Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 375014, 5 pages
doi:10.1155/2010/375014

Research Article

On the S-Invariance Property for S-Flows

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Received 3 August 2010; Revised 20 October 2010; Accepted 8 November 2010

Academic Editor: Allan C Peterson

We define an equivalence relation on a topological space which is acted by topological monoid S as a transformation semigroup. Then, we give some results about the S-invariant classes for this relation. We also provide a condition for the existence of relative S-invariant classes.

1. Introduction

The invariance theory is one of the principal concepts in the topological dynamics system, see [1, 2]. In [3], Colonius and Kliemann introduced the concept of a control set which is relatively invariant with respect to a subset of the phase space of the control system. From a more general point of view, the theory of control sets for semigroup actions was developed by San Martin and Tonelli in [4].

In this paper, we define an equivalence relation on a topological space which is acted by topological monoid S as a transformation semigroup. Then, we provide the necessary and sufficient conditions for the equivalence classes to be S-invariant classes which correspond with the control sets for control systems. Then, we study the S-invariant classes for this relation in X, in particular, and we provide the conditions for the existence and uniqueness of S-invariant classes.

Throughout this paper, $\mathrm{cl}(A)$ will denote the closure set of a set A, and $\mathrm{int}(A)$ will denote the interior set of A and all topological spaces involved Hausdorff.

Definition 1.1 (see [2]). Let S be a monoid with the identity element e and also a topological space. Then, S will be called a topological monoid if the multiplication operation of: $(s, t) \to st$ is continuous mapping from $S \times S$ to S.

Definition 1.2 (see [4]). Let S be a topological monoid and X a topological space. We say that *S acts on X as a transformation semigroup* if there is a continuous map $a : S \times X \to X$ between the product space $S \times X$ and X satisfying

$$a(st, x) = a(s, a(t, x)), \quad \forall s,\, t \in S,\, x \in X, \tag{1.1}$$

we further require that $a(e, x) = x$ for all $x \in X$. The triple (S, X, a) is called an S-flow; $s\bar{a}x$ will denote $a(s, x)$. In particular, an S-flow (S, X, a) is called *S-phase flow* if S is a compact space.

The *orbit* of $x \in X$ under S is the set $O_a(x) = \{s\bar{a}x : s \in S\}$. For a subset M of X, $S(M)$ denotes the set $\{s\bar{a}m : s \in S,\, m \in M\}$. And a subset M is called an *S-invariant* set if $M \neq \emptyset$ and $S(M) \subset M$. A *control* set for S on X is a subset C of X which satisfies

(1) $\mathrm{int}(C) \neq \emptyset$,

(2) for all $x \in C$, $C \subset \mathrm{cl}(O_a(x))$,

(3) C is a maximal with these properties.

Then, we say that a subset $M \subset X$, satisfies the *no-return condition* if $y \in \mathrm{cl}(O_a(x))$ for some $x \in M$ and $\mathrm{cl}(O_a(y)) \cap M \neq \emptyset$, then $y \in M$.

Lemma 1.3 (see [5, Zorn's Lemma]). *If each chain in a partially ordered set has an upper bound, then there is a maximal element of the set.*

2. S-Invariant Classes

Let (S, X, a) be an S-flow. From the action on X, we can define the relation $\sim$ on X by

$$x \sim y \quad \text{if } x \in O_a(y),\, y \in O_a(x),\, x,\, y \in X. \tag{2.1}$$

It is clear that the relation $\sim$ is an equivalence relation, and $[X]$ will denote the set of all equivalence classes induced by $\sim$ on X. We observe that $[x] \subset O_a(x)$ for all $x \in X$, and if $y \in O_a(x)$, then $O_a(y) \subset O_a(x)$ for all $x, y \in X$.

The following theorem shows that an equivalence class with nonempty interior set is a control set for S on X.

Theorem 2.1. *Let (S, X, a) be an S-phase flow. A class $[x] \in [X]$ with $\mathrm{int}_X([x]) \neq \emptyset$ is a control set for S on X.*

Proof. It is clear that $[x] \subset O_a(x) \subset O_a(y) \subset \mathrm{cl}(O_a(y))$ for all $y \in [x]$. Let C be a subset of X satisfying the property

$$C \subset \mathrm{cl}(O_a(z)), \quad \forall z \in C,\, [x] \subset C. \tag{2.2}$$

Now if $\omega \in C$ then $\omega \in \mathrm{cl}(O_a(z))$ for all $z \in C$. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $O_a(x)$ is a closed subset of X for all

$x \in X$ (i.e., $\mathrm{cl}(O_a(x)) = O_a(x)$ for all $x \in X$). Then, $\omega \in O_a(z)$ for all $z \in C$. Since $x \in C$, then $\omega \in O_a(x)$. On the other hand, since $x \in [x] \subset C \subset O_a(\omega)$, then $\omega \in [x]$. Hence, $C = [x]$. □

In the following lemma, we give necessary and sufficient conditions for the equivalence classes to be S-invariant classes.

Lemma 2.2. *Let (S, X, a) be an S-flow. A class $[x] \in [X]$ is an S-invariant class if and only if $[x] = O_a(x)$.*

Proof. Suppose that $[x] \in [X]$ is an S-invariant and let $y \in O_a(x)$, then $y = s\overline{a}x$ for some $s \in S$. Since $x \in [x]$, then $y \in S([x]) \subset [x]$. Hence, $O_a(x) \subset [x]$, and we have $[x] \subset O_a(x)$. Therefore, $[x] = O_a(x)$.

