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Research Article

The Study of Triple Integral Equations with Generalized Legendre Functions

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A method is developed for solutions of two sets of triple integral equations involving associated Legendre functions of imaginary arguments. The solution of each set of triple integral equations involving associated Legendre functions is reduced to a Fredholm integral equation of the second kind which can be solved numerically.

1. Introduction

Dual integral equations involving Legendre functions have been solved by Babloian [1]. He applied these equations to problems of potential theory and to a torsion problem. Later on Pathak [2] and Mandal [3] who considered dual integral equations involving generalized Legendre functions which have more general solution than the ones considered by Babloian [1]. Recently, Singh et al. [4] considered dual integral equations involving generalized Legendre functions, and their results are more general than those in [1–3].

In the analysis of mixed boundary value problems, we often encounter triple integral equations. Triple integral equations involving Legendre functions have been studied by Srivastava [5]. Triple integral equations involving Bessel functions have also been considered by Cooke [6–9], Tranter [10], Love and Clements [11], Srivastava [12], and most of these authors reduced the solution into a solution of Fredholm integral equation of the second kind. The relevant references for dual and triple integral equations are given in the book of Sneddon [13].

In this paper, a method is developed for solutions of two sets of triple integral equations involving generalized Legendre functions in Sections 3 and 4. Each set of triple integral equations is reduced to a Fredholm integral equation of the second kind which may be solved numerically. The aim of this paper is to find a more general solution for the type of integral equations given in [1–5] and to develop an easier method for solving triple integral equations in general.

2. Integral involved generalized Legendre functions and some useful results

We first summarize some known results needed in the paper.

We find from [14, equation (21), page 330] that

$$\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{1}{2}-\mu\right)\left[\sinh(\alpha c)\right]^{-\mu}\int_0^\infty P^{\mu}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right]\cos(\tau x)d\tau = c\left[\cosh(\alpha c)-\cosh(xc)\right]^{-\mu-1/2}H(\alpha-x), \tag{2.1}$$

where $\mu < 1/2$ and from [4], we obtain

$$\sqrt{2}\pi^{-3/2}\Gamma\left(\frac{1}{2}+\mu\right)\left[\sinh(\alpha c)\right]^{\mu} \times\int_0^\infty \Gamma\left(\frac{1}{2}-\mu+i\frac{\tau}{c}\right)\Gamma\left(\frac{1}{2}-\mu-i\frac{\tau}{c}\right)\sinh(f\tau)\sin(x\tau)P^{\mu}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right]d\tau = c\left[\cosh(xc)-\cosh(\alpha c)\right]^{\mu-1/2}H(x-\alpha), \tag{2.2}$$

where $\mu > -1/2$ and $H()$ denotes the Heaviside unit function. Furthermore, $c = \pi/f$, $f > 0$ and $P^{\mu}_{-1/2+i(\tau/c)}(\cosh \alpha c)$ is the generalized Legendre function defined in [15, page 370]. From [4, 16], the generalized Mehler-Fock transform is defined by

$$\psi(\cosh(\alpha c)) = \int_0^\infty P^{\mu}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right]F(\tau)d\tau, \tag{2.3}$$

and its inversion formula is

$$F(\tau) = \frac{f\tau}{\pi^2}\sinh(f\tau)\Gamma\left(\frac{1}{2}-\mu+i\frac{\tau}{c}\right)\Gamma\left(\frac{1}{2}-\mu-i\frac{\tau}{c}\right) \times\int_0^\infty P^{\mu}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right]\psi(\cosh(\alpha c))\sinh(\alpha c)d\alpha. \tag{2.4}$$

Equations (2.1) and (2.2) are of form (2.3). From the inversion formula given by (2.4), (2.1), and (2.2), it follows that

$$\frac{\cos(x\tau)}{\tau} = \frac{\sinh(f\tau)\Gamma(1/2-\mu+i(\tau/c))\Gamma(1/2-\mu-i(\tau/c))}{\sqrt{2\pi}\Gamma(1/2-\mu)} \times\int_x^\infty \frac{\left[\sinh(\alpha c)\right]^{1+\mu}P^{\mu}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right]d\alpha}{\left[\cosh(\alpha c)-\cosh(xc)\right]^{\mu+1/2}}, \quad \mu < \frac{1}{2}, \tag{2.5}$$

$$\frac{\sin(x\tau)}{\tau} = \sqrt{\frac{\pi}{2}}\frac{1}{\Gamma(1/2+\mu)}\int_0^x \frac{\left[\sinh(\alpha c)\right]^{1-\mu}P^{\mu}_{-1/2+i(\tau/c)}(\cosh\alpha c)d\alpha}{\left[\cosh(xc)-\cosh(\alpha c)\right]^{1/2-\mu}}, \quad \mu > -\frac{1}{2}. \tag{2.6}$$

