}(x,y) = (1 + x^r y^r)/(x^r + y^r)$, *where* $r \in (0,1]$. *Then,*

(1) $\mathcal{F}(x,y)$ *is nondecreasing in* x *if* $y \geq 1$ *and strictly decreasing in* x *if* $0 < y < 1$;

(2) $\mathcal{F}(x,y)$ *is nondecreasing in* y *if* $x \geq 1$ *and strictly decreasing in* y *if* $0 < x < 1$.

Proof. The results follow directly from the following fact:

$$\mathcal{F}(x, y) = y^r + \frac{1 - y^{2r}}{x^r + y^r} = x^r + \frac{1 - x^{2r}}{x^r + y^r}. \tag{2.1}$$

□

For simplicity, systems (1.2) and (1.3) can be respectively rewritten in the following forms:

$$z_n^{(i)} = \mathcal{F}\left(z_{n-l}^{(i)}, z_{n-m}^{(\theta(i+1))}\right), \quad i = 1, 2, \ldots, k, \ n \in \mathbb{N}_0, \tag{2.2}$$

$$z_n^{(i)} = \frac{1}{\mathcal{F}\left(z_{n-p}^{(i)}, z_{n-q}^{(\theta(i+1))}\right)}, \quad i = 1, 2, \ldots, k, \ n \in \mathbb{N}_0. \tag{2.3}$$

Lemma 2.2. *Denote a transformation*

$$\hat{y} = \begin{cases} y & \text{if } y \geq 1, \\ \dfrac{1}{y} & \text{if } 0 < y < 1. \end{cases} \tag{2.4}$$

Then for the mapping $\mathcal{F}(x, y)$ defined in Lemma 2.1, we have $\widehat{\mathcal{F}}(x, y) = \mathcal{F}(\hat{x}, \hat{y})$.

Proof. It is easy to see that $\mathcal{F}$ defined in Lemma 2.1 is invariant, if both arguments are replaced by the reciprocal ones, but it turns over into $1/\mathcal{F}$ if only one argument is replaced by its reciprocal value. Note that

$$\mathcal{F}(x, y) - 1 = \frac{(x^r - 1)(y^r - 1)}{x^r + y^r}, \tag{2.5}$$

from which the result directly follows by considering the next four cases $(x \geq 1, y \geq 1)$, $(x \geq 1, y < 1)$, $(x < 1, y \geq 1)$ and $(x < 1, y < 1)$. The proof is complete. □

The following lemma is a corollary of Lemma 2.2.

Lemma 2.3. *Let $\mathcal{B} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mathcal{B}(x, y) = 1/\mathcal{F}(x, y)$; then we have*

$$\widehat{\mathcal{B}}(x, y) = \frac{1}{\mathcal{B}(\hat{x}, \hat{y})} = \mathcal{F}(\hat{x}, \hat{y}). \tag{2.6}$$

Let $(\mathbf{Z}_n)_{n=-\alpha}^{\infty}$ be a positive solution to system (2.2), where $\mathbf{Z}_n = (z_n^{(1)}, z_n^{(2)}, \ldots, z_n^{(k)})^T$, $\alpha = \max\{l, m\}$; then by the transformation (2.4), we can define a transformed sequence $(\widehat{\mathbf{Z}}_n)_{n=-\alpha}^{\infty}$, where $\widehat{\mathbf{Z}}_n = (\widehat{z}_n^{(1)}, \widehat{z}_n^{(2)}, \ldots, \widehat{z}_n^{(k)})^T$, and

$$\widehat{z}_n^{(i)} = \begin{cases} z_n^{(i)} & \text{if } z_n^{(i)} \geq 1, \\ \dfrac{1}{z_n^{(i)}} & \text{if } 0 < z_n^{(i)} < 1, \end{cases} \quad i = 1, 2, \ldots, k. \tag{2.7}$$

Note that transformation (2.7) is a natural extension of the transformation in papers [1, 2]. Then by Lemma 2.2 and the transformation (2.4), we derive the following corollary.

