INTERIOR POINT CONTROL AND OBSERVATION FOR THE WAVE EQUATION

A. Y. KHAPALOV

ABSTRACT. This paper is concerned with the approximate and exact controllability properties of the wave equation with interior point controls entering via the concentrated force, the velocity of the displacement and the moment. The emphasis is given to the moving point controls and their dual observations whose advantages and disadvantages, versus the static ones, are analyzed with respect to the space dimension, the duration of the control time interval and the function spaces involved.

1. INTRODUCTION

We consider the following control problem for the wave equation:

$$y_{tt} = \Delta y + \mathbf{L}(\hat{x}(\cdot))v \quad \text{in } \Omega \times (0,T) = Q, \quad v \in V, \tag{1.1}$$

$$y = 0 \quad \text{in } \partial\Omega \times (0,T),$$

$$y\mid_{t=0} \; = \; y_t \mid_{t=0} \; = \; 0 \quad \text{in } \Omega,$$

where Ω is a bounded domain in R^n with boundary $\partial\Omega$, v is a control and V is a control space. The structure of the linear operator $\mathbf{L}(\hat{x}(\cdot))$ is associated with a spatial curve $(0,T) \ni t \to \hat{x}(t) \in \Omega$. In particular, when $\hat{x}(\cdot) \equiv \bar{x}$ one deals with the static point control. System (1.1) is said to be exactly controllable at time T in the Hilbert space H if its reachable set at time T, namely,

$$Y_T = \{ \{y\mid_{t=T},\; y_t\mid_{t=T}\} \mid y \text{ satisfies (1.1) with some } v \in V\}$$

coincides with H. If Y_T is dense in H, then (1.1) is said to be approximately controllable at time T in H.

1991 *Mathematics Subject Classification.* Primary 93B07; Secondary 35B37, 93C20.

Key words and phrases. Wave equation, controllability, observability, point control.

Research supported in part by NSF Grant ECS 9312745. This paper was presented at the Annual SIAM Meeting, San-Diego, California, July 25-29, 1994.

Received: September 20, 1995.

The aim of this paper is to study the exact and approximate controllability properties of (1.1) with the following control operators:

$$\mathbf{L}(\hat{x}(\cdot))v = \delta_{\hat{x}(\cdot)} \circ v, \quad \delta_{\hat{x}(\cdot)} = \delta(x - \hat{x}(t)), \tag{1.2}$$

$$\mathbf{L}(\hat{x}(\cdot))v = \nabla(\delta_{\hat{x}(\cdot)} \circ v), \quad \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \tag{1.3}$$

and

$$\mathbf{L}(\hat{x}(\cdot))v = \frac{\partial}{\partial t}(\delta_{\hat{x}(\cdot)} \circ v), \tag{1.4}$$

where the symbol "$\circ$" indicates the duality associated with V. Three spaces are considered below for the controls (1.2), (1.4): $V = L^2(0,T)$, $[L^\infty(0,T)]'$ and $[C((0,T)\backslash\{t_i\}_{i=1}^\infty)]'$, where $\{t_i\}_{i=1}^\infty \subset (0,T)$ are preassigned isolated points. (1.3) is associated with the n-dimensional versions of these spaces.

The issues of regularity and controllability for the wave equation with interior point control have received considerable attention in the literature mostly in the context of the static control (1.2). A thorough account of the regularity properties of (1.1), (1.2) when $V = L^2(0,T)$, $\hat{x}(\cdot) \equiv \bar{x}$ is given for $n = 3$ by Y. Meyer [14] and J.-L. Lions [11], and for $n = 1,2,3$ by R. Triggiani [18], [20]. Among early works on controllability in one space dimension we mention A. Butkovski [1]. We refer to I.M. Lasiecka and R. Triggiani [10] on the comprehensive account of the use of static point controls in the framework of the optimal control theory with quadratic performance index for different types of linear partial differential equations.

Recent studies exposed the lack of exact controllability of (1.1) with static $L^2(0,T)$-control (1.2) in the spaces where the solutions are continuous in time. In particular, the Hilbert Uniqueness Method, introduced by J.-L. Lions in [11, 12], pointed out at the space F' for exact controllability which is defined as the dual of the completion in the norm $(\int_0^T \phi^2(\bar{x},t)dt)^{1/2}$ of the space of smooth initial conditions $\{\phi_0, \phi_1\}$ with $\phi_0 = 0$ on $\partial\Omega$ and ϕ being the corresponding solution of the wave equation. On the other hand, in [20, 21] it was noticed that for $n = 2,3$ in the spaces of optimal regularity, exact controllability is not possible when using the aforementioned static control. An analogous negative result for the boundary controls of finite range was given in [19] for $n \geq 2$.

The just-described situation is reflected in the set-up of this paper. Namely, the emphasis below is given to the study of exact controllability in the spaces where the solutions to (1.1) can be discontinuous in time and to the moving point controls (1.2)-(1.4). In applications these can also describe temporal activation over preassigned location-fixed actuators, or, in the dual setting, scanning over location-fixed sensors. It is worth noticing that in the multi-dimensional case the moving point controls can cope with the negative effect of "poor" asymptotic properties of the corresponding eigenvalues as well as with their unlimited (or, unknown) multiplicities. The latter makes the treatment of the controllability problem under static controls of any finite range impossible.