The inversion theorem for Fourier cosine transforms and the results (2.1) and (2.2) lead to

$$P^{\mu}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right] = \sqrt{\frac{2}{\pi}}c\frac{\sinh^{\mu}(\alpha c)}{\Gamma(1/2-\mu)}\int_0^{\alpha}\frac{\cos(\tau s)ds}{\left[\cosh(\alpha c)-\cosh(sc)\right]^{\mu+1/2}}, \quad \mu < \frac{1}{2}, \tag{2.7}$$

$$P^{\mu}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right] = \frac{\sqrt{2\pi}c}{\left[\sinh^{\mu}(\alpha c)\right]\Gamma(1/2+\mu)\sin(f\tau)\Gamma(1/2-\mu+i(\tau/c))\Gamma(1/2-\mu-i(\tau/c))} \times\int_{\alpha}^{\infty}\frac{\sin(\tau s)ds}{\left[\cosh(sc)-\cosh(\alpha c)\right]^{1/2-\mu}}, \quad \mu > -\frac{1}{2}. \tag{2.8}$$

If $h(t)$ is monotonically increasing and differentiable for $a < t < b$ and $h'(t) \neq 0$ in this interval, then the solutions of the equations

$$\int_a^t \frac{f(x)dx}{\left[h(t)-h(x)\right]^{\alpha}} = g(t), \quad a < t < b,\ 0 < \alpha < 1, \tag{2.9}$$

$$\int_t^b \frac{f(x)dx}{\left[h(x)-h(t)\right]^{\alpha}} = g(t), \quad a < t < b,\ 0 < \alpha < 1, \tag{2.10}$$

are given by Sneddon [13] as

$$f(x) = \frac{\sin(\pi\alpha)}{\pi}\frac{d}{dx}\int_a^x \frac{h'(t)g(t)dt}{\left[h(x)-h(t)\right]^{1-\alpha}}, \quad a < x < b, \tag{2.11}$$

$$f(x) = \frac{-\sin(\pi\alpha)}{\pi}\frac{d}{dx}\int_x^b \frac{h'(t)g(t)dt}{\left[h(t)-h(x)\right]^{1-\alpha}}, \quad a < x < b, \tag{2.12}$$

respectively, where the prime denotes the derivative with respect to t.

3. Triple integral equations with generalized Legendre functions: set I

In this section, we will find solution of the following triple integral equations:

$$\int_0^{\infty}\tau A(\tau)\sinh(\tau f)\Gamma\left(\frac{1}{2}-\mu_1+i\frac{\tau}{c}\right)\Gamma\left(\frac{1}{2}-\mu_1-i\frac{\tau}{c}\right)P^{\mu_1}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right]d\tau = 0, \quad 0 < \alpha < a, \tag{3.1}$$

$$\int_0^{\infty}A(\tau)P^{\mu_2}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right]d\tau = f(\alpha), \quad a < \alpha < b, \tag{3.2}$$

$$\int_0^{\infty}\tau A(\tau)\sinh(\tau f)\Gamma\left(\frac{1}{2}-\mu_3+i\frac{\tau}{c}\right)\Gamma\left(\frac{1}{2}-\mu_3-i\frac{\tau}{c}\right)P^{\mu_3}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right]d\tau = 0, \quad b < \alpha < \infty, \tag{3.3}$$

where $A(\tau)$ is an unknown function to be determined, $f(\alpha)$ is a known function, and $P^{\mu}_{-1/2+i(\tau/c)}[\cosh(\alpha c)]$ is the generalized Legendre function defined in Section 2 and $-1/2 < \mu_1 < 1/2$, $-1/2 < \mu_2 < 1/2$, $\mu_3 > -1/2$.

The trial solution of (3.1), (3.2), and (3.3) can be written as

$$A(\tau) = \int_0^b \psi(t)\cos(\tau t)dt, \tag{3.4}$$

where $\psi(t)$ is an unknown function to be determined. On integrating (3.4) by parts, we get

$$A(\tau) = \frac{\psi(b)\sin(\tau b)}{\tau} - \frac{1}{\tau}\int_0^b \psi'(t)\sin(\tau t)dt, \tag{3.5}$$

where the prime denotes the derivative with respect to t.

