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Research Article

State Trajectories Analysis for a Class of Tubular Reactor Nonlinear Nonautonomous Models

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The existence and uniqueness of global mild solutions are proven for a class of semilinear nonautonomous evolution equations. Moreover, it is shown that the system, under considerations, has a unique steady state. This analysis uses, essentially, the dissipativity, a subtangential condition, and the positivity of the related C_0-semigroup.

1. Introduction

Several chemical and biochemical processes are typically described by nonlinear coupled partial differential equations "PDE" and hence by distributed parameter models (see [1] and the references within). The source of nonlinearities is essentially the kinetics of the reactions involved in the process. For numerical simulation as well as for control design problems, many authors approximate those distributed parameter systems by lumped parameter models [1–5]. However, an important number of questions remained unsolved. In particular, to study the stability of the tubular reactor, the trajectory must exist on the whole real positive time interval $[0,+\infty[$. In our previous works [6, 7], we have proven the global state trajectories existence for a class of nonlinear systems arising from convection-dispersion-reaction systems, assuming that the inlet concentrations are independent of time. In this paper, we investigate the question in the case where the involved inlet concentrations are functions of time t. The considered class of models correspond to the following chemical reaction:

$$nA + mB \longrightarrow P, \tag{1.1}$$

whose kinetic is given by $r = (-k_1C^mL^n, -k_2C^mL^n)^T$, where C and L are the concentrations of the reactants A and B, respectively, k_1 and k_2 are the kinetic constants and m, n are the order of the reaction to A and B, respectively. More precisely, we study the global existence and uniqueness of the trajectories of the models which describe the evolution of two reactant concentrations C and L:

$$\frac{\partial C}{\partial t} = -\nu\frac{\partial C}{\partial \xi} + D_1\frac{\partial^2 C}{\partial \xi^2} - k_1C^mL^n, \tag{1.2}$$

$$\frac{\partial L}{\partial t} = -\nu\frac{\partial L}{\partial \xi} + D_2\frac{\partial^2 L}{\partial \xi^2} - k_2C^mL^n, \tag{1.3}$$

for $\xi \in]0, l[$ and $t > 0$, with the following boundary and initial conditions:

$$D_1\frac{\partial C}{\partial \xi}(0,t) - \nu C(0,t) + \nu C_{\text{in}}(t) = 0 = D_1\frac{\partial C}{\partial \xi}(l) \quad \forall t > 0 \tag{1.4}$$

$$D_2\frac{\partial L}{\partial \xi}(0,t) - \nu L(0,t) + \nu L_{\text{in}}(t) = 0 = D_2\frac{\partial L}{\partial \xi}(l,t) \quad \forall t > 0, \tag{1.5}$$

$$C(\xi,0) = C_0(\xi), \quad L(\xi,0) = L_0(\xi) \quad \text{for } \xi \in]0,l[. \tag{1.6}$$

Additionally, the existence and uniqueness of the corresponding equilibrium profile will be proven.

In the above equations, D_1, D_2 are the dispersion coefficients, ν is the superficial fluid velocity, t, ξ denote the time and space independent variables, respectively, l is the length of the reactor, m and n are two positive integers, C_{in} and L_{in} are the inlet concentration. For further discussion of parameters, we refer to [3].

Comment 1. (i) The nonlinear models considered in this paper have been studied in a qualitative manner by several authors. In the case, $\nu = 0$, [8] established the asymptotic behavior of solutions for the second-order reaction (i.e., $n = m = 1$). N. Alikakos [9] established global existence and L^∞ bounds of positive solutions, when $m = 1$ and $1 < n < 3/2$. This latter result has been generalized by [10] for the case $m = 1$ and $n > 1$.

In practice, the special cases $m = n = 1, 2, 3$ have been used as an industrial pulp bleaching model, where the two reactants are chlorine dioxide (C) and lignin (L). In particular, [3] studied approximate solutions by using several methods (orthogonal collocation, finite elements, and finite difference methods), when $n = m$ and $D_1 = D_2$. The reader can find another model with $D_1 \neq D_2$ in [11], where the numerical analysis has been done for $m = n = 1$ and $D_2 = 4D_1$, $D_2 = 16D_1$ (see also [12]).

Recently, the existence of global solutions for problems such as (1.2)–(1.6) has been extensively studied in [6, 7] with constant inlet concentrations.

(ii) For technological limitations and economical considerations, the following saturation conditions are usually fulfilled for all $0 \le \xi \le l$ and for all $t \ge 0$:

$$0 \le C \le \overline{C}, \qquad 0 \le L \le \overline{L}, \tag{1.7}$$

$$C_{\text{in}}(t) \le \overline{C}, \qquad L_{\text{in}}(t) \le \overline{L}, \tag{1.8}$$

where $\overline{C}$ and $\overline{L}$ are positive constants.

