MULTIPLE PERIODIC SOLUTIONS FOR A CLASS OF SECOND-ORDER NONLINEAR NEUTRAL DELAY EQUATIONS

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Received 14 October 2004; Accepted 12 December 2004

By means of a variational structure and Z_2-group index theory, we obtain multiple periodic solutions to a class of second-order nonlinear neutral delay equations of the form $x''(t-\tau)+\lambda(t)f(t,x(t),x(t-\tau),x(t-2\tau)) = x(t)$, $\lambda(t)>0$, $\tau>0$.

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1. Introduction

Recently, the existence and multiplicity of periodic solutions for second-order neutral delay equations have received a good deal of attention (see, e.g., [3, 4, 7]). In [4], Wang and Yan studied the second-order neutral delay equation

$$\left[x(t)+cx(t-\tau)\right]''+g\left(t,x(t-\sigma)\right) = p(t), \tag{1.1}$$

where τ, σ, and c are real constants with $\tau \geq 0$, $\sigma \geq 0$, $|c|<1$, $g(t,x)$ is a T-periodic function for $t>0$ and, for an arbitrary bounded domain $D \subset R$, $g(t,x)$ is a Lipschitz-function on $[0,T]\times D$. Moreover, $p \in C(R,R)$, $p(t+T) = p(t)$ and $\int_0^T p(t)dt = 0$. They obtained sufficient conditions which guarantee the existence of at least one T-periodic solution for the above system.

However, for the existence of periodic solutions of functional differential equations, previous authors have used, mainly, fixed point theory, coincidence degree theory, Fourier analysis, and so forth. They have rarely used critical point theory. In [5, 6], the authors obtained multiple periodic solutions for a class of retarded differential equations by means of critical point theory and Z_p-group index theory. These results were obtained by reducing retarded differential equations to related ordinary differential equations.

The purpose of this paper is to establish a variational framework with delayed variables for a class of neutral differential equations. Unlike the papers [5, 6], our approach enables us to obtain by critical point theory and Z_2-group index theory the existence of

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2006, Article ID 10252, Pages 1–9
DOI 10.1155/AAA/2006/10252

nontrivial periodic solutions to such equations without reducing them to ordinary differential equations. To this end, we give below some preliminary material about Z_2-group index theory and critical points.

In what follows, E is a real Banach space with norm $\|\cdot\|$.

Definition 1.1. A "critical point" of $f \in C^1(E,R)$ is a point $x \in E$ for which $f'(x) = 0$. A "critical value" of f is a number c such that $f(x) = c$ for some critical point x. The set $K = \{x \in E \mid f'(x) = 0\}$ is the "critical set" of f. We denote by K_c the set $\{x \in E \mid f'(x) = 0, f(x) = c\}$. The "critical level" set f_c of f is defined by $f_c = \{x \in E \mid f(x) \le c\}$.

Definition 1.2. Let $f \in C^1(E,R)$. We say that f satisfies the "Palais-Smale" condition if every sequence $\{x_n\} \subset E$ such that $\{f(x_n)\}$ is bounded and $f'(x_n) \to \theta$ as $n \to \infty$ has a convergent subsequence.

We say that a closed symmetric set $A \subset E$ satisfies property $\mathcal{P}$ if, for some $n \in Z^+$, there exists an odd continuous function $\varphi : A \to R^n \setminus \{\theta\}$. Let $N_A \subset Z$ be defined as follows: $n \in N_A$ if and only if A satisfies property $\mathcal{P}$ with this n.

Definition 1.3. Let E be real Banach space, and $\sum = \{A \mid A \subset E \setminus \{\theta\}$ a closed, symmetric set$\}$. Define $\gamma : \sum \to Z^+ \bigcup \{+\infty\}$ as follows:

$$\gamma(A) = \begin{cases} \min N_A & \text{if } N_A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \\ +\infty & \text{if } A \neq \emptyset, \text{ but } N_A = \emptyset. \end{cases} \tag{1.2}$$

We say that "γ is the genus of $\sum$." We let $i_1(f) = \lim_{a \to -0} \gamma(f_a)$ and $i_2(f) = \lim_{a \to -\infty} \gamma(f_a)$.