Substituting (3.5) into (3.3), interchanging the order of integrations and using (2.2), we find that (3.3) is satisfied identically. Substituting (3.5) into (3.1) and using the integral defined by (2.2), we obtain

$$\frac{\psi(b)}{\left[\cosh(bc) - \cosh(\alpha c)\right]^{1/2-\mu_1}} - \int_\alpha^b \frac{\psi'(t)dt}{\left[\cosh(tc) - \cosh(\alpha c)\right]^{1/2-\mu_1}} = 0, \quad 0 < \alpha < a. \tag{3.6}$$

Equation (3.6) is equivalent to the following integral equation:

$$\frac{d}{d\alpha}\int_\alpha^b \frac{c\sinh(tc)\psi(t)dt}{\left[\cosh(tc) - \cosh(\alpha c)\right]^{1/2-\mu_1}} = 0, \quad 0 < \alpha < a. \tag{3.7}$$

By substituting (3.4) into (3.2), interchanging the order of integrations and using the integral defined by (2.1) we find that

$$c\int_0^\alpha \frac{\psi(t)dt}{\left[\cosh(\alpha c) - \cosh(tc)\right]^{1/2+\mu_2}} = \sqrt{\frac{2}{\pi}}\Gamma\left(\frac{1}{2} - \mu_2\right)\left[\sinh(\alpha c)\right]^{-\mu_2} f(\alpha), \quad a < \alpha < b,\ \mu_2 < \frac{1}{2}. \tag{3.8}$$

For obtaining the solution of the problem, we need to solve two Abel's type integral equations (3.7) and (3.8).

We assume that

$$\frac{d}{d\alpha}\int_\alpha^b \frac{c\sinh(tc)\psi(t)dt}{\left[\cosh(tc) - \cosh(\alpha c)\right]^{1/2-\mu_1}} = \phi(\alpha), \quad a < \alpha < b. \tag{3.9}$$

The above equation is of the same form as (3.7) and defined in a different region. Equation (3.9) is of form (2.12). Hence, the solution of the integral equation (3.9) can be written as

$$\psi(t) = -\frac{\cos(\pi\mu_1)}{\pi}\int_t^b \frac{\phi(\alpha)d\alpha}{\left[\cosh(c\alpha) - \cosh(tc)\right]^{1/2+\mu_1}}, \quad -\frac{1}{2} < \mu_1 < \frac{1}{2},\ a < t < b. \tag{3.10}$$

The solution of Abel's type integral equations (2.11) together with (3.7) and (3.9) leads to

$$\psi(t) = -\frac{\cos(\pi\mu_1)}{\pi}\int_a^b \frac{\phi(\alpha)d\alpha}{\left[\cosh(c\alpha) - \cosh(tc)\right]^{1/2+\mu_1}}, \quad -\frac{1}{2} < \mu_1 < \frac{1}{2},\ 0 < t < a. \tag{3.11}$$

Equations (3.10) and (3.11) mean that (3.7) is satisfied identically. Equation (3.8) can be rewritten in the form

$$\begin{aligned}&\int_0^a \frac{\psi(t)dt}{\left[\cosh(\alpha c) - \cosh(tc)\right]^{1/2+\mu_2}} + \int_a^\alpha \frac{\psi(t)dt}{\left[\cosh(\alpha c) - \cosh(tc)\right]^{1/2+\mu_2}}\\ &\quad = \frac{1}{c}\sqrt{\frac{2}{\pi}}\Gamma(1-\mu_2)f(\alpha)\left[\sinh(\alpha c)\right]^{\mu_2}, \quad a < \alpha < b.\end{aligned} \tag{3.12}$$

Substituting the expression for $\psi(t)$ from (3.11) and (3.10) into the first and second integral of (3.12) we obtain

$$\begin{aligned}&\int_a^\alpha \frac{S(t)dt}{\left[\cosh(\alpha c) - \cosh(tc)\right]^{1/2+\mu_2}}\\ &\quad = F(\alpha) - \int_0^a \frac{dt}{\left[\cosh(\alpha c) - \cosh(tc)\right]^{1/2+\mu_2}}\int_a^b \frac{\phi(u)dt}{\left[\cosh(cu) - \cosh(tc)\right]^{1/2+\mu_2}}, \quad a < t < b,\end{aligned} \tag{3.13}$$

where

$$S(t) = \int_t^b \frac{\phi(u)dt}{\left[\cosh(cu) - \cosh(tc)\right]^{1/2+\mu_1}}, \tag{3.14}$$

$$F(\alpha) = \frac{-\sqrt{2\pi}\Gamma(1-\mu_2)f(\alpha)\left[\sinh(\alpha c)\right]^{-\mu_2}}{c\cos(\mu_1\pi)}. \tag{3.15}$$