This paper is organized as follows. In Section 2, we will recall briefly some basic results and preliminary facts from semilinear nonautonomous evolution equations which will be used throughout Section 4. In Section 3, the problem (1.2)–(1.6) is converted through some transformations to a homogeneous form where the semigroup theory applies. In Section 4 we establish the main global existence result for system (1.2)–(1.6). We report the existence and uniqueness of equilibrium profiles results in Section 5. Finally, the main conclusions are outlined in Section 6. The background of our approach can be found in [13–16].

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$, $J = [a, b[$ $(a < b \leq +\infty)$, and let $\{\mathfrak{T}(t);\ t \geq 0\}$ be a linear contraction C_0-semigroup on X generated by $\mathcal{A}$. Let $\mathcal{B}$ be a nonlinear continuous operator form Ω into X, where Ω is a subset of $J \times X$. I and $\mathbb{I}$ denote, respectively, the identity operator of X and the function identically equal to 1.

This section is devoted to investigate sufficient conditions for the existence and uniqueness of global mild solutions to the following abstract Cauchy problem:

$$\begin{gathered} \dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}(t, x(t)), \qquad \tau < t < b, \\ x(\tau) = x_\tau \in \Omega(\tau), \end{gathered} \tag{2.1}$$

where $\Omega(\tau)$ denote the section of Ω at $\tau \in J$, given by $\Omega(t) = \{x \in X; (t, x) \in \Omega\}$. Assume that $\Omega(t) \neq \emptyset$ for all $t \in J$. Moreover, recall that $d(x; \mathfrak{D}) = \inf\{\|x - y\|, y \in \mathfrak{D}\}$, for $x \in X$ and $\mathfrak{D}$ is a subset of X.

The semilinear nonautonomous evolution equations have been treated by a number of authors [14, 15, 17–21]. However, one may find that in most cases Ω is cylindrical, that is, $\Omega = J \times \mathfrak{D}$ [14, 22]. More generally, the cylindrical case of Ω will not be convenient for the study of evolution system satisfying time-dependent constraints, that is, $x(t) \in \Omega(t)$ on J (see our problem in Section 3). A noncylindrical Ω case was studied in [16, 19].

The following result gives sufficient conditions for the existence and uniqueness of global mild solutions to the semilinear equations of type (2.1). It is a particular version of [16, Theorem 8.1], when the nonlinear $\mathcal{B}(t, \cdot)$ is $l_{\mathcal{B}}$-dissipative [16].

Theorem 2.1 (see [16]). *Suppose that the following conditions are fulfilled:*

(i) *Ω is closed from the left, that is, if $(t_n, x_n) \in \Omega$, $t_n \uparrow t$ in J, and $x_n \to x$ in X as $n \to \infty$, then $(t, x) \in \Omega$;*

(ii) *for all $(t, x) \in \Omega$, $\liminf_{h\downarrow 0} (1/h) d(\mathfrak{T}(h)x + h\mathcal{B}(t, x), \Omega(t + h)) = 0$;*

(iii) *$\mathcal{B}$ is continuous on Ω and there exists $l_{\mathcal{B}} \in \mathbb{R}^+$ such that the operator $(\mathcal{B}(t, \cdot) - l_{\mathcal{B}} I)$ is dissipative on $\Omega(t)$ for all $t \in J$.*

If Ω is a connected subset of $J \times X$ such that for all $t \in J$, $\Omega(t) \neq \emptyset$, then, for each $(\tau, x_\tau) \in \Omega$, (2.1) *has a unique mild solution on J.*

Comment 2. It is shown in [16] that the "subtangential condition" (ii) is a necessary condition for the existence of the mild solutions of (2.1). For more details on the conditions of Theorem 2.1, we refer to [16].

In the particular case when $\Omega(t)$ is $\mathfrak{T}(s)$-invariant, that is, $\mathfrak{T}(s)(\Omega(t)) \subset \Omega(t)$ for all $t, s \geq 0$, we have the following lemma.

Lemma 2.2. *Let $\mathcal{B} : \Omega \to X$ be continuous and let Ω be closed from the left. If $\Omega(t)$ is $\mathcal{T}(s)$-invariant for all $t, s \geq 0$, then the following subtangential condition*

$$\liminf_{h\downarrow 0} \frac{1}{h} d\big(x + h\mathcal{B}(t,x); \Omega(t+h)\big) = 0 \quad \forall (t,x) \in \Omega \tag{2.2}$$

implies the condition

$$\liminf_{h\downarrow 0} \frac{1}{h} d\big(\mathcal{T}(h)x + h\mathcal{B}(t,x), \Omega(t+h)\big) = 0, \quad \forall (t,x) \in \Omega. \tag{2.3}$$