In the recent paper [7] it was shown that for any given $T > 0$ there exists a class of curves continuous on $(0,T)$ which, regardless of the space-dimension, make (1.1) with the following controls:

$$\mathbf{L}(\hat{x}(\cdot))\{v_1, v_2\} = \nabla(\delta_{\hat{x}(\cdot)} \circ v_1) + \frac{\partial}{\partial t}(\delta_{\hat{x}(\cdot)} \circ v_2), \tag{1.5}$$

where $\{v_1, v_2\} \in [L^\infty(0,T; R^{n+1})]'$, exactly controllable at time T in $L^2(\Omega) \times H^{-1}(\Omega)$. This was achieved thanks to the *combined* structure of controls (1.5), which allows the direct employment of the conservation law in the derivation of the corresponding a priori estimate. The present paper focuses on the case of "*separate*" controls such as (1.2)-(1.4), a radically different case from (1.5).

The remainder of this paper is organized as follows. The next section discusses the main exact controllability results. Section 3 introduces the dual observability problems and states the main exact observability results. The case of the static observations is considered in Section 4. Section 5 discusses the techniques applied to obtain necessary a priori (exact observability) estimates for the moving point observations (3.3)-(3.5) for $n = 1$. These are then extended to the multidimensional case in Appendix A. Section 6 discusses the proofs of the main controllability results.

2. Main Controllability Results

Theorem 2.1. (The static case) *Let* $\Omega = (0,1)$, $V = L^2(0,T)$, $\hat{x}(\cdot) \equiv \bar{x}$, $\bar{x} \in (0,1)$.

1. Let $\bar{x} \in (0,1)$ *be an arbitrary algebraic number of degree 2 (see, e.g., [17], p. 18). Then (1.1) is exactly controllable at time* $T = 2$, *minimal possible, in* $(H^2(0,1) \bigcap H^1_0(0,1)) \times H^1_0(0,1)$ *with the static control (1.2), and in* $H^1_0(0,1) \times L^2(0,1)$ *with the static controls (1.3)/(1.4).*

2. System (1.1), (1.5), $v_1, v_2 \in V = L^2(0,T)$ *is exactly controllable at time* $T = 2 \times \max\{1-\bar{x}, \bar{x}\}$, *minimal possible, in* $L^2(0,1) \times H^{-1}(0,1)$, *regardless of the choice of* $\bar{x} \in (0,1)$.

Comments on the static case. (i) The static one dimensional case is a "milestone" for further study of the moving point controls. To our knowledge, though the former has often appeared in one context or another in control studies, little was asserted concerning the spaces of exact controllability and of the corresponding controls. For example, the algebraic points were pointed out in [1] in the context of static control (1.2), $n = 1$, but the related function spaces were not explicitly specified.

(ii) Theorem 2.1 distinguishes the algebraic numbers of degree 2 which are known as the "worst approximations" for the rational points. For the same reason these points are well known in the context of observability of the one-dimensional heat equation, see, e.g., Sz. Dolecki [2]. The assertion 1. in Theorem 2.1 (as well as Corollaries 2.1, 3.1 below) admits straightforward extensions to the algebraic points of any higher degree with respect to exact

controllability in more regular spaces (see also Remark 4.1 below).
(iii) (1.5) is the only control among (1.2)-(1.5) which ensures the corresponding exact controllability property in a *stable* way with respect to its allocation.

The following results deal with the moving point controls. Their proofs, given in Sections 5 and 6, focus on the one dimensional case, while Appendix A outlines how they can be extended to any space dimension. To make the formulation of Theorem 2.2 more compact, we will say further: "(1.1) is exactly (approximately) controllable .." meaning by that that "*there exist (measurable, or piecewise continuous) curves for which (1.1) is exactly (approximately) controllable.*"

Theorem 2.2. (Moving point controls) *Let* $T > 0$ *be given and* $\partial\Omega$ *be of class* $C^{[n/2]+1}$ *in the case of control (1.2) and of class* $C^{[n/2]+2}$ *for the controls (1.3)/(1.4). (Here and elsewhere* $[\alpha]$ *denotes for the largest non-negative integer such that* $[\alpha] \le \alpha$.)

1. Then (1.1), (1.2), with $V = [L^\infty(0,T)]'$ *is exactly controllable at time* T *in* $H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega)$. *If* $V = L^2(0,T)$, *then (1.1), (1.2) is approximately controllable at the same time in* $H_D^{-[n/2]}(\Omega) \times H_D^{-[n/2]-1}(\Omega)$.

2. Both systems (1.1), (1.3), with $V = [L^\infty(0,T)]'$ *and (1.1), (1.4), with* $V = [L^\infty(0,T;R^n)]'$ *are exactly controllable at time* T *in* $H_D^{[n/2]+1}(\Omega) \times H_D^{[n/2]}(\Omega)$. *Systems (1.1), (1.3), with* $V = L^2(0,T)$ *and (1.1), (1.4), with* $V = L^2(0,T;R^n)$ *are approximately controllable at the same time in* $H_D^{-[n/2]-1}(\Omega) \times H_D^{-[n/2]-2}(\Omega)$.