Lemma 1.4 (Chang [1]). *Let $f \in C^1(E,R)$ be an even functional which satisfies the Palais-Smale condition and $f(\theta) = 0$. Then*

(F_1) *if there exists an m-dimensional subspace X of E and $\rho > 0$ such that*

$$\sup_{x \in X \bigcap S_\rho} f(x) < 0, \tag{1.3}$$

then we have $i_1(f) \ge m$;

(F_2) *if there exists a j-dimensional subspace $\widetilde{X}$ of E such that*

$$\inf_{x \in \widetilde{X}^\perp} f(x) > -\infty, \tag{1.4}$$

we have $i_2(f) \le j$.

If $m \ge j$ and (F_1), (F_2) hold, then f has at least $2(m - j)$ distinct critical points.

In this paper, we use Lemma 1.4 to show the existence of multiple periodic solutions of the following second-order neutral delay equations:

$$x''(t-\tau) + \lambda(t) f\big(t, x(t), x(t-\tau), x(t-2\tau)\big) = x(t-\tau), \quad \lambda(t) > 0,\ \tau > 0. \tag{1.5}$$

Our basic assumptions are the following:

(A_1) $f(t,x_1,x_2,x_3) \in C(R^4,R)$, and $\partial f(t,x_1,x_2,x_3)/\partial t \neq 0$;

(A_2) there exists a continuously differentiable function $F \in C^1(R^3, R)$ with such that

$$F'_{u_2}(t,u_1,u_2) + F'_{u_2}(t,u_2,u_3) = f(t,u_1,u_2,u_3); \tag{1.6}$$

(A_3) $F(t+\tau,u_1,u_2) = F(t,u_1,u_2)$ for all $u_1,u_2 \in R$, and $\lambda(t)$ is τ-periodic in t.
(A_4) F satisfies: $F(t,-u_1,-u_2) = F(t,u_1,u_2)$, and

$$f(t,-u_1,-u_2,-u_3) = -f(t,u_1,u_2,u_3). \tag{1.7}$$

By assumption (A_2), we have

$$F'_{u_1}(t,x(t-\tau),x(t-2\tau)) + F'_{u_2}(t,x(t),x(t-\tau)) = f(t,x(t),x(t-\tau),x(t-2\tau)). \tag{1.8}$$

Thus, under assumptions (A_1)–(A_4), we only need to study the following equation:

$$x''(t-\tau) - x(t-\tau) + \lambda(t)\left[F_{u_1}(t,x(t-\tau),x(t-2\tau)) + F_{u_2}(t,x(t),x(t-\tau))\right] = 0. \tag{1.9}$$

2. Variational structure

Fix $\gamma > 0$, $\tau > 0$, and consider

$$H^1_{2\gamma\tau} = \{x \in L^2[0,2\gamma\tau] \mid x(t) \text{ is a continuously differentiable } 2\gamma\tau\text{-periodic function in } t\}. \tag{2.1}$$

It is obvious that $H^1_{2\gamma\tau}$ is a Sobolev space with inner product and norm defined by

$$\begin{aligned} \langle x,y\rangle_{H^1_{2\gamma\tau}} &= \int_0^{2\gamma\tau} \left[x(t)y(t) + x'(t)y'(t)\right]dt, \\ \|x\|_{H^1_{2\gamma\tau}} &= \left|\int_0^{2\gamma\tau} \left[|x(t)|^2 + |x'(t)|^2\right]dt\right|^{1/2}, \quad \forall x,y \in H^1_{2\gamma\tau}. \end{aligned} \tag{2.2}$$

Moreover, $x \in H^1_{2\gamma\tau}$ can be expressed as follows;

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi}{\gamma\tau}t + b_k \sin\frac{k\pi}{\gamma\tau}t\right). \tag{2.3}$$