Proof. Let $(t,x) \in \Omega$, given $\epsilon > 0$, from condition (2.2) it follows, by [23, Lemma 3] (see also [24, Lemma 1]), that there is $h \in (0,\epsilon]$ and $y \in \Omega(t+h)$ such that $\|y - x - h\mathcal{B}(t,x)\| \leq h\epsilon$. Let now $u = y - x - h\mathcal{B}(t,x)$ and $v = (1/h)u$. We get $\|v\| \leq \epsilon$ such that $y = x + h(B(t,x) + v) \in \Omega(t+h)$. By the invariance properties of $\Omega(t)$, we have $\mathcal{T}(h)y \in \Omega(t+h)$. Consequently,

$$\begin{aligned} d\big(\mathcal{T}(h)x + h\mathcal{B}(t,x); \Omega(t+h)\big) &\leq \big\|\mathcal{T}(h)x + h\mathcal{B}(t,x) - \mathcal{T}(h)y\big\|, \\ &\leq \big\|h\mathcal{B}(t,x) - h\mathcal{T}(h)\mathcal{B}(t,x) - h\mathcal{T}(h)v\big\|, \\ &\leq h\big\|\mathcal{T}(h)\mathcal{B}(t,x) - \mathcal{B}(t,x)\big\| + h\big\|\mathcal{T}(h)v\big\|, \\ &\leq h\big\|\mathcal{T}(h)\mathcal{B}(t,x) - \mathcal{B}(t,x)\big\| + h\epsilon. \end{aligned} \tag{2.4}$$

By using the continuity of C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$, the desired result (2.3) is obtained. □

Theorem 2.1 with Lemma 2.2 obviously imply the following.

Corollary 2.3. *Suppose that the following conditions are fulfilled:*

(i) *Ω is closed from the left, that is, if $(t_n, x_n) \in \Omega$, $t_n \uparrow t$ in J, and $x_n \to x$ in X as $n \to \infty$, then $(t,x) \in \Omega$;*

(ii) *$\Omega(t)$ is $\mathcal{T}(s)$-invariant, for all t, $s \geq 0$;*

(iii) *for all $(t,x) \in \Omega$, $\liminf_{h\downarrow 0} (1/h) d(x + h\mathcal{B}(t,x), \Omega(t+h)) = 0$;*

(iv) *$\mathcal{B}$ is continuous on Ω and there exists $l_{\mathcal{B}} \in \mathbb{R}^+$ such that the operator $(\mathcal{B}(t,\cdot) - l_{\mathcal{B}} I)$ is dissipative on $\Omega(t)$, for all $t \in J$.*

If Ω is a connected subset of $J \times X$ such that for all $t \in J$, $\Omega(t) \neq \emptyset$, then, for each $(\tau, x_\tau) \in \Omega$, (2.1) has a unique mild solution on J.

3. Abstract semigroup formulation

Throughout the sequel, we assume $H = L^2[0,1] \oplus L^2[0,1]$, the Hilbert space with the usual inner product

$$\big\langle (x_1,x_2), (y_1,y_2) \big\rangle = \langle x_1, y_1 \rangle_{L^2} + \langle x_2, y_2 \rangle_{L^2} \tag{3.1}$$

and the induced norm

$$\left\|(x_1, x_2)\right\| = \left(\|x_1\|_{L^2}^2 + \|x_2\|_{L^2}^2\right)^{1/2} \tag{3.2}$$

for all (x_1, x_2) and (y_1, y_2) in H.

Clearly, the Hilbert space H is a real Banach lattice, where for all given $x = (x_1, x_2) \in H,\ y = (y_1, y_2) \in H$,

$$x \leq y \quad \text{iff } x_1(z) \leq y_1(z), \quad x_2(z) \leq y_2(z) \quad \text{for a.e. } z \in [0,1]. \tag{3.3}$$

Recall that for every pair $x, y \in H$, the set

$$[x, y] = \{w \in H :\ x_1 \leq w_1 \leq y_1,\ x_2 \leq w_2 \leq y_2\} = [x_1, y_1] \times [x_2, y_2] \tag{3.4}$$

is called the order interval between x and y. Clearly, $[x, y]$ is nonempty if $x \leq y$ (for more details, see, e.g., [25]). A bounded linear operator $\mathcal{T}$ on H is said to be positive if $0 \leq \mathcal{T}x$ for all $0 \leq x$. Similarly, a family of bounded linear operators $(\mathcal{T}(t))_{t\geq 0}$ of H is said to be a positive C_0-semigroup on H if $\mathcal{T}(t)$ is a C_0-semigroup on H and $\mathcal{T}(t)$ is a positive operator for all $t \geq 0$.