Here, with s being a positive integer,

$$H_D^s(\Omega) = D(A^{s/2}) \quad (\text{where } A\varphi = -\Delta\varphi,\ D(A) = H^2(\Omega)\bigcap H_0^1(\Omega)) =$$

$$= \{\phi \mid \phi \in H^s(\Omega),\ \phi\mid_{\partial\Omega} = \ldots = \Delta^{[(s-1)/2]}\phi\mid_{\partial\Omega} = 0\},\quad s \ge 1,$$

$$H^2(\Omega)\bigcap H_0^1(\Omega) = H_D^2(\Omega), \quad [H_D^s(\Omega)]' = H_D^{-s}(\Omega), \quad [H_0^1(\Omega)]' = H^{-1}(\Omega).$$

Everywhere in this paper $L^2(\Omega)$ is identified with its dual space, whence one can write $H_D^s(\Omega) \subset L^2(\Omega) \subset H_D^{-s}(\Omega)$.

Remark 2.1. For each of the controls (1.2)/(1.4) and (1.3) there exists a class of curves $\hat{x}(\cdot)$ continuous everywhere on $(0,T)$, except (maybe) for a countable number of isolated points $\{t_i\}_{i=1}^\infty$, for which the corresponding system (1.1) with $V = [C((0,T)\backslash\{t_i\}_{i=1}^\infty)]'$ or $[C((0,T)\backslash\{t_i\}_{i=1}^\infty; R^n)]'$ is exactly controllable at time T in the spaces specified in Theorem 2.2.

The following assertion exposes the role of the algebraic numbers in the context of the moving point controls.

Corollary 2.1. *Let* $\Omega = (0,1)$. *Given* $T > 0$, *for each of the controls (1.2) or (1.3)/(1.4) there exists a class of curves* $\hat{x}(\cdot)$ *continuous everywhere on* $(0,T)$, *except (maybe) for the only point* t^* *such that* $t_a = T - t^*$ *is an* algebraic *number of degree 2, for which the corresponding system (1.1),* *with* $V=[C((0,T)\backslash\{t^*\})]'$ *is exactly controllable at time* T *accordingly in* $H^3_D(0,1)\times H^2_D(0,1)$ *and in* $H^2_D(0,1)\times H^1_0(0,1)$.

Comments on moving point controls. (i) The techniques used in this paper for the construction of control curves are new. They allow one to extend the approach of [6], [7] to the case of the separate controls (1.2)-(1.4) of finite range. In [6], [7] the invariance of the energy in time was employed - via the dual observation (3.6) - to evaluate directly the energy norm of the solution to the dual system. This resulted in a construction of control curves continuous on $(0,T)$. In contrast to that, this paper considers "separate" controls. We successively evaluate the Fourier coefficients of the solution to the dual equation expanded along the eigenfunctions, while constructing the curves which admit a countable number of discontinuities. These techniques are aimed at the space variable and focus on the properties of the series along the eigenfunctions rather than on time-dependent series usually involved in analogous studies. Such a "permutation" of variables leads to "time-compression," and, consequently, to the introduction of non-Hilbert spaces for controls/observations.
(ii) In the one dimensional case the moving point controls (from suitable spaces) yield exact controllability in the same spaces as in the static case, but at an arbitrary time, specified in advance.

Remark 2.2. *Details about the spaces* $[L^\infty(0,T)]'$, $[C((0,T)\backslash\{t_i\}_{i=1}^\infty)]'$ *can be found in [4]. There is an isometric isomorphism between the former space and the space of bounded additive functions on measurable subsets of* $(0,T)$ *which vanish on sets of zero-measure, see [4], p. 296. The latter space can be regarded as the space of functions of bounded variation defined on* $(0,T)\backslash\{t_i\}_{i=1}^\infty$, *see [4], p. 262.*

3. Dual Observability Problems

It is well-understood now that the issue of controllability is strictly connected with the observability properties of an associated dual system. Accordingly, we shall further approach the problem (1.1) by studying the following system:

$$\varphi_{tt} = \Delta\varphi \quad \text{in } Q, \tag{3.1}$$

$$\varphi=0 \qquad \text{in } \partial\Omega\times(0,T),$$

$$\varphi\mid_{t=T}=\varphi_0,\ \ \varphi_t\mid_{t=T}=\varphi_1 \qquad \text{in } \Omega,$$

$$z(t)=\mathbf{G}(\hat{x}(t))\varphi, \quad t\in(0,T) \tag{3.2}$$

with the observation operators $\mathbf{G}(\hat{x}(\cdot))$ dual of the control operators (1.2)-(1.5), namely:

$$\mathbf{G}(\hat{x}(\cdot))\varphi=\varphi(\hat{x}(\cdot),\cdot), \tag{3.3}$$

$$\mathbf{G}(\hat{x}(\cdot))\varphi=\nabla\varphi(\hat{x}(\cdot),\cdot), \tag{3.4}$$

$$\mathbf{G}(\hat{x}(\cdot))\varphi=\varphi_t(\hat{x}(\cdot),\cdot), \tag{3.5}$$

$$\text{(3.6)} \qquad \mathbf{G}(\hat{x}(\cdot))\varphi \;=\; \{\nabla\varphi(\hat{x}(\cdot),\cdot), \varphi_t(\hat{x}(\cdot),\cdot)\}.$$