Corollary 2.4. *For system* (2.2)*, we have*

$$\widehat{z}_n^{(i)} = \mathcal{F}\left(\widehat{z}_{n-l}^{(i)}, \widehat{z}_{n-m}^{(\theta(i+1))}\right) = \frac{1 + \left(\widehat{z}_{n-l}^{(i)}\right)^r \left(\widehat{z}_{n-m}^{(\theta(i+1))}\right)^r}{\left(\widehat{z}_{n-l}^{(i)}\right)^r + \left(\widehat{z}_{n-m}^{(\theta(i+1))}\right)^r}, \quad i = 1, 2, \ldots, k, \ n \in \mathbb{N}_0. \tag{2.8}$$

From Corollary 2.4, we easily see that $(\widehat{\mathbf{Z}}_n)_{n=-\alpha}^{\infty}$ is also a positive solution to the system (2.2).

Lemma 2.5. *For the function $\mathcal{F}(x, y)$ defined in Lemma 2.1, there hold*

$$1 \leq \widehat{\mathcal{F}}(x, y) \leq \max\{\widehat{x}, \widehat{y}\}, \quad x, y > 0. \tag{2.9}$$

Proof. By (2.4), it immediately follows that $\widehat{\mathcal{F}}(x, y) \geq 1$.

Let $\lambda = \max\{\widehat{x}, \widehat{y}\} \geq 1$, which indicates that $1 \leq \widehat{x} \leq \lambda$ and $1 \leq \widehat{y} \leq \lambda$. Employing Lemma 2.1 two times and Lemma 2.2, we have

$$\widehat{\mathcal{F}}(x, y) = \mathcal{F}(\widehat{x}, \widehat{y}) \leq \mathcal{F}(\widehat{x}, \lambda) \leq \mathcal{F}(\lambda, \lambda) = \frac{1 + \lambda^r \lambda^r}{\lambda^r + \lambda^r} \leq \lambda. \tag{2.10}$$

The proof is complete. □

The following corollary follows directly from Corollary 2.4 and Lemma 2.5.

Corollary 2.6. *For the transformed sequence $(\widehat{\mathbf{Z}}_n)_{n=-\alpha}^{\infty}$, we get*

$$1 \leq \widehat{z}_n^{(i)} \leq \max\left\{\widehat{z}_{n-l}^{(i)}, \widehat{z}_{n-m}^{(\theta(i+1))}\right\}, \quad i = 1, 2, \ldots, k, \ n \in \mathbb{N}_0. \tag{2.11}$$

Let

$$H_n = \max_{n-\alpha \leq j \leq n-1}\left\{\max_{1 \leq i \leq k}\left\{\widehat{z}_j^{(i)}\right\}\right\}, \quad n \in \mathbb{N}_0, \tag{2.12}$$

for a positive solution $(\mathbf{Z}_n)_{n=-\alpha}^{\infty}$ to system (2.2).

Lemma 2.7. *The sequence $(H_n)_{n=0}^{\infty}$ is monotonically nonincreasing.*

Proof. The proof is a straightforward consequence of Corollary 2.6 and (2.12), and hence is omitted. □

3. Attractivity of (1.2) and (1.3)

In this section, we will formulate and prove the main results of this paper developing the methods and ideas from [1].

Lemma 3.1. *Both systems* (2.2) *and* (2.3) *have the unique positive equilibrium point* $\overline{\mathbf{Z}} = (\underbrace{1,1,\ldots,1}_{k})^T$.

Proof. Let $\overline{\mathbf{Z}} = (e_1, e_2, \ldots, e_k)^T$ be a positive equilibrium point of system (2.2); then we have

$$e_1 = \mathcal{F}(e_1, e_2), e_2 = \mathcal{F}(e_2, e_3), \ldots, e_k = \mathcal{F}(e_k, e_1). \tag{3.1}$$

Through certain calculations, we obtain

$$e_1 - 1 = \frac{(e_1^r - 1)(e_2^r - 1)}{e_1^r + e_2^r}, e_2 - 1 = \frac{(e_2^r - 1)(e_3^r - 1)}{e_2^r + e_3^r}, \ldots, e_k - 1 = \frac{(e_k^r - 1)(e_1^r - 1)}{e_k^r + e_1^r}. \tag{3.2}$$

If there exists some $s \in \{1,2,\ldots,k\}$ such that $e_s = 1$, then from the previous system easily follows that $e_j = 1$ for all $j = 1,2,\ldots,k$. Hence, suppose that $e_j \neq 1$ for all $j = 1,2,\ldots,k$; then by comparing the signs in the last system it is easy to get that it must be $e_j > 1,\ j = 1,2,\ldots,k$. However, since $r \in (0,1]$, then

$$e_1 - 1 \geq e_1^r - 1 > \frac{(e_1^r - 1)(e_2^r - 1)}{e_1^r + e_2^r}, \tag{3.3}$$

which contradicts $e_1 = \mathcal{F}(e_1, e_2)$. Therefore, $e_j = 1$ for all $j \in \{1,2,\ldots,k\}$.