In the following, we will assume that $C_{\text{in}}(t)$ and $L_{\text{in}}(t)$ are positive $\mathcal{C}^1([0, \infty[)$-functions. Let us consider the following state transformation:

$$z = \frac{\xi}{l}, \quad x_1 = C - C_{\text{in}}, \quad x_2 = L - L_{\text{in}}, \quad x_{01} = C_0 - C_{\text{in}}, \quad x_{02} = L_0 - L_{\text{in}}. \tag{3.5}$$

Then, we obtain the new equivalent system for all $z \in]0, 1[$ and $t > 0$:

$$\frac{\partial x_1}{\partial t} = -v\frac{\partial x_1}{\partial z} + d_1\frac{\partial^2 x_1}{\partial z^2} - k_1\left(x_1 + C_{\text{in}}(t)\right)^m\left(x_2 + L_{\text{in}}(t)\right)^n - \dot{C}_{\text{in}}(t), \tag{3.6}$$

$$\frac{\partial x_2}{\partial t} = -v\frac{\partial x_2}{\partial z} + d_2\frac{\partial^2 x_2}{\partial z^2} - k_2\left(x_1 + C_{\text{in}}(t)\right)^m\left(x_2 + L_{\text{in}}(t)\right)^n - \dot{L}_{\text{in}}(t), \tag{3.7}$$

with

$$d_i\frac{\partial x_i}{\partial z}(0, t) - v x_i(0, t) = 0 = d_i\frac{\partial x_i}{\partial z}(1, t) \quad \forall t > 0\ i = 1; 2, \tag{3.8}$$

$$x_i(z, 0) = x_{0i}(z) \quad \text{for } z \in]0, 1[,\ i = 1; 2, \tag{3.9}$$

where

$$d_1 = \frac{D_1}{l^2}, \qquad d_2 = \frac{D_2}{l^2}, \qquad v = \frac{\nu}{l}. \tag{3.10}$$

This PDEs describing the reactor dynamics may be formally written in the abstract form as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\left(t, x(t)\right),\\ x(0) &= x_0 \in \Omega(0),\end{aligned} \tag{3.11}$$

where $\Omega(t)$ denote the section of Ω at $t \in \mathbb{R}^+$, which is given in view of (1.7) by

$$\begin{aligned} \Omega = \Big\{ \big(t, (x_1, x_2)\big)^T \in \mathbb{R}^+ \times H : -C_{\text{in}}(t) \le x_1(z) \le \overline{C} - C_{\text{in}}(t), \\ - L_{\text{in}}(t) \le x_2(z) \le \overline{L} - L_{\text{in}}(t) \ \text{ a.e. } z \in [0,1] \Big\}. \end{aligned} \tag{3.12}$$

The linear operator A is defined by

$$\begin{aligned} D(A) = \Big\{ x = (x_1, x_2)^T \in H \ : \ x, \frac{dx}{dz} \in H \text{ are absolutely continuous}, \\ \frac{d^2x}{dz^2} \in H,\ d_i \frac{dx_i}{dz}(0) - \upsilon x_i(0) = 0 = d_i \frac{dx_i}{dz}(1);\ i = 1;2 \Big\}, \end{aligned} \tag{3.13}$$

$$Ax = \begin{pmatrix} d_1 \dfrac{d^2x_1}{dz^2} - \upsilon \dfrac{dx_1}{dz} & 0 \\ 0 & d_2 \dfrac{d^2x_2}{dz^2} - \upsilon \dfrac{dx_2}{dz} \end{pmatrix} = \begin{pmatrix} A_1 x_1 & 0 \\ 0 & A_2 x_2 \end{pmatrix}. \tag{3.14}$$

The nonlinear operator B is defined on Ω by

$$\begin{aligned} B(t,x) = \big(- k_1 \big(x_1 + C_{\text{in}}(t)\mathbb{I}\big)^m \big(x_2 + L_{\text{in}}(t)\mathbb{I}\big)^n - \dot{C}_{\text{in}}(t)\mathbb{I}, \\ - k_2 \big(x_1 + C_{\text{in}}(t)\mathbb{I}\big)^m \big(x_2 + L_{\text{in}}(t)\mathbb{I}\big)^n - \dot{L}_{\text{in}}(t)\mathbb{I}\big)^T. \end{aligned} \tag{3.15}$$

It is shown in [7] that the linear operator A given by (3.14) is the infinitesimal generator of contraction semigroup on H

$$T(t) = \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix}, \tag{3.16}$$

where $T_1(t)$ and $T_2(t)$ are the C_0-semigroups generated, respectively, by A_1 and A_2.

4. Global existence

This section is concerned with the existence and the uniqueness of mild solution for our problem given by (3.6)–(3.9) In order to be able to apply Corollary 2.3, we need the following lemmas.