Given a normed space H, (3.1), (3.2) is said to be observable at time T on H if for any solution φ of the system (1.1) such that $\{\varphi(\cdot,T), \varphi_t(\cdot,T)\} \in H$, the pair $\{\varphi(\cdot,T), \varphi_t(\cdot,T)\}$ can be uniquely determined from the observation $z(\cdot)$ in (3.2) over the time interval $(0,T)$. Given normed spaces B, $H_1 \subseteq H_2$ we shall say that (3.1), (3.2) is B- exactly observable at time T on H_1 with respect the H_2-norm if

$$\exists \nu > 0 \text{ such that } \|\mathbf{G}(\hat{x}(\cdot))\varphi\|_B \;\geq\; \nu \|\{\varphi(\cdot,T), \varphi_t(\cdot,T)\}\|_{H_2}$$

for any solution φ of the system (1.1) such that $\{\varphi(\cdot,T), \varphi_t(\cdot,T)\} \in H_1$. This definition takes into account the situation typically arising in the context of infinite dimensional studies, namely: the domain of the observation operator may not match the desired regularity of the solutions of the system considered (while being, e.g., densely defined).

The main observability results of this paper are as follows.

Theorem 3.1. (The static case) *Let* $\Omega = (0,1)$, $\hat{x}(\cdot) \equiv \bar{x}$, $\bar{x} \in (0,1)$.

1. For the algebraic points $\bar{x} \in (0,1)$ *of degree 2 system (3.1), (3.2) is* $L^2(0,T)$*-exactly observable at* $T = 2$, *minimal possible, on the space* $H_0^1(0,1) \times L^2(0,1)$ *with respect to the* $H^{-1}(0,1) \times H_D^{-2}(0,1)$*-norm for the static observation (3.3), and on* $H_D^2(0,1) \times H_0^1(0,1)$ *with respect to the* $L^2(0,1) \times H^{-1}(0,1)$*-norm for the static observations (3.4) or (3.5).*

2. Regardless of the choice of $\bar{x} \in (0,1)$, *system (3.1), (3.2), (3.6) is* $L^2(0,T;R^2)$*-exactly observable at* $T = 2\times\max\{\bar{x}, (1-\bar{x})\}$, *minimal possible, on* $H_D^2(0,1) \times H_0^1(0,1)$ *with respect to the* $H_0^1(0,1) \times L^2(0,1)$*-norm.*

Theorem 3.2. (Moving observations when $n = 1$) *Let* $\Omega = (0,1)$ *and* $T > 0$ *be given.*

1. Then (3.1), (3.2) is $L^\infty(0,T)$*-exactly observable at time* T *(in the sense that there exists a suitable class of measurable curves) for the moving observation (3.3) on* $H_0^1(0,1) \times L^2(0,1)$ *with respect to the* $H^{-1}(0,1) \times H_D^{-2}(0,1)$*-norm, and for the moving observations (3.4) or (3.5) on* $H_D^2(0,1)\times H_0^1(0,1)$ *with respect to the* $L^2(0,1) \times H^{-1}(0,1)$*-norm.*

2. The observation curves satisfying the above requirements can be selected to be continuous everywhere on $(0,T)$ *except, maybe, for a countable number of isolated points* $\{t_i\}_{i=1}^\infty$. *For these curves the assertions of 1. in the above hold true with respect to* $C((0,T)\backslash\{t_i\}_{i=1}^\infty)$*-exact observability property.*

The following assertion is dual of Corollary 2.1.

Corollary 3.1. *Let* $\Omega = (0,1)$. *Given* $T > 0$, *for each of the observations (3.3)-(3.5) there exists a class of curves* $\hat{x}(\cdot)$ *continuous everywhere on* $(0,T)$, *except (maybe) for the only instant* t^* *such that* $t_a = T - t^*$ *is an* algebraic *number of degree 2, for which the corresponding system (3.1), (3.2) is* $C((0,T)\backslash t^*)$*-exactly observable at time* T *for the observation (3.3) on* $H_0^1(0,1) \times L^2(0,1)$ *with respect to the* $H_D^{-2}(0,1) \times H_D^{-3}(0,1)$*-norm, and for the observation (3.4) or (3.5) on* $H_D^2(0,1) \times H_0^1(0,1)$ *with respect to the* $H^{-1}(0,1) \times H_D^{-2}(0,1)$*-norm.*

Theorem 3.3. (The general case) *Let* $T > 0$ *be given and* $\partial\Omega$ *be of class* $C^{[n/2]+1}$ *in the case of observations (3.3) and of class* $C^{[n/2]+1}$ *in the case of observations (3.4)/(3.5). Then all the assertions of Theorem 3.2 hold true (recall only that (3.4) is an* n*-dimensional vector) accordingly on* $H_D^{[n/2]+1}(\Omega) \times H_D^{[n/2]}(\Omega)$ *with respect to the* $H_D^{-[n/2]-1}(\Omega) \times H_D^{[-n/2]-2}(\Omega)$*-norm for the observation (3.3) and on* $H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega)$ *with respect to the* $H_D^{-[n/2]}(\Omega) \times H_D^{-[n/2]-1}(\Omega)$*-norm for the observations (3.4)/(3.5).*