The uniqueness of equilibrium of system (2.3) can be analogously proved and thus is omitted. □

Theorem 3.2. *The unique equilibrium point of system* (1.2) *is a global attractor.*

Proof. Let $(\mathbf{Z}_n)_{n=-\alpha}^{\infty}$ be an arbitrary positive solution to system (1.2), where $\mathbf{Z}_n = (z_n^{(1)}, z_n^{(2)}, \ldots, z_n^{(k)})^T$; then we need to prove that

$$\lim_{n\to\infty} \mathbf{Z}_n = \overline{\mathbf{Z}}. \tag{3.4}$$

Define a transformed sequence $(\widehat{\mathbf{Z}}_n)_{n=-\alpha}^{\infty}$ by (2.4) and (2.7), where $\widehat{\mathbf{Z}}_n = (\widehat{z}_n^{(1)}, \widehat{z}_n^{(2)}, \ldots, \widehat{z}_n^{(k)})^T$; then it suffices to confirm that

$$\lim_{n\to\infty} \widehat{\mathbf{Z}}_n = \overline{\mathbf{Z}}. \tag{3.5}$$

Let the sequence $(H_n)_{n=0}^{\infty}$ be defined by (2.12); then using Lemma 2.7 we know that there is a finite limit of H_n as $n \to \infty$, say H. Note that $H \geq 1$. By Corollary 2.6, we have $\widehat{z}_j^{(i)} \in [1, H_j]$, for all $i = 1,2,\ldots,k$ and $j = 0,1,2,\ldots$. It suffices to show that $H = 1$.

Assume that $H > 1$; then by Lemma 2.7 and (2.12), for arbitrary $\varepsilon > 0$, there exists a sufficiently large $N \in \mathbb{N}$ such that $\hat{z}_N^{(\beta)} \in [H, H+\varepsilon]$, for some $\beta \in \{1, 2, \ldots, k\}$ and

$$\hat{z}_n^{(i)} \in [1, H+\varepsilon], \quad n \geq N-\alpha, \ i = 1, 2, \ldots, k. \tag{3.6}$$

Employing Lemma 2.1 two times and (3.6), we get

$$H \leq \hat{z}_N^{(\beta)} = \digamma\left(\hat{z}_{N-l}^{(\beta)}, \hat{z}_{N-m}^{(\theta(\beta+1))}\right) \leq \digamma\left(\hat{z}_{N-l}^{(\beta)}, H+\varepsilon\right) \leq \digamma(H+\varepsilon, H+\varepsilon) = \frac{1+(H+\varepsilon)^r(H+\varepsilon)^r}{(H+\varepsilon)^r+(H+\varepsilon)^r}. \tag{3.7}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$H \leq \lim_{\varepsilon \to 0^+} \digamma(H+\varepsilon, H+\varepsilon) = \digamma(H, H) = \frac{1+H^r H^r}{H^r + H^r}, \tag{3.8}$$

which implies $H^{2r} - 2H^{r+1} + 1 \geq 0$ for all $H > 1$.

On the other hand, let $J(x) = x^{2r} - 2x^{r+1} + 1, \ x > 1$; then the derivative of $J(x)$ is

$$J'(x) = 2rx^{2r-1} - 2(r+1)x^r = 2x^r\left(rx^{r-1} - r - 1\right) < 0. \tag{3.9}$$

Since $J(1) = 0$, we get $H^{2r} - 2H^{r+1} + 1 < 0$ which contradicts $H^{2r} - 2H^{r+1} + 1 \geq 0$, for all $H > 1$.

Therefore $H = 1$. The proof is complete. □

By Theorem 3.2, the following corollary easily follows.