Lemma 4.1. *For each* $(t,x) \in \Omega$,

$$\lim_{h \downarrow 0} \frac{1}{h} d\big(x + hB(t,x); \Omega(t+h)\big) = 0. \tag{4.1}$$

Proof. Let $(t,x) \in \Omega$. Observe that $\Omega(t)$ is given by $\Omega(t) = \Omega_1(t) \times \Omega_2(t)$, where

$$\begin{aligned} \Omega_1(t) &= \big[- C_{\text{in}}(t)\mathbb{I}, \big(\overline{C} - C_{\text{in}}(t)\big)\mathbb{I} \big], \\ \Omega_2(t) &= \big[- L_{\text{in}}(t)\mathbb{I}, \big(\overline{L} - L_{\text{in}}(t)\big)\mathbb{I} \big]. \end{aligned} \tag{4.2}$$

Denote

$$X_1(t) = x_1 + C_{\text{in}}(t)\mathbb{I}, \qquad X_2(t) = x_2 + L_{\text{in}}(t)\mathbb{I}, \tag{4.3}$$

we have, for $x \in \Omega(t)$,

$$X(t) = \left(X_1(t), X_2(t)\right)^T \in [0, \overline{C}\mathbb{I}] \times [0, \overline{L}\mathbb{I}]. \tag{4.4}$$

Let $h_0 > 0$ be sufficiently small such that $h_0 k_1 \overline{C}^{m-1} \overline{L}^n \leq 1$.

Let, now, $h \in (0, h_0)$, then

$$X_1(t)\left(\mathbb{I} - hk_1 X_1^{m-1}(t) X_2^n(t)\right) \in [0, \overline{C}\mathbb{I}]. \tag{4.5}$$

Hence

$$f_1(t, X(t)) = X_1(t)\left(\mathbb{I} - hk_1 X_1^{m-1}(t) X_2^n(t)\right) - C_{\text{in}}(t+h)\mathbb{I} \in \Omega_1(t+h). \tag{4.6}$$

By using the regularity of the inlet function C_{in}, we get

$$\begin{aligned} d\left(x_1 + hB_1(t,x), \Omega_1(t+h)\right) &\leq d\Big(X_1(t) - hk_1 X_1^m(t) X_2^n(t) - C_{\text{in}}(t+h)\mathbb{I}, \Omega_1(t+h)\Big) + h\epsilon(h) \\ &\leq d\left(f_1(t, X(t)), \Omega_1(t+h)\right) + h\epsilon(h) = h\epsilon(h), \end{aligned} \tag{4.7}$$

where $\epsilon(h) \to 0$ as $h \to 0$. Whence

$$\lim_{h \downarrow 0} \frac{1}{h} d\left(x_1 + hB_1(t,x);\ \Omega_1(t+h)\right) = 0. \tag{4.8}$$

By similar considerations as above, taking into account the regularity of the function L_{in}, we also get

$$\lim_{h \downarrow 0} \frac{1}{h} d\left(x_2 + hB_2(t,x); \Omega_2(t+h)\right) = 0. \tag{4.9}$$

Observe, now, that

$$d\left(x + B(t,x), \Omega(t+h)\right) \leq d\left(x_1 + B_1(t,x), \Omega_1(t+h)\right) + d\left(x_2 + B_2(t,x), \Omega_2(t+h)\right), \tag{4.10}$$

combining the latter with (4.8)-(4.9) we get the desired result (4.1). □

The following lemma is useful to establish the dissipativity property.

Lemma 4.2. *There exists $l_B \in \mathbb{R}^+$ such that the operator $(B(t, \cdot) - l_B I)$ is dissipative on $\Omega(t)$ for each $t \geq 0$.*

Proof. Let $t \geq 0$ and let x, y be in $\Omega(t)$. Denote

$$g_i(t,x) = -k_i\left(x_1 + C_{\text{in}}(t)\mathbb{I}\right)^m\left(x_2 + L_{\text{in}}(t)\mathbb{I}\right)^n \quad \text{for } i = 1,2, \tag{4.11}$$

and let also

$$X_1(t) = x_1 + C_{\text{in}}(t)\mathbb{I}; \quad X_2(t) = x_2 + L_{\text{in}}(t)\mathbb{I}; \quad Y_1(t) = y_1 + C_{\text{in}}(t)\mathbb{I}, \quad Y_2(t) = y_2 + L_{\text{in}}(t)\mathbb{I}. \tag{4.12}$$

Observe that, for each $x, y \in \Omega(t)$, $(X_i(t), Y_i(t))^T \in [0, \overline{C}\mathbb{I}] \times [0, \overline{L}\mathbb{I}]$ for $i = 1,2$. Hence, by applying the mean value theorem, for $i = 1,2$, we get

$$\begin{aligned}\left\|g_i(t,x) - g_i(t,y)\right\|_{L^2} &\leq k_i\left(\overline{C}^{2m}\left\|X_2^n(t) - Y_2^n(t)\right\|_{L^2}^2 + \overline{L}^{2n}\left\|X_1^m(t) - Y_1^m(t)\right\|_{L^2}^2\right)^{1/2}\\ &\leq k_i\left(n^2\overline{C}^{2m}\overline{L}^{2n-2}\left\|x_2 - y_2\right\|_{L^2}^2 + m^2\overline{L}^{2n}\overline{C}^{2m-2}\left\|x_1 - y_1\right\|_{L^2}^2\right)^{1/2}\\ &\leq k_i\overline{C}^{m-1}\overline{L}^{n-1}\max\left(n\overline{C}; m\overline{L}\right)\|x - y\|.\end{aligned} \tag{4.13}$$