Remark 3.1. (i) The arguments of Theorems 3.1-3.3 (except for the assertion 3.1.2) make use of the Fourier expansion of the solution to (3.1) along the corresponding eigenfunctions and of the asymptotic behavior of the (multiple) eigenvalues. Corollary 3.1 also employs the explicit formula for the latter in one space dimension.
(ii) Exact observability of (3.1), (3.2), (3.6), stated as a part of assertion 3.1.2, was shown by L.F. Ho [5] for $T > 2 \times \max\{\bar{x}, 1 - \bar{x}\}$ by using the multipliers techniques. Our argument is based on d'Alembert's formula (4.8), which is due to the wave reflection principle. It allows one to calculate precisely the energy of the solution to (3.1) via its output (3.6), see (4.12) below.
(iii) An application of the assertion 2. in Theorem 3.1 to the issue of pointwise stabilization is discussed in [8].

Given $T > 0$, let a sequence $\{x_k, t_k\}_{k=1}^{\infty} \subset \Omega \times (0, T)$ be given. Consider the following discrete-time observations:

$$\mathbf{G}_k\varphi = \varphi(x_k, t_k), \quad k = 1 \ldots, \tag{3.7}$$

$$\mathbf{G}_k\varphi = \nabla\varphi(x_k, t_k), \quad k = 1 \ldots, \tag{3.8}$$

$$\mathbf{G}_k\varphi = \varphi_t(x_k, t_k), \quad k = 1 \ldots. \tag{3.9}$$

The arguments of Theorems 3.2, 3.3 and Corollary 3.1 are linked below to the existence of skeletons $\{x_k, t_k\}_{k=1}^{\infty}$ such that any curve passing through them provides a desirable exact observability estimate. This yields the following reformulation of the aforementioned exact observability results.

Theorem 3.4. (Discrete time observations) *The results of Theorems 3.2, 3.3 and Corollary 3.1 remain true for the observations (3.7)-(3.9) with the replacement of the space* $L^\infty(0, T)$ *for observations by its sequential analogue* l^∞*. Suitable sequences for observations are described in Steps 2-4 in Section 5 and in Appendix A, and in the proof of Corollary 3.1.*

Remark 3.2. If one has more than one sensor, i.e., if the observation becomes vector-valued, the measurement instants t_k, $k = 1, \ldots$ in (3.7)-(3.9) can be selected to coincide. In particular, for a countable set of sensors in the case of assertions of Theorem 3.4 corresponding to Theorems 3.2, 3.3 one can take only the instants $\{t_k^i\}_{i=1,k=1}^{2\infty}$ pointed out in (5.3) and in Step 2 of Appendix A. In the case of Corollary 3.1 it is sufficient to have only two observation instants T and t^*.

4. Observability With Static Observations: the Case $\Omega = (0,1)$.

It is well known that the general solution of (3.1) for $n = 1$ admits the following representation:

$$\varphi(x,t) = \sqrt{2}\sum_{k=1}^{\infty}(\varphi_{0k}\cos\pi k(t-T) + \frac{\varphi_{1k}}{\pi k}\sin\pi k(t-T))\sin\pi kx, \tag{4.1}$$

where

$$\varphi_{0k} = \sqrt{2}\int_0^1 \varphi_0(x)\sin\pi kx\,dx, \quad \varphi_{1k} = \sqrt{2}\int_0^1 \varphi_1(x)\sin\pi kx\,dx.$$

The series in (4.1) with $\{\varphi_0, \varphi_1\}$ satisfying

$$\{\varphi_0, \varphi_1\} \in H_0^1(0,1)\times L^2(0,1) \tag{4.2}$$

converges in $C[0,1]$ uniformly over $t \in [0,T]$ which ensures well-posedness of (3.3), and the following estimate holds ([13], [15], pp. 155, 307):

$$\max_{t\in[0,T]}\|\varphi(\cdot,t)\|_{C[0,1]} \leq \text{const}\,(\|\varphi_0\|^2_{H^1(0,1)} + \|\varphi_1\|^2_{L^2(0,1)})^{1/2}.$$

The observations (3.4), (3.5) in their turn are well-defined if

$$\{\varphi_0, \varphi_1\} \in H_D^2(0,1)\times H_0^1(0,1) \tag{4.3}$$

and the series in (4.1) converges then with its first derivatives with respect to x and t in $C[0,1]$ uniformly over $t \in [0,T]$. The following estimate is verified (e.g., [15, pp. 155, 307]):

$$\max_{t\in[0,T]}\{\|\varphi(\cdot,t),\ \varphi_x(\cdot,t),\ \varphi_t(\cdot,t)\|_{C[0,1]})\}$$
$$\leq \text{const}\,(\|\varphi_0\|^2_{H^2(0,1)} + \|\varphi_1\|^2_{H^1(0,1)})^{1/2}.$$