Corollary 3.3. *The unique positive equilibrium point* $\overline{\mathbf{X}} = (\underbrace{\sqrt{A}, \sqrt{A}, \ldots, \sqrt{A}}_{k})^T$ *of the following difference equation system:*

$$x_n^{(i)} = \frac{A + x_{n-l}^{(i)} x_{n-m}^{(\theta(i+1))}}{x_{n-l}^{(i)} + x_{n-m}^{(\theta(i+1))}}, \quad i = 1, 2, \ldots, k, \ n \in \mathbb{N}_0, \tag{3.10}$$

where the parameter $A \in \mathbb{R}_+$, $r \in (0, 1]$, $k, l, m \in \mathbb{N}$, $k \geq 2$, $l \neq m$, *and* $\alpha = \max\{l, m\}$, *is a global attractor.*

Theorem 3.4. *The unique positive equilibrium point of system* (1.3) *is a global attractor.*

Proof. Let $(\mathbf{Z}_n)_{n=-\mu}^{\infty}$ be any positive solution to system (1.3), where $\mathbf{Z}_n = (z_n^{(1)}, z_n^{(2)}, \ldots, z_n^{(k)})^T$, $\mu = \max\{p, q\}$. By using Lemma 2.3 and (1.3), we obtain

$$\widehat{z}_n^{(i)} = \mathcal{F}\left(\widehat{z}_{n-p}^{(i)}, \widehat{z}_{n-q}^{(\theta(i+1))}\right) = \frac{1 + \left(\widehat{z}_{n-p}^{(i)}\right)^r \left(\widehat{z}_{n-q}^{(\theta(i+1))}\right)^r}{\left(\widehat{z}_{n-p}^{(i)}\right)^r + \left(\widehat{z}_{n-q}^{(\theta(i+1))}\right)^r}, \quad i = 1, 2, \ldots, k, \ n \in \mathbb{N}_0, \tag{3.11}$$

which indicates that $(\widehat{\mathbf{Z}}_n)_{n=-\mu}^{\infty}$ is a positive solution to system (1.2), where $\widehat{\mathbf{Z}}_n = (\widehat{z}_n^{(1)}, \widehat{z}_n^{(2)}, \ldots, \widehat{z}_n^{(k)})^T$. Hence by Theorem 3.2, we have

$$\lim_{n \to \infty} \widehat{\mathbf{Z}}_n = \overline{\mathbf{Z}} \tag{3.12}$$

and then by the transformation (2.4) we easily get

$$\lim_{n \to \infty} \mathbf{Z}_n = \overline{\mathbf{Z}}. \tag{3.13}$$

The proof is complete. □

4. Attractivity of (1.4) and (1.5)

Similar to the proofs of Lemmas 2.1–2.7, we can get the following lemmas. We omit their proofs.

Lemma 4.1. *Define a mapping* $\mathcal{M} : (\mathbb{R}^+)^3 \to \mathbb{R}$ *by* $\mathcal{M}(x_0, x_1, x_2) = (1 + x_0^r x_1^r + x_0^r x_2^r + x_1^r x_2^r)/(x_0^r + x_1^r + x_2^r + x_0^r x_1^r x_2^r)$, *where* $r \in (0, 1]$. *Then for* $j = 0, 1, 2$ *we have*

(1) $\mathcal{M}(x_0, x_1, x_2)$ *is nonincreasing in* x_j *if* $0 < x_{\omega(j+1)}, x_{\omega(j+2)} \le 1$ *or* $x_{\omega(j+1)}, x_{\omega(j+2)} \ge 1$;

(2) $\mathcal{M}(x_0, x_1, x_2)$ *is nondecreasing in* x_j *if* $x_{\omega(j+1)} \ge 1$, $0 < x_{\omega(j+2)} \le 1$ *or* $x_{\omega(j+2)} \ge 1$, $0 < x_{\omega(j+1)} \le 1$, *where* $\omega(s) = s (\mathrm{mod}\ 3)$.