Conversely, let $[x] = O_a(x)$ and $y \in S([x])$, then $y = s\overline{a}z$ for some $s \in S$, $z \in [x]$. Hence, $z \in O_a(x)$. Take $z = s'\overline{a}x$ for some $s' \in S$. Hence

$$y = s\overline{a}z = s\overline{a}(s'\overline{a}x) = ss'\overline{a}x \in O_a(x) = [x]. \tag{2.3}$$

Therefore, $[x]$ is an S-invariant class. □

Theorem 2.3. *Let (S, X, a) be an S-phase flow. Then, for all $x \in X$, there exists an S-invariant class $[y] \subset O_a(x)$.*

Proof. For $x \in X$, consider the family of subsets

$$E_x = \{z : O_a(z) \subset O_a(x)\}. \tag{2.4}$$

We can define the relation $\preceq$ on E_x by

$$x_1 \preceq x_2, \quad \text{if } O_a(x_2) \subset O_a(x_1) \quad \text{for } x_1, x_2 \in E_x. \tag{2.5}$$

Then, it is clear that the family E_x with $\preceq$ is a partially order set. Let $\{z_i : i \in \wedge\}$ be a linearly ordered subset of E_x, where $\wedge$ is an index set. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $O_a(x)$ is a compact closed subset of X for all $x \in X$. Hence we have a chain $\{O_a(z_i) : i \in \wedge\}$ of closed subsets of a compact $O_a(x)$. Hence the intersection

$$\bigcap_{i \in \wedge} O_a(z_i) \neq \emptyset. \tag{2.6}$$

Take $r \in O_a(z_i)$ for all $i \in \wedge$. Then, $O_a(r) \subset O_a(z_i)$ for all $i \in \wedge$, implies that $O_a(r)$ is a lower bound of the chain $\{O_a(z_i) : i \in \wedge\}$ (i.e., r is an upper bound of the linearly order subset $\{z_i : i \in \wedge\}$ of E_x). Hence, Zorn's lemma implies that the family E_x has a maximal element, say y. Then, $[y] \subset O_a(y) \subset O_a(x)$.

Now, we show that $[y]$ is an S-invariant. Let $z \in O_a(y)$, then $z \in O_a(z) \subset O_a(x)$ and $y \preceq z$, but by the maximality of y, we get that $z \preceq y$, this implies $y \in O_a(z)$. Hence, $z \in [y]$ (i.e., $O_a(y) \subset [y]$) and we have that $[y] \subset O_a(y)$. Then, by Lemma 2.2, $[y]$ is an S-invariant class. □

Now, we propose an open problem that whether S-invariant class is unique?

Theorem 2.4. *Let (S, X, a) be an S-phase flow. Every $[x] \in [X]$ satisfies the no-return condition for all $x \in X$.*

Proof. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $O_a(x)$ is a compact closed subset of X for all $x \in X$ (i.e., $\mathrm{cl}(O_a(x)) = O_a(x)$ for all $x \in X$). Let $z \in O_a(y)$ for some $y \in [x]$ and $O_a(z) \cap [x] \neq \emptyset$. Take $\omega \in O_a(z)$ and $\omega \in [x]$. Hence,

$$x \in O_a(x) \subset O_a(\omega) \subset O_a(z). \tag{2.7}$$

On the other hand, $z \in O_a(y)$ for some $y \in [x]$, we have

$$z \in O_a(z) \subset O_a(y) \subset O_a(x). \tag{2.8}$$

Hence, $z \in [x]$. □

The next theorem states that if M has the no-return condition, then any class $[x]$ is entirely contained in M or M^c. Further M is also an S-invariant if $[x]$ an S-invariant class for all $x \in M$.

Theorem 2.5. *Let (S, X, a) be S-phase flow and M be a subset of X has no-return condition. Then, M is an S-invariant set if $[x]$ is an S-invariant class for all $x \in M$.*

Proof. It is clear that $M \subset \bigcup_{x\in M}[x]$ because $x \in [x]$. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $O_a(x)$ is a compact closed subset of X for all $x \in X$ (i.e., $\mathrm{cl}(O_a(x)) = O_a(x)$ for all $x \in X$). Let $y \in \bigcup_{x\in M}[x]$, then $y \in [x]$ for some $x \in M$. Hence, $[x] = [y]$ (i.e., $x \in O_a(y)$ and $y \in O_a(x)$). Since $x \in M$, then $O_a(y) \cap M \neq \emptyset$. By the no-return condition, we have that $y \in M$. Hence,

$$M = \bigcup_{x\in M}[x]. \tag{2.9}$$

Now, we show that M is an S-invariant set. Let $y \in S(M)$. Then, $y = s\bar{a}x$ for some $x \in M$. Hence, $y \in O_a(x)$. Since $[x]$ is an S-invariant class then by Lemma 2.2, $[x] = O_a(x)$ and by (2.9), we get that $y \in [x] \subset M$. Hence, M is an S-invariant. □

Acknowledgments

The authors wish to express their sincere gratitude to anonyms referees for their very helpful comments and suggestions which improved the paper. The authors would also acknowledge that this research was partially supported by the University Putra Malaysia under the Research University Grant Scheme (RUGS) 05-01-09-0720RU and Fundamental Research Grant Scheme 01-11-09-723FR.

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