Let us consider the functional $I(x)$ defined on $H^1_{2\gamma\tau}$ as follows:

$$I(x) = \int_0^{2\gamma\tau}\left[\frac{1}{2}\left(|x'(t)|^2 + |x(t)|^2\right) - \lambda(t)F(t,x(t),x(t-\tau))\right]dt. \tag{2.4}$$

For all $x, y \in H^1_{2\gamma\tau}$ and $\varepsilon > 0$, we know that

$$\begin{aligned} I(x+\varepsilon y) = I(x) + \varepsilon \int_0^{2\gamma\tau} \big[& x(t)y(t) + x'(t)y'(t) \\ & - \lambda(t)\big(F'_{u_1}(t, x(t)+\varepsilon\theta(t)y(t), x(t-\tau))y(t) \\ & + F'_{u_2}(t, x(t), x(t-\tau)+\varepsilon\theta(t)y(t-\tau))y(t-\tau)\big)\big]dt \\ & + \frac{\varepsilon^2}{2}\int_0^{2\gamma\tau}\Big[y^2(t) + |y'(t)|^2\Big]dt \quad \forall x, y \in H^1_{2\gamma\tau},\ 0 < \theta(t) < 1. \end{aligned} \tag{2.5}$$

It is easy to see that

$$\begin{aligned} \langle I'(x), y\rangle = \int_0^{2\gamma\tau} \big[& x'(t)y'(t) + x(t)y(t) - \lambda(t)F'_{u_1}(t, x(t), x(t-\tau))y(t) \\ & - \lambda(t)F'_{u_2}(t, x(t), x(t-\tau))y(t-\tau)\big]dt. \end{aligned} \tag{2.6}$$

By the periodicity of $F(t, u_1, u_2)$, $\lambda(t)$, $x(t)$, $x(t-\tau)$, and $y(t)$, we get

$$\begin{aligned} \int_0^{2\gamma\tau} \lambda(t)F'_{u_2}(t, x(t), x(t-\tau))y(t-\tau)dt &= \int_{-\tau}^{(2\gamma-1)\tau} \lambda(t+\tau)F'_{u_2}(t+\tau, x(t+\tau), x(t))y(t)dt \\ &= \int_0^{2\gamma\tau} \lambda(t)F'_{u_2}(t, x(t+\tau), x(t))y(t)dt. \end{aligned} \tag{2.7}$$

Hence,

$$\begin{aligned} \langle I'(x), y\rangle = \int_0^{2\gamma\tau} \big[& -x''(t) + x(t) - \lambda(t)\big(F'_{u_1}(t, x(t), x(t-\tau)) \\ & + F'_{u_2}(t, x(t+\tau), x(t))\big)\big]y(t)dt. \end{aligned} \tag{2.8}$$

Therefore, the Euler equation corresponding to the functional $I(x)$ is

$$x''(t) - x(t) + \lambda(t)\big[F'_{u_1}(t, x(t), x(t-\tau)) + F'_{u_2}(t, x(t+\tau), x(t))\big] = 0. \tag{2.9}$$

It is easy to see that (2.9) is equivalent to (1.9). Thus, system (1.9) is the Euler equation of the functional $I(x)$. It follows that it is possible to obtain $2\gamma\tau$-periodic solutions of system (1.5) by seeking critical points of the functional $I(x)$.