Assuming that the right-hand side of (3.13) is a known function of α it has the form of (2.9), whose solution is given by

$$S(t) = \frac{\cos(\pi\mu_2)}{\pi}\frac{d}{dt}\int_a^t \frac{c\sinh(c\alpha)F(\alpha)d\alpha}{\left[\cosh(ct) - \cosh(c\alpha)\right]^{1/2-\mu_2}} - I(t), \quad a < t < b,\ -\frac{1}{2} < \mu_2 < \frac{1}{2}, \tag{3.16}$$

where

$$I(t)=\frac{\cos(\pi\mu_2)}{\pi}\frac{d}{dt}\int_a^t\frac{c\sinh(c\alpha)d\alpha}{[\cosh(ct)-\cosh(c\alpha)]^{1/2-\mu_2}}\int_0^a\frac{dp}{(\cosh(c\alpha)-\cosh(cp))^{1/2+\mu_2}}\times\int_a^b\frac{\phi(u)du}{(\cosh(cu)-\cosh(cp))^{1/2+\mu_2}},\quad a<t<b,\ -\frac{1}{2}<\mu_2<\frac{1}{2}. \tag{3.17}$$

From the integral

$$\frac{d}{dt}\int_a^t\frac{c\sinh(c\alpha)d\alpha}{[\cosh(ct)-\cosh(c\alpha)]^{1/2-\mu_2}[\cosh(c\alpha)-\cosh(cp)]^{1/2+\mu_2}}=\frac{c\sinh(ct)}{[\cosh(ct)-\cosh(cp)]}\frac{[\cosh(ca)-\cosh(cp)]^{1/2-\mu_2}}{[\cosh(ct)-\cosh(ca)]^{1/2-\mu_2}},\quad p<a<t,\ -\frac{1}{2}<\mu_2<\frac{1}{2}, \tag{3.18}$$

we then obtain

$$I(t)=\frac{c\cos(\mu_2\pi)\sinh(ct)}{\pi[\cosh(ct)-\cosh(ca)]^{1/2-\mu_2}}\int_0^a\frac{(\cosh(ca)-\cosh(cp))^{1/2-\mu_2}dp}{[\cosh(ct)-\cosh(cp)]}\times\int_a^b\frac{\phi(u)du}{[\cosh(cu)-\cosh(cp)]^{1/2+\mu_1}}. \tag{3.19}$$

Equation (3.14) is an Abel-type equation. Hence, its solution is

$$\phi(u)=-\frac{\cos(\mu_1\pi)}{\pi}\frac{d}{du}\int_u^b\frac{c\sinh(cv)S(v)dv}{[\cosh(vc)-\cosh(uc)]^{1/2-\mu_1}},\quad a<u<b,\ -\frac{1}{2}<\mu_1<\frac{1}{2}, \tag{3.20}$$

$$R(p)=\int_a^b\frac{\phi(u)du}{[\cosh(cu)-\cosh(cp)]^{1/2+\mu_1}}. \tag{3.21}$$

Substituting the expression for $\phi(u)$ from (3.20) into (3.21), integrating by parts, and finally interchanging the order of integrations in second integral, we arrive at

$$R(p)=\frac{c\cos(\mu_1\pi)}{\pi}\left[\frac{1}{[\cosh(ca)-\cosh(cp)]^{1/2+\mu_1}}\int_a^b\frac{S(v)\sinh(cv)dv}{[\cosh(cv)-\cosh(ca)]^{1/2-\mu_1}}-\left(\frac{1}{2}+\mu_1\right)\int_a^b S(v)\sinh(cv)dv\times\int_a^v\frac{c\sinh(cu)du}{[\cosh(cu)-\cosh(cp)]^{3/2+\mu_1}[\cosh(cv)-\cosh(cu)]^{1/2-\mu_1}}\right]. \tag{3.22}$$