Proof of Theorem 3.1. **1.** Note first that, by standard results from harmonic analysis, see, e.g., [16], the observation time $T = 2$ cannot be improved. Furthermore, if $\bar{x}$ is an algebraic number of degree l, then by Liouville's theorem [17], p. 21:

$$|\,k\bar{x} - m\,| \geq \frac{\text{const}}{k^{l-1}} \tag{4.4}$$

for any integers m and k, $k > 0$. When $l = 2$, this yields

$$|\sin\pi k\bar{x}\,| \geq \frac{\text{const}}{k}, \quad k = 1, \dots . \tag{4.5}$$

We proceed with the proof of exact observability at $T = 2$ by the analysis of the system (3.1)-(3.3). Since the system $\{\sin\pi k(t-2), \cos\pi k(t-2)\}_{k=1}^{\infty}$ is orthonormalized in $L^2(0,2)$, from (4.1), (3.2), (3.3) it follows:

$$|\,\varphi_{0k}\,| = \frac{z_{0k}}{\sqrt{2}\sin\pi k\bar{x}}, \quad |\,\varphi_{1k}\,| = \frac{\pi k\,z_{1k}}{\sqrt{2}\sin\pi k\bar{x}}, \quad k = 1, \dots, \tag{4.6}$$

where

$$z_{0k} = \int_0^2 z(t)\sin\pi k(t-2)\,dt, \quad z_{1k} = \int_0^2 z(t)\cos\pi k(t-2)\,dt.$$

Recall now that, if $\partial\Omega$ is of class C^s (where s is a positive integer), then the usual norm of $H^s_D(\Omega)$ is equivalent to the following one ([15], p. 230):

$$\|\varphi\| \;=\; \left(\sum_{k=1}^{\infty}(\lambda_k)^s \;\left(\int_{\Omega}\varphi(x)\,\omega_k(x)\,dx\right)^2\right)^{1/2}. \tag{4.7}$$

Here $\{\lambda_k\}_{k=1}^{\infty}$ $(\lambda_{k+1} \;\geq\; \lambda_k;\;\; \lambda_k \to +\infty)$, $\{\omega_k\}_{k=1}^{\infty}$ are the eigenvalues and respective eigenfunctions (orthonormalized in $L^2(\Omega)$) of the spectral problem: $\Delta\omega = -\lambda\omega,\ \omega \in H^s_D(\Omega)$.

Take any pair $\{\psi_0, \psi_1\} \in H^2_D(0,1) \times H^1_0(0,1)$. Then (4.7) along with Parseval's formula yield

$$\sum_{k=1}^{\infty}\varphi_{0k}\psi_{1k} \;\leq\; \text{const}\;\|z\|_{L^2(0,2)}\|\psi_1\|_{H^1(0,1)}$$

and

$$\sum_{k=1}^{\infty}\varphi_{1k}\psi_{0k} \;\leq \text{const}\;\|z\|_{L^2(0,2)}\|\psi_1\|_{H^2(0,1)},$$

where

$$\psi_{0k} = \;\sqrt{2}\int_0^1 \psi_0(x)\;\sin\pi kx\;dx, \qquad \psi_{1k} = \;\sqrt{2}\int_0^1 \psi_1(x)\;\sin\pi kx\;dx.$$

From the latter the first assertion of Theorem 3.1 follows immediately. The second assertion can be established analogously.

2. The general solution of the system (3.1), (4.3) can also be represented by d'Alembert's formula:

$$\varphi(x,t) = \frac{1}{2}(\varphi_0(x+T-t) \;+\; \varphi_0(x-T+t)) \;-\; \frac{1}{2}\int_{x-T+t}^{x+T-t}\varphi_1(\tau)\;d\tau, \tag{4.8}$$

where the domains of the functions $\varphi_0(x)$ and $\varphi_1(x)$ are extended to R as follows:

$$\varphi_i(x) = -\varphi_i(-x), \quad \varphi_i(x) = -\varphi_i(2-x), \quad x \in (-\infty,+\infty),\ i = 0,1. \tag{4.9a}$$

In particular,

$$\varphi_0'(x) \;=\; +\,\varphi_0'(-x), \quad x \in (-\infty,+\infty). \tag{4.9b}$$

Observe now that for the observations (3.6) we have

$$\varphi_x(\bar{x},t) = \frac{1}{2}(\varphi_0'(\bar{x}+T-t)+\varphi_0'(\bar{x}-T+t)) \;-\; \frac{1}{2}(\varphi_1(\bar{x}+T-t)-\varphi_1(\bar{x}-T+t)),$$

$$\varphi_t(\bar{x},t) = \frac{1}{2}(-\varphi_0'(\bar{x}+T-t)+\varphi_0'(\bar{x}-T+t)) \;-\; \frac{1}{2}(-\varphi_1(\bar{x}+T-t)-\varphi_1(\bar{x}-T+t)).$$

Hence,

$$\varphi_x(\bar{x},t) + \varphi_t(\bar{x},t) = \varphi_0'(\bar{x}-T+t) \;+\; \varphi_1(\bar{x}-T+t). \tag{4.10a}$$

$$\varphi_x(\bar{x},t) - \varphi_t(\bar{x},t) = \varphi_0'(\bar{x}+T-t) \; - \; \varphi_1(\bar{x}+T-t). \tag{4.10b}$$