3. Main results

Theorem 3.1. *Let assumptions (A_1)–(A_4) be satisfied and assume, further, that the function $F(t, u_1, u_2)$ satisfies the following:*

(C_1) $F(t, 0, 0) = 0$, $\forall t \in [0, \tau]$;

(C_2)

$$\lim_{|u|\to 0} \frac{F(t, u_1, u_2)}{|u|^2} = 1, \tag{3.1}$$

where $|u| = \sqrt{u_1^2 + u_2^2}$;

(C_3) *there exists* $\alpha > 0$ *such that* $F(t,u_1,u_2) < 0$ *whenever* $u_1^2 + u_2^2 > \alpha$, $t \in [0,\tau]$.
Let $m = \min_{t\in[0,\tau]} \lambda(t) > 0$. *Then, for*

$$m > \frac{n^2(\pi^2 + \gamma^2\tau^2)}{4\gamma\tau^2}, \tag{3.2}$$

problem (1.5) has at least $2n$ *nontrivial* $2\gamma\tau$*-periodic solutions.*

It is not difficult to see that $x(t)$ is a solution of system (1.5), then $-x(t)$ is also a solution of system (1.5) by assumption (A_4), that is, the solutions of system (1.5) is a set that symmetric with respect to the origin in $H^1_{2\gamma\tau}$. On the other hand , if we let $\eta(t,x) = F(t,x(t),x(t-\tau))$, it is easy to see that $\eta(t,x)$ is an even function in x, so $I(x)$ an even function in x and we can show that Theorem 3.1 holds by Lemma 1.4.

In order to exploit Lemma 1.4 to find the critical points of function $I(x)$ in (2.4), one need to verify all the assumptions. First of all, we point out that the functional $I(\cdot)$ defined on $H^1_{2\gamma\tau}$ satisfies the Palais-Smale condition, that is, we have the following lemma.

Lemma 3.2. *Under assumptions (A_1)–(A_3) and the conditions (C_1)–(C_3),* $I(u)$ *satisfies the P.S. condition.*

Proof. Let $\{u_n\} \subset H^1_{2\gamma\tau}$ and the constants c_1, c_2 satisfy

$$c_1 \le I(u_n) \le c_2, \tag{3.3}$$

$$I'(u_n) \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.4}$$

The above equality is equivalent to

$$c_1 \le \int_0^{2\gamma\tau} \left[\frac{1}{2}\left(|u_n'(t)|^2 + |u_n(t)|^2\right) - \lambda(t)F(t,u_n(t),u_n(t-\tau))\right]dt \le c_2, \tag{3.5}$$

$$\begin{aligned} \sup\bigg| \int_0^{2\gamma\tau} [u_n'(t)v'(t) + u_n(t)v(t)]dt &- \int_0^{2\gamma\tau} \lambda(t)F'_{u_1}(t,u_n(t),u_n(t-\tau))v(t)dt \\ &- \int_0^{2\gamma\tau} \lambda(t)F'_{u_2}(t,u_n(t+\tau),u_n(t))v(t)dt\bigg| \longrightarrow 0 \quad (n \longrightarrow \infty), \end{aligned} \tag{3.6}$$

where $v(t) \in H^1_{2\gamma\tau}$, $\|v\| = 1$. Inequality (3.6) is equivalent to

$$\begin{aligned} \sup\bigg| \int_0^{2\gamma\tau} [u_n'(t)v'(t) + u_n(t)v(t)]dt &- \int_0^{2\gamma\tau} \lambda(t)F'_{u_1}(t,u_n(t),u_n(t-\tau))v(t)dt \\ &- \int_0^{2\gamma\tau} \lambda(t)F'_{u_2}(t,u_n(t),u_n(t-\tau))v(t-\tau)dt\bigg| \longrightarrow 0 \quad (n \longrightarrow \infty) \end{aligned} \tag{3.7}$$

with $v(t) \in H^1_{2\gamma\tau}$, $\|v\| = 1$.

Letting $z_n = I'(u_n)$ and $\varepsilon_n = \|z_n\|$, we have $\varepsilon_n \to 0$ as $n \to \infty$.