The integral

$$
\begin{aligned}
&\int_a^v \frac{c\sinh(cu)du}{\left[\cosh(cu)-\cosh(cp)\right]^{3/2+\mu_1}\left[\cosh(cv)-\cosh(cu)\right]^{1/2-\mu_1}} \\
&= \frac{\left[\cosh(cv)-\cosh(ca)\right]^{1/2+\mu_1}}{(\mu_1+1/2)\left[\cosh(cv)-\cosh(cp)\right]\left[\cosh(ca)-\cosh(cp)\right]} \\
&\qquad p<a<v, \quad -\frac{1}{2}<\mu_1<\frac{1}{2}
\end{aligned}
\tag{3.23}
$$

together with (3.22) leads to

$$
\begin{aligned}
R(p) &= \frac{c\cos(\pi\mu_1)}{\pi}\left(\cosh(ac)-\cosh(pc)\right)^{1/2-\mu_1} \\
&\quad\times\int_a^b \frac{S(\nu)\sinh(c\nu)d\nu}{\left[\cosh(\nu c)-\cosh(pc)\right]\left[\cosh(\nu c)-\cosh(ac)\right]^{1/2-\mu_1}}.
\end{aligned}
\tag{3.24}
$$

From (3.19), (3.21), and (3.24), we obtain

$$
I(t) = \int_a^b S(\nu)K(\nu,t)d\nu, \tag{3.25}
$$

where

$$
\begin{aligned}
K(\nu,t) &= \frac{c^2\cos(\pi\mu_1)\cos(\pi\mu_2)\sinh(ct)\sinh(c\nu)}{\pi^2\left[\cosh(ct)-\cosh(ca)\right]^{1/2-\mu_2}\left[\cosh(c\nu)-\cosh(ca)\right]^{1/2-\mu_1}} \\
&\quad\times\int_0^a \frac{\left(\cosh(ca)-\cosh(cp)\right)^{1-\mu_1-\mu_2}dp}{\left[\cosh(ct)-\cosh(cp)\right]\left[\cosh(c\nu)-\cosh(cp)\right]}.
\end{aligned}
\tag{3.26}
$$

From (3.25), (3.16) can be written as

$$
S(t)+\int_a^b S(\nu)K(\nu,t)d\nu = \frac{\cos(\pi\mu_2)}{\pi}\frac{d}{dt}\int_a^t \frac{c\sinh(c\alpha)F(\alpha)d\alpha}{\left[\cosh(ct)-\cosh(c\alpha)\right]^{1/2-\mu_2}}, \quad a<t<b. \tag{3.27}
$$

Equation (3.27) is a Fredholm integral equation of the second kind with kernel $K(\nu,t)$. The kernel is defined by (3.26). The integral in (3.26) cannot be solved analytically, but for particular values of μ_1 and μ_2 the values of $K(\nu,t)$ can be found numerically. Hence, the numerical solution of Fredholm integral equation (3.27) can be obtained for particular value of $f(\alpha)$, μ_1, and μ_2 to find numerical values of $S(t)$. Making use of (3.20), (3.11), and (3.10), the numerical results for $\psi(t)$ can be obtained. Finally, making use of (3.4) the numerical results for $A(\tau)$ can be obtained.

4. Triple integral equations with generalized Legendre functions: set II

In this section, we will find the solution of the following triple integral equations:

$$\int_0^\infty \tau A(\tau) P^{\mu_1}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right] d\tau = 0, \quad 0 < \alpha < a, \tag{4.1}$$

$$\int_0^\infty \sinh(\tau f)\Gamma\left(\frac{1}{2} - \mu_2 + i\frac{\tau}{c}\right)\Gamma\left(\frac{1}{2} - \mu_2 - i\frac{\tau}{c}\right) A(\tau) P^{\mu_2}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right] d\tau = f(\alpha), \quad a < \alpha < b, \tag{4.2}$$

$$\int_0^\infty \tau A(\tau) P^{\mu_3}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right] d\tau = 0, \quad b < \alpha, \tag{4.3}$$

where $\mu_1 > -1/2$, $-1/2 < \mu_2 < 1/2$, $-1/2 < \mu_3 < 1/2$.