Lemma 4.2. *For the mapping* $\mathcal{M}(x_0, x_1, x_2)$ *defined in Lemma 4.1, we have*

$$\widehat{\mathcal{M}}(x_0, x_1, x_2) = \frac{1}{\mathcal{M}(\widehat{x_0}, \widehat{x_1}, \widehat{x_2})}. \tag{4.1}$$

Lemma 4.3. *Let* $\mathcal{N} : (\mathbb{R}^+)^3 \to \mathbb{R}^+$ *such that* $\mathcal{N}(x_0, x_1, x_2) = 1/\mathcal{M}(x_0, x_1, x_2)$, *then we have*

$$\widehat{\mathcal{N}}(x_0, x_1, x_2) = \mathcal{N}(\widehat{x_0}, \widehat{x_1}, \widehat{x_2}). \tag{4.2}$$

Let $(\mathbf{Z}_n)_{n=-\delta}^{\infty}$ be a positive solution to the system (1.5), where $\mathbf{Z}_n = (z_n^{(1)}, z_n^{(2)}, \ldots, z_n^{(k)})^T$, $\delta = \max\{v, p, q\}$; then by Lemma 4.2 and the transformation (2.7), we get the following corollary.

Corollary 4.4. *For system* (1.5)*, we have*

$$\widehat{z}_n^{(i)} = \frac{1}{\mathcal{M}\left(\widehat{z}_{n-v}^{(i)}, \widehat{z}_{n-p}^{(\theta(i+1))}, \widehat{z}_{n-q}^{(\theta(i+2))}\right)}, \quad i = 1, 2, \ldots, k, \ n \in \mathbb{N}_0. \tag{4.3}$$

Lemma 4.5. *For the function* $\mathcal{M}(x_0, x_1, x_2)$ *defined in Lemma 4.1, there hold*

$$1 \leq \widehat{\mathcal{M}}(x_0, x_1, x_2) \leq \max\{\widehat{x_0}, \widehat{x_1}, \widehat{x_2}\}, \quad x_0, x_1, x_2 > 0. \tag{4.4}$$

The following corollary follows directly from Corollary 4.4 and Lemma 4.5.

Corollary 4.6. *For the transformed sequence* $(\widehat{\mathbf{Z}}_n)_{n=-\delta}^{\infty}$*, we get*

$$1 \leq \widehat{z}_n^{(i)} \leq \max\left\{\widehat{z}_{n-v}^{(i)}, \widehat{z}_{n-p}^{(\theta(i+1))}, \widehat{z}_{n-q}^{(\theta(i+2))}\right\}, \quad i = 1, 2, \ldots, k, \ n \in \mathbb{N}_0. \tag{4.5}$$

Let

$$Q_n = \max_{n-\delta \leq j \leq n-1}\left\{\max_{1 \leq i \leq k}\left\{\widehat{z}_j^{(i)}\right\}\right\}, \quad n \in \mathbb{N}_0, \tag{4.6}$$

for a positive solution $(\mathbf{Z}_n)_{n=-\delta}^{\infty}$ to the system (1.5), where $\delta = \max\{v, p, q\}$.

Lemma 4.7. *The sequence* $(Q_n)_{n=0}^{\infty}$ *is monotonically nonincreasing.*

Note that, by the proof of Lemma 3.1 we can similarly confirm that both system (1.4) and (1.5) have the same unique positive equilibrium $\overline{\mathbf{Z}} = (\underbrace{1, 1, \ldots, 1}_{k})^T$.

Theorem 4.8. *The unique equilibrium point of system* (1.5) *is a global attractor.*

Proof. Let $(\mathbf{Z}_n)_{n=-\delta}^{\infty}$ be an arbitrary positive solution to system (1.5), where $\mathbf{Z}_n = (z_n^{(1)}, z_n^{(2)}, \ldots, z_n^{(k)})^T$, then we need to prove that

$$\lim_{n \to \infty} \mathbf{Z}_n = \overline{\mathbf{Z}}. \tag{4.7}$$

Define a transformed sequence $(\widehat{\mathbf{Z}}_n)_{n=-\delta}^{\infty}$ by (2.4) and (2.7), where $\widehat{\mathbf{Z}}_n = (\widehat{z}_n^{(1)}, \widehat{z}_n^{(2)}, \ldots, \widehat{z}_n^{(k)})^T$, then it suffices to confirm that

$$\lim_{n \to \infty} \widehat{\mathbf{Z}}_n = \overline{\mathbf{Z}}. \tag{4.8}$$

Let the sequence $(Q_n)_{n=0}^{\infty}$ be defined by (4.6); then by using Lemma 4.7 we know that the limit of $(Q_n)_{n=0}^{\infty}$ exists, say Q. Note that $Q \geq 1$. By Corollary 4.6 we get $\widehat{z}_j^{(i)} \in [1, Q_j]$, for all $i = 1, 2, \ldots, k$ and $j = 0, 1, 2, \ldots$. Obviously, it suffices to show that $Q = 1$.