Replacing v by u_n in (3.7), we have

$$\begin{aligned} \|u_n\|^2_{H^1_{2\gamma\tau}} = \int_0^{2\gamma\tau} &[\lambda(t)F'_{u_1}(t,u_n(t),u_n(t-\tau))u_n(t) \\ &+ \lambda(t)F'_{u_2}(t,u_n(t),u_n(t-\tau))u_n(t-\tau)]dt + \langle z_n, u_n\rangle. \end{aligned} \tag{3.8}$$

By condition (C_3), we know that $F(t,u_n(t),u_n(t-\tau))$ has an upper bound. Thus, by

$$\max F(t,u_n(t),u_n(t-\tau)) = \max_{(t,u_1,u_2)\in[0,\tau]\times[-\alpha,\alpha]\times[-\alpha,\alpha]} F(t,u_1,u_2) = R > 0, \tag{3.9}$$

we get that $\int_0^{2\gamma\tau} F(t,u_n(t),u_n(t-\tau))dt \le 2\gamma\tau R$. Let $M = 2\gamma\tau R$, $Q = \max_{t\in[0,\tau]}\lambda(t) > 0$, then

$$\begin{aligned} I(u_n) &= \int_0^{2\gamma\tau}\left[\frac{1}{2}\left(|u_n'(t)|^2 + |u_n(t)|^2\right) - \lambda(t)F(t,u_n(t),u_n(t-\tau))\right]dt \\ &\ge \frac{1}{2}||u_n||^2_{H^1_{2\gamma\tau}} - QM. \end{aligned} \tag{3.10}$$

By (3.5) and (3.10), it is easy to see

$$||u_n||_{H^1_{2\gamma\tau}} \le \sqrt{2QM + 2c_2}, \tag{3.11}$$

that is, $\|u_n\|_{H^1_{2\gamma\tau}}$ is bounded.

Since $H^1_{2\gamma\tau}$ is the Hilbert space of all continuously differentiable $2\gamma\tau$-periodic functions and for a continuously $2\gamma\tau$-periodic functions convergence sequences $\{x_n\}$ converges to a $2\gamma\tau$-periodic function, it is not difficult to proof that conjugate space of $H^1_{2\gamma\tau}$ is

$$\begin{aligned} H^{1*}_{2\gamma\tau} = \{x(t) \in L^2[0,2\gamma\tau] \mid\, & x(t) \text{ is continuously} \\ & \text{differentiable } 2\gamma\tau\text{-periodic function in } t\}. \end{aligned} \tag{3.12}$$

Since $H^1_{2\gamma\tau}$ is a reflexive Banach space, that there exists a subsequence of $\{u_n\}$ which is weakly convergent in $H^1_{2\gamma\tau}$. We denote, again, by $\{u_n\}$ and suppose that $u_n \rightharpoonup u_0$ in $H^1_{2\gamma\tau}$ as $n \to \infty$.

By (3.8) and the boundedness of $\|u_n\|$, we get

$$\begin{aligned} ||u_n||^2_{H^1_{2\gamma\tau}} &- \int_0^{2\gamma\tau}\lambda(t)F'_{u_1}(t,u_n(t),u_n(t-\tau))u_n(t)dt \\ &- \int_0^{2\gamma\tau}\lambda(t)F'_{u_2}(t,u_n(t),u_n(t-\tau))u_n(t-\tau)dt \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{3.13}$$

On the other hand, the weak convergence of $\{u_n\}$ of $H^1_{2\gamma\tau}$ implies the uniform convergence of $\{u_n\}$ in $C([0,1],R)$(see [2]). Hence,

$$\begin{aligned} ||u_n||^2_{H^1_{2\gamma\tau}} \longrightarrow &\int_0^{2\gamma\tau}\lambda(t)F'_{u_1}(t,u_n(t),u_n(t-\tau))u_n(t)dt \\ &+ \int_0^{2\gamma\tau}\lambda(t)F'_{u_2}(t,u_n(t),u_n(t-\tau))u_n(t-\tau)dt \quad (n \longrightarrow \infty). \end{aligned} \tag{3.14}$$

This means that $\|u_n\|$ is convergent in $H^1_{2\gamma\tau}$, that is, the function I satisfy P.S. condition. □

Lemma 3.3. *Under assumptions* (A_1)–(A_4) *and conditions* (C_1)–(C_3), *there exists an* n-*dimensional subspace* E_n *of* $H^1_{2\gamma\tau}$ *and* $\rho > 0$ *such that*

$$\sup_{x\in E_n\bigcap S_\rho} I(x) < 0, \tag{3.15}$$

that is, we have $i_1(f) \geq n$.