We assume that

$$\int_0^\infty \tau A(\tau) P^{\mu_3}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right] d\tau = M(\alpha), \quad 0 < \alpha < b. \tag{4.4}$$

The inversion formula for generalized Mehler-Fock transforms (2.4) together with (4.3) and (4.4) implies that

$$\begin{aligned} A(\tau) = {} & \frac{f}{\pi^2}\sinh(f\tau)\Gamma\left(\frac{1}{2} - \mu_3 + i\frac{\tau}{c}\right)\Gamma\left(\frac{1}{2} - \mu_3 - i\frac{\tau}{c}\right) \\ & \times \int_0^b \sinh(uc) P^{\mu_3}_{-1/2+i(\tau/c)}\left[\cosh(uc)\right] M(u)\, du. \end{aligned} \tag{4.5}$$

Multiplying (4.1) by $[\sinh(\alpha c)]^{1-\mu_1}/[\cosh(xc) - \cosh(\alpha c)]^{1/2-\mu_1}$, integrating both sides from 0 to x and with respect to α, and then using (2.6) we obtain

$$\int_0^\infty A(\tau)\sin(x\tau)\, d\tau = 0, \quad 0 < x < a. \tag{4.6}$$

Substituting the value of $A(\tau)$ from (4.5) into (4.6), interchanging the order of integrations, and using the integral (2.2), we get

$$\int_0^x \frac{\sinh(uc) M(u)\, du}{\left[\cosh(xc) - \cosh(uc)\right]^{1/2-\mu_3}} = 0, \quad \mu_3 > -\frac{1}{2},\ 0 < x < a. \tag{4.7}$$

Substituting the value of $A(\tau)$ from (4.5) into (4.2) and interchanging the order of integrations we arrive at

$$\int_0^b \sinh(uc) M(u) K_2(u, \alpha)\, du = f(\alpha), \quad a < \alpha < b, \tag{4.8}$$

where

$$K_2(u,\alpha) = \int_0^\infty \frac{f}{\pi^2}\Gamma\left(\frac{1}{2}-\mu_2-i\frac{\tau}{c}\right)\Gamma\left(\frac{1}{2}-\mu_2+i\frac{\tau}{c}\right)\Gamma\left(\frac{1}{2}-\mu_3+i\frac{\tau}{c}\right)\Gamma\left(\frac{1}{2}-\mu_3-i\frac{\tau}{c}\right) \times \sinh^2(f\tau) P^{\mu_3}_{-1/2+i(\tau/c)}\left[\cosh(uc)\right] P^{\mu_2}_{-1/2+i(\tau/c)}\left[\cosh(\alpha c)\right] d\tau, \tag{4.9}$$

and then (2.8) and (2.2) imply that

$$K_2(u,\alpha) = \frac{c\pi}{\Gamma(1/2+\mu_2)\Gamma(1/2+\mu_3)\left[\sinh(\alpha c)\right]^{\mu_2}\left[\sinh(uc)\right]^{\mu_3}} \times \int_{\max(\alpha,u)}^\infty \frac{ds}{\left[\cosh(sc)-\cosh(\alpha c)\right]^{1/2-\mu_2}\left[\cosh(sc)-\cosh(uc)\right]^{1/2-\mu_3}}, \quad \mu_3 > -\frac{1}{2}, \quad \mu_2 > -\frac{1}{2}. \tag{4.10}$$

Equation (4.7) is an Abel-type equation and has the form (2.9). Hence, the solution of (4.7) is

$$M(u) = 0, \quad 0 < u < a. \tag{4.11}$$

Using (4.10) and (2.5), (4.8) can be written in the form

$$\int_a^b \left[\sinh(uc)\right]^{1-\mu_3} M(u)\,du \int_{\max(\alpha,u)}^\infty \frac{ds}{\left[\cosh(sc)-\cosh(\alpha c)\right]^{1/2-\mu_2}\left[\cosh(sc)-\cosh(uc)\right]^{1/2-\mu_3}} = \frac{\Gamma(1/2+\mu_2)\Gamma(1/2+\mu_2)\left[\sinh(\alpha c)\right]^{\mu_3}}{c\pi} f(\alpha) = F_1(\alpha), \quad \text{say, } a < \alpha < b. \tag{4.12}$$