Finally,

$$\begin{aligned}\|B(t,x) - B(t,y)\| &= \left(\left\|g_1(t,x) - g_1(t,y)\right\|_{L^2}^2 + \left\|g_2(t,x) - g_2(t,y)\right\|_{L^2}^2\right)^{1/2}\\ &\leq \max\left(k_1, k_2\right)\overline{C}^{m-1}\overline{L}^{n-1}\max\left(n\overline{C}; m\overline{L}\right)\|x - y\|.\end{aligned} \tag{4.14}$$

Consequently, $B(t,\cdot)$ is an l_B-dissipative operator on $\Omega(t)$ [14, page 245], where

$$l_B = \max\left(k_1, k_2\right)\overline{C}^{m-1}\overline{L}^{n-1}\max\left(n\overline{C}; m\overline{L}\right). \tag{4.15}$$

Finally, we state the invariance properties of the state trajectories of the model given by (3.6)–(3.9). □

Proposition 4.3. *One has that*

$$\Omega(t) \text{ is } \mathcal{T}(s)\text{-invariant} \quad \forall t, s \geq 0. \tag{4.16}$$

Proof. Let $t, s \geq 0$ and $(x,y)^T \in \Omega(t)$. We have

$$\left(-C_{\text{in}}(t)\mathbb{I}, -L_{\text{in}}(t)\mathbb{I}\right)^T \leq (x,y)^T \leq \left(\left(\overline{C} - C_{\text{in}}(t)\right)\mathbb{I}, \left(\overline{L} - L_{\text{in}}(t)\right)\mathbb{I}\right)^T. \tag{4.17}$$

Hence, by using the positivity of $(T(t))_{t\geq 0}$ [26], we have

$$\begin{aligned}\left(-C_{\text{in}}(t)T_1(s)\mathbb{I}, -L_{\text{in}}(t)T_2(s)\mathbb{I}\right)^T &\leq T(s)(x,y)^T\\ &\leq \left(\left(\overline{C} - C_{\text{in}}(t)\right)T_1(s)\mathbb{I}, \left(\overline{C} - C_{\text{in}}(t)\right)T_2(s)\mathbb{I}\right)^T.\end{aligned} \tag{4.18}$$

Since, $T_i(t)\mathbb{I} \leq \mathbb{I}$ for $i = 1;2$ (see [26]) and by using the inequalities (1.8) (i.e., $\overline{C} \geq C_{\text{in}}$ and $\overline{L} \geq L_{\text{in}}$), the invariance of $\Omega(t)$ holds for all $t \geq 0$. Thus, $(T_1(s)x, T_2(s)y)^T \in \Omega(t)$ for all $t, s \geq 0$. □

Now, we are in a position to state and prove our global existence result for problem (3.6)–(3.9).

Theorem 4.4. *Let $C_{in}(t)$ and $L_{in}(t)$ be positive $C^1([0,+\infty[)$-functions. Then, for every $x_0 \in \Omega(0)$, the problem* (3.6)–(3.9) *has a unique global mild solution.*

Proof. Since B is continuous function in Ω, by Corollary 2.3, it is sufficient to prove the condition (i) in Corollary 2.3 and to check that the subset Ω is connected

(a) *Let us first show that Ω is closed from the left.*

Let $t_n \nearrow t$ and $x_n \in \Omega(t_n)$ with $x_n \to x \in H$, then there exists a subsequence of (x_n) which is also denoted by (x_n) such that $x_n(z) \to x(z)$, that is, on $[0,1]$ which implies, by continuity of C_{in} and L_{in}, that $x(z) \in [-C_{\text{in}}(t), \overline{C} - C_{\text{in}}(t)] \times [-L_{\text{in}}(t), \overline{L} - L_{\text{in}}(t)]$, that is, on $[0,1]$, hence $x \in \Omega(t)$ for each $t \geq 0$.

(b) *Let us, now, check that Ω is connected in* $[0,+\infty[\times H$*:*

Let $K = [0,\overline{C}\mathbb{I}] \times [0,\overline{L}\mathbb{I}]$ and define $G : [0,+\infty[\times K \to \Omega$ such that for all $(t,x) \in [0,+\infty[\times K, G(t,x) = (t, x_1 - C_{\text{in}}(t)\mathbb{I}, x_2 - L_{\text{in}}(t)\mathbb{I})^T$. Since C_{in} and L_{in} are continuous functions in $[0,+\infty[$, it follows that G in $[0,+1[\times K$ is also a continuous function. Observe that G is surjective; since $[0,\overline{C}\mathbb{I}] \times [0,\overline{L}\mathbb{I}]$ is connected in H, we get that $\Omega = G([0,+\infty[\times K)$ is also connected in $[0,+\infty[\times H$.