The relations (4.10) yield

$$\begin{aligned}\int_0^T (\varphi_x(\bar{x},t) + \varphi_t(\bar{x},t))^2 dt = &\int_{\bar{x}-T}^{\bar{x}} (\varphi_0'^2(x) + \varphi_1^2(x))dx \\ &+ 2\int_{\bar{x}-T}^{\bar{x}} \varphi_0'(x)\varphi_1(x)dx,\end{aligned} \tag{4.11a}$$

$$\begin{aligned}\int_0^T (\varphi_x(\bar{x},t) - \varphi_t(\bar{x},t))^2 dt = &\int_{\bar{x}}^{\bar{x}+T} (\varphi_0'^2(x) + \varphi_1^2(x))dx \\ &- 2\int_{\bar{x}}^{\bar{x}+T} \varphi_0'(x)\varphi_1(x)dx.\end{aligned} \tag{4.11b}$$

The relations (4.9) ensure the cancellation of the last term in the right-hand side of (4.11a) for $T = 2\bar{x}$ and for $T = 2(1-\bar{x})$ in (4.11b). This provides the following exact formula for the energy:

$$\begin{aligned}\int_0^1 (\varphi_0'^2(x) + \varphi_1^2(x))dx = &\frac{1}{2}\int_0^{2\bar{x}} (\varphi_x(\bar{x},t) + \varphi_t(\bar{x},t))^2 dt \\ &+ \frac{1}{2}\int_0^{2(1-\bar{x})} (\varphi_x(\bar{x},t) - \varphi_t(\bar{x},t))^2 dt,\end{aligned} \tag{4.12}$$

which gives us the time for observability as required by Theorem 3.1. Finally, (4.11) allows us to construct an example of a sequence $\{\varphi_0^i, \varphi_1^i\}_{i=1}^{\infty}$ which can prove the minimality of time $T = 2 \times \max\{1-\bar{x}, \bar{x}\}$. Indeed, if, say, $\bar{x} < 1/2$, such a sequence can be taken to satisfy: $\varphi_{0i}^{i\prime} \equiv \varphi_1^i$ and distinct from zero only on the sequence of intervals $(\bar{x}, \bar{x}+\delta_i)$, $i = 1, \ldots$ $\delta_i \to 0+$, $i \to \infty$. This completes the proof of Theorem 3.1. ∎

Remark 4.1. The algebraic numbers are countable [17] p. 19. Along the lines (4.4)-(4.7) Theorem 3.1 can immediately be extended to all of them. On the other hand, the transcendental Liouville's numbers give us an opposite example. Indeed, as it was noticed in [1], these numbers are associated with the "worst" locations of the static point control (1.2). For example, if $\bar{x} = \sum_{j=1}^{\infty} 10^{-j!}$, then the series $\sum_{k=1}^{\infty} \frac{1}{(\pi k)^{2s} \sin^2 \pi k \bar{x}}$ diverges for any positive integer s. The latter does not allow one to extend the argument of Theorem 3.1 to all the irrational numbers.

5. Observability With Moving Observations: the Case $\Omega = (0,1)$

Proof of Theorem 3.2. We deal below with the observation (3.3). The cases (3.4) and (3.5) can be treated analogously.

Step 1: Basic auxiliary estimate. Fix T. Due to Parseval's formula, (4.1) implies (if one excludes the trivial case):

$$\|\varphi(\cdot,t)\|^2_{L^2(0,1)} \geq (\varphi^2_{0k} + \frac{\varphi^2_{1k}}{(\pi k)^2}) \sin^2(\pi k(t-T) + \alpha_k), \quad \forall t \in [0,T],\ k = 1,\dots, \tag{5.1}$$

where $\alpha_k \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$ and

$$|\sin\alpha_k| = \frac{|\varphi_{0k}|}{(\varphi^2_{0k} + \frac{\varphi^2_{1k}}{(\pi k)^2})^{1/2}}, \quad |\cos\alpha_k| = \frac{|\frac{\varphi_{1k}}{\pi k}|}{(\varphi^2_{0k} + \frac{\varphi^2_{1k}}{(\pi k)^2})^{1/2}}.$$

Since $\Omega = (0,1)$, this gives us the following basic estimate:

$$\sin^2(\pi k(t-T) + \alpha_k)(\varphi^2_{0k} + \frac{\varphi^2_{1k}}{(\pi k)^2}) \leq \|\varphi(\cdot,t)\|^2_{C[0,1]}, \quad \forall t \in [0,T],\ k = 1,\dots. \tag{5.2}$$

Step 2: Selection of observation instants. Given $\varepsilon \in (0,\pi/4)$, put

$$t^1_k = -\frac{1}{2k} + T, \quad t^2_k = -\frac{1+2\varepsilon/\pi}{2k} + T, \quad k = 1,\dots. \tag{5.3}$$

It is readily seen that

$$\sin(\pi k(t^1_k - T) + \alpha_k) = -\cos\alpha_k, \quad \sin(\pi k(t^2_k - T) + \alpha_k) = -\cos(-\varepsilon + \alpha_k).$$

Hence, $\exists \gamma_* = \gamma_*(\varepsilon) > 0$ such that

$$\max_{i=1,2} |\sin(\pi k(t^i_k - T) + \alpha_k)| \geq \gamma_* \quad \forall\, \alpha_k \in [-\frac{\pi}{2}, \frac{\pi}{2}],\ k = 1,\dots. \tag{5.4}$$

Without loss of generality, we can assume further that all $t^1_k, t^2_k \in (0,T)$, $k = 1,\dots$.