Using the formula

$$\int_a^b du \int_{\max(\alpha,u)}^\infty ds = \int_\alpha^b ds \int_a^s du + \int_b^\infty ds \int_a^b du, \tag{4.13}$$

we can write (4.12) in the form

$$\int_\alpha^b \frac{S_1(s)\,ds}{\left[\cosh(sc)-\cosh(\alpha c)\right]^{1/2-\mu_2}} = F_1(\alpha) - \int_b^\infty \frac{ds}{\left[\cosh(sc)-\cosh(\alpha c)\right]^{1/2-\mu_2}} \times \int_a^b \frac{M(u)\left[\sinh(uc)\right]^{1-\mu_3} du}{\left[\cosh(sc)-\cosh(uc)\right]^{1/2-\mu_2}}, \quad a < \alpha < b, \tag{4.14}$$

where

$$S_1(s) = \int_a^s \frac{M(u)\left[\sinh(uc)\right]^{1-\mu_3} du}{\left[\cosh(sc) - \cosh(\alpha c)\right]^{1/2-\mu_2}}, \quad a < s < b. \tag{4.15}$$

Assuming that the right-hand side of (4.14) is known function equation and (4.14) has the form of (2.10), hence the solution of (4.14) can be written as

$$S_1(s) = -\frac{c}{\pi} \cos(\pi\mu_2) \frac{d}{ds} \int_s^b \frac{F_1(\alpha)\sinh(\alpha c) d\alpha}{\left[\cosh(\alpha c) - \cosh(sc)\right]^{1/2+\mu_2}} + I_1(s), \quad a < s < b, \ -\frac{1}{2} < \mu_2 < \frac{1}{2}, \tag{4.16}$$

where

$$\begin{aligned} I_1(s) = {} & \frac{c}{\pi} \cos(\pi\mu_2) \frac{d}{ds} \int_s^b \frac{\sinh(\alpha c) d\alpha}{\left[\cosh(\alpha c) - \cosh(sc)\right]^{1/2+\mu_2}} \\ & \times \int_b^\infty \frac{dp}{\left[\cosh(pc) - \cosh(\alpha c)\right]^{1/2-\mu_2}} \int_a^b \frac{M(u)\left[\sinh(cu)\right]^{1-\mu_3} d\alpha}{\left[\cosh(pc) - \cosh(uc)\right]^{1/2-\mu_3}}, \quad a < s < b. \end{aligned} \tag{4.17}$$

Equation (4.17) is simplified to

$$\begin{aligned} I_1(s) = {} & \frac{c \cos(\pi\mu_2)}{\pi} \frac{\sinh(sc)}{\left[\cosh(bc) - \cosh(sc)\right]^{1/2+\mu_2}} \int_b^\infty \frac{\left[\cosh(cp) - \cosh(bc)\right]^{1/2+\mu_2} dp}{\left[\cosh(sc) - \cosh(cp)\right]} \\ & \times \int_a^b \frac{M(u)\left[\sinh(cu)\right]^{1-\mu_3} du}{\left[\cosh(cp) - \cosh(cu)\right]^{1/2-\mu_3}}, \quad a < s < b. \end{aligned} \tag{4.18}$$

Let

$$R_1(p) = \int_a^b \frac{M(u)\left[\sinh(cu)\right]^{1-\mu_3} du}{\left[\cosh(pc) - \cosh(uc)\right]^{1/2-\mu_3}}. \tag{4.19}$$

Equation (4.15) is of the form of (2.9). Hence, its solution is

$$M(u)\left[\sinh(cu)\right]^{1-\mu_3} = \frac{c \cos(\pi\mu_3)}{\pi} \frac{d}{du} \int_a^u \frac{S_1(s)\sinh(sc) ds}{\left[\cosh(uc) - \cosh(sc)\right]^{1/2+\mu_3}}, \quad a < u < b. \tag{4.20}$$

Substituting the expression for $M(u)$ from (4.20) into (4.19) and integrating by parts and then using the following integral:

$$\int_s^b \frac{c\sinh(uc)du}{(\cosh(pc)-\cosh(uc))^{3/2-\mu_3}(\cosh(uc)-\cosh(sc))^{1/2+\mu_3}} = \frac{-[\cosh(bc)-\cosh(cs)]^{1/2-\mu_3}}{(1/2-\mu_3)[\cosh(cs)-\cosh(cp)][\cosh(cp)-\cosh(cb)]^{1/2-\mu_3}}, \quad s<b<p, \quad -\frac{1}{2}<\mu_2<\frac{1}{2}, \tag{4.21}$$

we find that

$$R_1(p) = \frac{-c\cos(\mu_3\pi)}{\pi}(\cosh(cp)-\cosh(bc))^{1/2+\mu_3} \times \int_a^b \frac{S_1(u)\sinh(cu)du}{[\cosh(cp)-\cosh(uc)][\cosh(bc)-\cosh(cu)]^{1/2+\mu_3}}. \tag{4.22}$$