Assume that $Q > 1$; then by Lemma 4.7 and (4.6), for arbitrary $\varepsilon > 0$, there exists a sufficiently large $N \in \mathbb{N}$ such that $\hat{z}_N^{(\beta)} \in [Q, Q+\varepsilon]$, for some $\beta \in \{1, 2, \ldots, k\}$ and

$$\hat{z}_n^{(i)} \in [1, Q+\varepsilon], \quad n \geq N-\delta,\ i = 1, 2, \ldots, k. \tag{4.9}$$

Employing Lemma 4.1 three times and (4.9), we get

$$\begin{aligned} Q \leq \hat{z}_N^{(\beta)} &= \frac{1}{\mathcal{M}\left(\hat{z}_{N-v}^{(\beta)}, \hat{z}_{N-p}^{(\theta(\beta+1))}, \hat{z}_{N-q}^{(\theta(\beta+2))}\right)} \leq \frac{1}{\mathcal{M}\left(Q+\varepsilon, \hat{z}_{N-p}^{(\theta(\beta+1))}, \hat{z}_{N-q}^{(\theta(\beta+2))}\right)} \\ &\leq \frac{1}{\mathcal{M}\left(Q+\varepsilon, Q+\varepsilon, \hat{z}_{N-q}^{(\theta(\beta+2))}\right)} \\ &\leq \frac{1}{\mathcal{M}(Q+\varepsilon, Q+\varepsilon, Q+\varepsilon)} = \frac{3(Q+\varepsilon)^r + (Q+\varepsilon)^{3r}}{1+3(Q+\varepsilon)^{2r}}. \end{aligned} \tag{4.10}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$Q \leq \lim_{\varepsilon \to 0^+} \frac{1}{\mathcal{M}(Q+\varepsilon, Q+\varepsilon, Q+\varepsilon)} = \frac{1}{\mathcal{M}(Q, Q, Q)} = \frac{3Q^r + Q^{3r}}{1+3Q^{2r}}. \tag{4.11}$$

Since $Q > 1$; then by Lemma 2.2 in [3], we have that

$$\frac{3Q^r + Q^{3r}}{1+3Q^{2r}} < Q \tag{4.12}$$

which contradicts (4.11). Therefore $Q = 1$. The proof is complete. □

Theorem 4.9. *The unique positive equilibrium point of system* (1.4) *is a global attractor.*

Proof. The proof is similar to that of Theorem 4.8 and hence is omitted. □

5. Conclusions

Following the proofs in this paper line by line, we can also similarly confirm the following two results.

Remark 5.1. Every positive solution to the difference equation system

$$z_n^{(i)} = \frac{1 + \left(z_{n-l}^{(\theta(i-1))}\right)^r \left(z_{n-m}^{(i)}\right)^r}{\left(z_{n-l}^{(\theta(i-1))}\right)^r + \left(z_{n-m}^{(i)}\right)^r}, \quad i = 1, 2, \ldots, k,\ n \in \mathbb{N}_0, \tag{5.1}$$

where $r \in (0, 1],\ k, l, m \in \mathbb{N},\ k \geq 2$ and $\theta(0) = k$, converges to the unique positive equilibrium point $\overline{\mathbf{Z}} = (\underbrace{1, 1, \ldots, 1}_{k})^T$.

Remark 5.2. Every positive solution to the difference equation system

$$z_n^{(i)} = \frac{\left(z_{n-p}^{(\theta(i-1))}\right)^r + \left(z_{n-q}^{(i)}\right)^r}{1 + \left(z_{n-p}^{(\theta(i-1))}\right)^r \left(z_{n-q}^{(i)}\right)^r}, \quad i = 1, 2, \ldots, k, \ n \in \mathbb{N}_0, \tag{5.2}$$

where $r \in (0, 1]$, $k, p, q \in \mathbb{N}$ and $k \geq 2$, converges to the unique positive equilibrium point $\overline{\mathbf{Z}} = (\underbrace{1, 1, \ldots, 1}_{k})^T$.

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