Proof. Let $\beta_k(t) = (\gamma\tau/k\pi)\cos,(k\pi/\gamma\tau)t$, $k = 1,2,3,\ldots,n,\ldots$, then

$$\int_0^{2\gamma\tau} |\beta_k(t)|^2 dt = \frac{\gamma^2\tau^2}{k^2\pi^2}\gamma\tau, \qquad \int_0^{2\gamma\tau} |\beta'_k(t)|^2 dt = \gamma\tau. \tag{3.16}$$

We define the n-dimensional linear space E_n as follows:

$$E_n = \operatorname{span}\{\beta_1(t),\beta_2(t),\ldots,\beta_n(t)\}. \tag{3.17}$$

Obviously, E_n is symmetric. Suppose that $\rho > 0$. Then

$$E_n\bigcap S_\rho = \left\{\sum_{k=0}^{n} b_k\beta_k \,\middle|\, \sum_{k=0}^{n} b_k^2\gamma\tau\left(1+\frac{\gamma^2\tau^2}{k^2\pi^2}\right) = \rho^2\right\}. \tag{3.18}$$

On the other hand, we may choose $\varepsilon > 0$ such that $0 < \varepsilon < (mn^2\pi^2/\gamma^2\tau^2)(2\gamma^2\tau^2/n^2 - (\pi^2 + \gamma^2\tau^2)/m)$. Then, by condition (F_2), we know that there exists $\delta > 0$ such that

$$\begin{aligned} \lambda(t)F(t,x(t),x(t-\tau)) &\geq (\lambda(t)-\varepsilon)\left[|x(t)|^2 + |x(t-\tau)|^2\right] \\ &\geq (m-\varepsilon)\left[|x(t)|^2 + |x(t-\tau)|^2\right] \quad \forall t\in[0,2\gamma\tau], \end{aligned} \tag{3.19}$$

whenever $(\|x_n(t)\|_C^2 + \|x(t-\tau\|_C^2) \leq \delta$, where $\|x_n(t)\|_C^2 = \max_{0\leq t\leq 2\gamma\tau}|x(t)|$. Thus, when we choose $\rho = \delta$, we get, by the periodicity of $x(t)$, $x(t-\tau)$,

$$\begin{aligned} I(x) &= \int_0^{2\gamma\tau}\left[\frac{1}{2}\left(|x'(t)|^2 + |x(t)|^2\right) - \lambda(t)F(t,x(t),x(t-\tau))\right]dt \\ &\leq \frac{1}{2}\sum_{k=0}^{n}\gamma\tau b_k^2\left(1+\frac{\gamma^2\tau^2}{k^2\pi^2}\right) - (m-\varepsilon)\int_0^{2\gamma\tau}\left[|x(t)|^2 + |x(t-\tau)|^2\right]dt \\ &\leq \frac{1}{2}\sum_{k=0}^{n}\gamma\tau b_k^2\left(1+\frac{\gamma^2\tau^2}{k^2\pi^2}\right) - 2(m-\varepsilon)\int_0^{2\gamma\tau}|x(t)|^2 dt \\ &\leq \frac{1}{2}\sum_{k=0}^{n}\gamma\tau b_k^2\left(1+\frac{\gamma^2\tau^2}{k^2\pi^2}\right) - 2(m-\varepsilon)\sum_{k=0}^{n}\gamma\tau b_k^2\frac{\gamma^2\tau^2}{k^2\pi^2} \\ &\leq \frac{1}{2}\sum_{k=0}^{n}\gamma\tau b_k^2\left(1+\frac{\gamma^2\tau^2}{\pi^2}\right) - 2(m-\varepsilon)\sum_{k=0}^{n}\gamma\tau b_k^2\frac{\gamma^2\tau^2}{n^2\pi^2} \\ &\leq \frac{m\gamma\tau}{2\pi^2}\left(\frac{\pi^2+\gamma^2\tau^2}{m} - \frac{4\gamma^2\tau^2}{n^2} + \varepsilon\frac{\gamma^2\tau^2}{mn^2\pi^2}\right) < 0 \quad \forall x\in E_n\bigcap S_\rho, \end{aligned} \tag{3.20}$$