Making use of (4.18), (4.19), and (4.22), we find that

$$I_1(s) = -\int_a^b S_1(u)K_2(u,s)du, \tag{4.23}$$

where

$$K_2(u,s) = \frac{c^2\cos(\pi\mu_2)\cos(\pi\mu_3)\sinh(sc)\sinh(uc)}{\pi^2[\cosh(bc)-\cosh(sc)]^{1/2+\mu_2}[\cosh(bc)-\cosh(cu)]^{1/2+\mu_3}} \times \int_b^\infty \frac{[\cosh(cp)-\cosh(bc)]^{1+\mu_2+\mu_3}dp}{[\cosh(sc)-\cosh(cp)][\cosh(cp)-\cosh(cu)]}. \tag{4.24}$$

Using (4.17) and (4.23), (4.16) can be written in the form

$$S_1(s) + \int_a^b S_1(u)K_2(u,s)du = \frac{-c}{\pi}\cos(\pi\mu_2)\frac{d}{ds}\int_s^b \frac{F_1(\alpha)\sinh(\alpha c)d\alpha}{[\cosh(\alpha c)-\cosh(sc)]^{1/2+\mu_2}}, \quad a<s<b. \tag{4.25}$$

Equation (4.25) is a Fredholm integral equation of the second kind with kernel defined by (4.24). The Fredholm integral equation (4.25) may be solved to find numerical values of $S_1(s)$ for particular values of $f(\alpha)$. And hence from (4.20) and (4.5), the numerical values for $A(\tau)$ can be obtained for particular values of $f(\alpha)$, μ_2, and μ_3.

5. Conclusions

The solution of the two sets of triple integral equations involving generalized Legendre functions is reduced to the solution of Fredholm integral equations of the second kind which can be solved numerically.

References

[1] A. A. Babloian, "Solutions of certain dual integral equations," *Prikladnaya Matematika i Mekhanika*, vol. 28, no. 6, pp. 1015–1023, 1964, English translation in *Journal of Applied Mathematics and Mechanics*, vol. 28, no. 6, pp. 1227–1236, 1964.

[2] R. S. Pathak, "On a class of dual integral equations," *Koninklijke Nederlandse Akademie van Wetenschappen*, vol. 40, no. 4, pp. 491–501, 1978.

[3] B. N. Mandal, "A note on dual integral equations involving associated Legendre function," *International Journal of Mathematics and Mathematical Sciences*, vol. 15, no. 3, pp. 601–604, 1992.

[4] B. M. Singh, J. Rokne, and R. S. Dhaliwal, "The study of dual integral equations with generalized Legendre functions," *Journal of Mathematical Analysis and Applications*, vol. 304, no. 2, pp. 725–733, 2005.

[5] K. N. Srivastava, "On some triple integral equations involving Legendre functions of imaginary argument," *Journal of Maulana Azad College of Technology*, vol. 1, pp. 54–67, 1968.

[6] J. C. Cooke, "Triple integral equations," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 16, no. 2, pp. 193–203, 1963.

[7] J. C. Cooke, "Some further triple integral equation solutions," *Proceedings of the Edinburgh Mathematical Society*, vol. 13, pp. 303–316, 1963.

[8] J. C. Cooke, "The solution of triple integral equations in operational form," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 18, no. 1, pp. 57–72, 1965.

[9] J. C. Cooke, "The solution of triple and quadruple integral equations and Fourier-Bessel series," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 25, no. 2, pp. 247–263, 1972.

[10] C. J. Tranter, "Some triple integral equations," *Proceedings of the Glasgow Mathematical Association*, vol. 4, pp. 200–203, 1960.

[11] E. R. Love and D. L. Clements, "A transformation of Cooke's treatment of some triple integral equations," *Journal of the Australian Mathematical Society. Series B*, vol. 19, no. 3, pp. 259–288, 1976.

[12] N. Srivastava, "On triple integral equations involving Bessel function as kernel," *Journal of Maulana Azad College of Technology*, vol. 21, pp. 39–50, 1988.

[13] I. N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*, North-Holland, Amsterdam, The Netherlands, 1966.

[14] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms. Vol. II*, McGraw Hill, New York, NY, USA, 1954.

[15] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms. Vol. I*, McGraw Hill, New York, NY, USA, 1954.

[16] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Die Grundlehren der Mathematischen Wissenschaften. 52, Springer, New York, NY, USA, 1966.