Thus the proof of the theorem is complete. □

The next section deals with the existence and uniqueness results of equilibrium profile solutions for a nonlinear model given by (3.6)–(3.9).

5. Equilibrium profiles

In the steady-state solution analysis, the inlet functions C_{in} and L_{in} are independent of time t, which implies that the domain $\Omega(t)$ is also independent of t. If we denote by $\overline{C_{\text{in}}}$ and $\overline{L_{\text{in}}}$ the values of C_{in} and L_{in} which correspond to the steady-state solutions, the corresponding steady-state system to the models (3.6)–(3.9) is given by the following equations:

$$-v\frac{dx_1}{dz} = d_1\frac{d^2x_1}{dz^2} - k_1\left(x_1 + \overline{C_{\text{in}}}\right)^m\left(x_2 + \overline{L_{\text{in}}}\right)^n = 0, \tag{5.1}$$

$$-v\frac{dx_2}{dz} = d_2\frac{d^2x_2}{dz^2} - k_2\left(x_1 + \overline{C_{\text{in}}}\right)^m\left(x_2 + \overline{L_{\text{in}}}\right)^n = 0, \tag{5.2}$$

with

$$d_i\frac{dx_i}{dz}(0) - vx_i(0) = 0 = d_i\frac{dx_i}{dz}(1), \quad i = 1;2, \tag{5.3}$$

$$\begin{aligned}\Omega(t) = \Delta = \Big\{ &(x_1,x_2)^T \in H : -\overline{C_{\text{in}}} \leq x_1(z) \leq \overline{C} - \overline{C_{\text{in}}}, \\ &-\overline{L_{\text{in}}} \leq x_2(z) \leq \overline{L} - \overline{L_{\text{in}}} \text{ for almost all } z \in [0,1]\Big\}.\end{aligned} \tag{5.4}$$

The following existence result can be proven as in the case where C_{in} and L_{in} are independent of time.

Theorem 5.1 (see [7, 27]). *The tubular reactor modelled by the nonlinear coupled partial differential equations given by* (3.6)–(3.9) *has at least one equilibrium profile in* Δ.

The sequel of this paper will deal with the uniqueness analysis of steady states in the important case where $d_1 = d_2 = d$.

First, since $d_1 = d_2 = d$, we denote $\mathcal{A} = d(d^2/dz^2) - v(d/dz) = A_i$ with $D(\mathcal{A}) = D(A_i)$ for $i = 1; 2$.

Now, we derive a positivity lemma, which will play a fundamental role in the proof of the uniqueness result of steady states.

Lemma 5.2. *Let* $b(\cdot)$ *be a bounded nonnegative function defined in* $[0,1]$. *If* $u \in L^2([0,1])$ *satisfies the equations*

$$\mathcal{A}u = bu \quad \text{in }]0,1[, u \in D(\mathcal{A}), \tag{5.5}$$

then $u = 0$ *in* $[0,1]$.

Proof. Let u be the solution of problem (5.5), then

$$\langle \mathcal{A}u, u\rangle_{L^2} = \langle bu, u\rangle_{L^2}. \tag{5.6}$$

We have,

$$\begin{aligned}
\langle \mathcal{A}u, u\rangle_{L^2} &= \int_0^1 \left[d\frac{d^2u}{dz^2}(z) - v\frac{du}{dz}(z)\right]u(z)dz, \\
&= -\int_0^1 d\left[\frac{du}{dz}(z)\right]^2 dz + d\left[\frac{du}{dz}(1)u(1) - \frac{du}{dz}(0)u(0)\right] - \frac{1}{2}v\left[u^2(1) - u^2(0)\right], \qquad (5.7)\\
&= -d\left\|\frac{du}{dz}\right\|_{L^2}^2 - \frac{1}{2}vu^2(1) - \frac{1}{2}vu^2(0), \\
&\leq 0. \qquad (5.8)
\end{aligned}$$

Since $b(z)$ is nonnegative function in $[0,1]$, then by (5.8) and taking into account (5.6)

$$\langle bu, u\rangle_{L^2} = \int_0^1 b(z)u^2(z)dz = 0. \tag{5.9}$$

Which implies, in view of (5.6)-(5.7), that

$$\langle \mathcal{A}u, u\rangle_{L^2} = 0 = d\left\|\frac{du}{dz}\right\|_{L^2}^2 + \frac{1}{2}vu^2(1) + \frac{1}{2}vu^2(0). \tag{5.10}$$

Then, we get

$$\begin{gathered}
\frac{du}{dz}(z) = 0 \quad \text{a.e. } z \in [0,1], \\
u(0) = 0 = u(1).
\end{gathered} \tag{5.11}$$

Clearly, by using the Sobolev imbedding theorem, $D(\mathcal{A}) \subset \mathcal{C}([0,1])$. Therefore, $u = 0$ since $u \in D(\mathcal{A})$. □

Theorem 5.3. *For $d_1 = d_2 = d$, the steady-state problem given by* (5.1)–(5.3) *has a unique solution in* Δ.