that is, Lemma 3.3 holds true. □

Remark 3.4. The above equality makes use of

$$m > \frac{n^2(\pi^2+\gamma^2\tau^2)}{4\gamma\tau^2}, \qquad 0<\varepsilon<\frac{mn^2\pi^2}{\gamma^2\tau^2}\left(\frac{2\gamma^2\tau^2}{n^2}-\frac{\pi^2+\gamma^2\tau^2}{m}\right). \tag{3.21}$$

From (3.10) we know that $I(x)$ has a lower bound, that is, $i_2(I)=0$. On the other hand, by condition (C_1), we get $I(\theta)=0$. So, by Lemmas 3.2 and 3.3, we obtain Theorem 3.1.

Example 3.5. Let

$$F(t,u_1,u_2)=u_1^2+u_2^2-\left[1+\sin^2\frac{\pi t}{\tau}\right](u_1^2+u_2^2)^2. \tag{3.22}$$

Then

$$\begin{aligned}
&F_{u_1}(t,x(t-\tau),x(t-2\tau))+F_{u_2}(t,x(t),x(t-\tau))\\
&\quad=4x(t-\tau)-\left[1+\sin^2\frac{\pi t}{\tau}\right]\big(4(x^2(t-\tau)+x^2(t-2\tau))x(t-\tau)\\
&\qquad+4(x^2(t)+x^2(t-\tau))x(t-\tau)\big)\\
&\quad=4x(t-\tau)-4\left[1+\sin^2\frac{\pi t}{\tau}\right](x^2(t)+2x^2(t-\tau)+x^2(t-2\tau))x(t-\tau).
\end{aligned} \tag{3.23}$$

Let

$$\begin{aligned}
&f_1(t,x(t),x(t-\tau),x(t-2\tau))\\
&\quad=4x(t-\tau)-4\left[1+\sin^2\frac{\pi t}{\tau}\right](x^2(t)+2x^2(t-\tau)+x^2(t-2\tau))x(t-\tau).
\end{aligned} \tag{3.24}$$

It is easy to see that $f_1(t,-u_1,-u_2,-u_3)=-f_1(t,u_1,u_2,u_3)$ and

$$\lim_{|u|\to 0}\frac{F(t,u_1,u_2)}{|u|^2}=\lim_{|u|\to 0}\frac{u_1^2+u_2^2-[1+\sin^2(\pi t/\tau)](u_1^2+u_2^2)^2}{u_1^2+u_2^2}=1. \tag{3.25}$$

Since

$$F(t,u_1,u_2)<0, \quad \forall t\in[0,\tau], \tag{3.26}$$

whenever $u_1^2+u_2^2>1$, all the conditions of Theorem 3.1 hold true. By Theorem 3.1, we get that the problem

$$x''(t-\tau)+\lambda(t)f_1(t,x(t),x(t-\tau),x(t-2\tau))=x(t-\tau), \tag{3.27}$$

with $\lambda(t)$ continuous and positive, has at least $2n$ nontrivial $2\gamma\tau$-periodic solutions when $m>n^2(\pi^2+\gamma^2\tau^2)/4\gamma\tau^2$. Where $m=\min_{t\in[0,\tau]}\lambda(t)$.

Acknowledgment

This project was supported by NNSF of China (10471155), the Foundation of the Guangdong Province Natural Science Committee (031608) and a specific Foundation for Ph.D. Specialities of Educational Department of China (20020558092).

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