Proof. Let $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ be solutions to (5.1)–(5.3) on $[0,1]$. To obtain the desired result, we will be showing that $x = y$. Let

$$\begin{gathered} g(x_1, x_2) = -\left(x_1 + \overline{C_{\text{in}}\mathbb{I}}\right)^m \left(x_2 + \overline{L_{\text{in}}\mathbb{I}}\right)^n, \\ w_1 = y_1 - x_1 \in D(\mathcal{A}), \qquad w_2 = x_2 - y_2 \in D(\mathcal{A}). \end{gathered} \tag{5.12}$$

Then

$$\begin{aligned} -\mathcal{A}w_1 &= k_1\left(g(y_1, y_2) - g(x_1, x_2)\right) \\ &= k_1\left(y_1 + \overline{C_{\text{in}}}\right)^m \left[\left(x_2 + \overline{L_{\text{in}}}\right)^n - \left(y_2 + \overline{L_{\text{in}}}\right)^n\right] \\ &\quad + k_1\left(x_2 + \overline{L_{\text{in}}}\right)^n \left[\left(x_1 + \overline{C_{\text{in}}}\right)^m - \left(y_1 + \overline{C_{\text{in}}}\right)^m\right] \end{aligned} \tag{5.13}$$

Hence, by applying the mean value theorem, we get

$$-\mathcal{A}w_1 = k_1 n\left(y_1 + \overline{C_{\text{in}}}\right)^m \xi_2^{n-1} w_2 - mk_1\left(x_2 + \overline{L_{\text{in}}}\right)^n \xi_1^{m-1} w_1, \tag{5.14}$$

where (ξ_1, ξ_2) are some intermediate values between $(0,0)$ and $(\overline{C}, \overline{L})$.

By similar considerations as above, we also get

$$\begin{aligned} -\mathcal{A}w_2 &= -k_2\left(g(y_1, y_2) - g(x_1, x_2)\right) \\ &= -k_2 n\left(y_1 + \overline{C_{\text{in}}}\right)^m \xi_2^{n-1} w_2 + mk_2\left(x_2 + \overline{L_{\text{in}}}\right)^n \xi_1^{m-1} w_1, \end{aligned} \tag{5.15}$$

for the same ξ_1 and ξ_2.

Now, we have the following system:

$$-\mathcal{A}w_1 = -a_1 w_1 + b_1 w_2, \tag{5.16}$$

$$-\mathcal{A}w_2 = a_2 w_1 - b_2 w_2, \tag{5.17}$$

where, for $i = 1; 2$,

$$\begin{aligned} a_i(z) &= mk_i\left(x_2(z) + \overline{L_{\text{in}}}\right)^n \xi_1^{m-1}(z), \\ b_i(z) &= nk_i\left(y_1(z) + \overline{C_{\text{in}}}\right)^m \xi_2^{n-1}(z). \end{aligned} \tag{5.18}$$

Multiplying (5.16) by k_2 and (5.17) by k_1, we get by addition of both equations that

$$\mathcal{A}w = 0, \quad w \in D(\mathcal{A}), \tag{5.19}$$

where $w = k_2 w_1 + k_1 w_2$. By Lemma 5.2, this system has a unique solution $w = 0$ in $[0,1]$. Now, let

$$-\mathcal{A}w_2 = a_2 w_1 - b_2 w_2 \tag{5.20}$$

and substituting the expression

$$w_1 = -k_2^{-1} k_1 w_2 \tag{5.21}$$

yields

$$\mathcal{A}w_2 = c w_2, \tag{5.22}$$

where $c(z) = a_1(z) + b_2(z)$. Observe that, for $i = 1; 2$,

$$0 \le a_i(z) \le mk_i\overline{L}^n\overline{C}^{m-1}, \qquad 0 \le b_i(z) \le nk_i\overline{C}^m\overline{L}^{n-1}. \tag{5.23}$$

Let $\lambda = \max(m\overline{L}, n\overline{C})\max(k_1, k_2)\overline{C}^{m-1}\overline{L}^{n-1}$, then we have $0 \le c(z) \le 2\lambda$. By Lemma 5.2 we get $w_2 = 0$. Thus it follows, by (5.21), that $w_1 = 0$, which ensures the desired result, that is, $x = y$. □

6. Conclusion

In this paper, we have studied the existence and uniqueness of the global mild solution for a class of tubular reactor nonlinear nonautonomous models. It has also been proven that the trajectories are satisfying time-dependent constraints, that is, $x(t) \in \Omega(t)$. Moreover, the set of physically meaningful admissible states $\Omega(t)$ is invariant under the dynamics of the reactions. In addition, the existence and uniqueness results of equilibrium profiles are reported.

An important open question is the stability analysis of equilibrium profile for system (1.2)–(1.6). This question is under investigation.

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