For any positive integers k,m select in an arbitrary way two distinct (as well as with respect to different k,m, which is due to our aim to employ a single-point sensor) monotone sequences $\{s_l(k,m)\}^\infty_{l=1}$, $\{\tau_l(k,m)\}^\infty_{l=1} \subset (0,T)$ such that:

(i)

$$\lim_{l\to\infty} s_l(k,m) = t^1_k, \quad \lim_{l\to\infty} \tau_l(k,m) = t^2_k, \quad k,m = 1,\dots; \tag{5.5a}$$

(ii) the sequences $\{s_1(k,m)\}^\infty_{m=1}$ and $\{\tau_1(k,m)\}^\infty_{m=1}$, $k = 1,\dots$ are monotone;

(iii)

$$\lim_{m\to\infty} s_1(k,m) = t^1_k, \quad \lim_{m\to\infty} \tau_1(k,m) = t^2_k \quad k = 1,\dots; \tag{5.5b}$$

(iv) $T, \{t^i_k\}^\infty_{k=1}$, $i = 1,2$ are the *only* possible limit points of the set $\{s_l(k,m), \tau_l(k,m) \mid l,k,m = 1,\dots\}$.

Step 3: Net. Denote by $S[0,1]$ the closed linear manifold in $C[0,1]$ spanned by $\{\sin\pi kx\}^\infty_{k=1}$, $S[0,1] = \{p(x) \mid p(x) = \sum^\infty_{k=1} p_k \sin\pi kx\} \subset C[0,1]$. In particular, all the solutions of (3.1), (4.2) lie in $S[0,1]$ $\forall t \in [0,T]$.

Fix an arbitrary $\delta > 0$. By making use of separability of $C[0,1]$ (or of Lemma 5 in [4], p. 50), select in its topological subset $S[0,1]$ a countable δ-net $\{p_l\}_{l=1}^{\infty} \subset S[0,1]$ (this can be done in infinitely many ways). In other words, for any element $p \in S[0,1]$ there exists an element p_{l*} such that $\|p - p_{l*}\|_{C[0,1]} \leq \delta$.

Step 4: Selection of an observation curve. Consider any function $\hat{x}(t)$, $t \in (0,T)$, which satisfies the following requirements: (i) it is continuous everywhere in $(0,T)$ except, maybe, for $t = t_k^i$, $i = 1,2$, $k = 1,\dots$; (ii) $\hat{x}(t) \in (0,1)$, $t \in (0,T)\backslash\{t_k^i\}_{i=1,k=1}^{2,\infty}$; (iii):

$$\hat{x}(s_l(m,k)) = \hat{x}(\tau_l(m,k)) = x_l, \quad l,k,m = 1,\dots, \tag{5.6}$$

where

$$x_l = \arg\max_{x \in [0,1]} | \, p_l(x) \, |, \quad l = 1,\dots . \tag{5.7}$$

The last optimization problem may have several solutions. If so, we take any of them. Clearly, if $p_l \neq 0$, then $x_l \neq 0, 1$ either.

Step 5: Verification. We show now that any curve satisfying the requirements of Step 4 satisfies the assertion 1. in Theorem 3.2. Fix any positive integer k. Take an arbitrary solution φ of the system (3.1), (4.2). It is readily seen that there exists $\gamma\ (= \gamma(\varphi,k)) > 0$ such that

$$\|\varphi(\cdot,t) - \varphi(\cdot,t_k^i)\|_{C[0,1]} \leq \delta \quad \forall\, t \in (t_k^i - \gamma, t_k^i), \ \ i = 1,2. \tag{5.8}$$

Assume that for our particular solution the maximum in the left-hand side of (5.4) is achieved for $i = 1$. Find next an element p_{l*} in the δ-net constructed in Step 3 such that

$$\|p_{l*}(\cdot) - \varphi(\cdot,t_k^1)\|_{C[0,1]} \leq \delta. \tag{5.9}$$

Take any instant $s_{l*}(k,m^*) \in (t_k^1 - \gamma, t_k^1)$. Due to (5.5b), such an instant always exists for m^* big enough. Combining (5.2), (5.4), (5.6)-(5.9) yields the following chain of estimates:

$$\begin{aligned}
\gamma_* \left(\varphi_{0k}^2 + \frac{\varphi_{1k}^2}{(\pi k)^2} \right)^{1/2} &\leq \|\varphi(\cdot,t_k^1)\|_{C[0,1]} \\
&\leq \|p_{l*}(\cdot)\|_{C[0,1]} + \delta \\
&= | \, p_{l*}(x_{l*}) \, | + \delta \\
&= | \, p_{l*}(\hat{x}(s_{l*}(k,m^*)) \, | + \delta \\
&\leq | \, \varphi(\hat{x}(s_{l*}(k,m^*), t_k^1) \, | + 2\delta \\
&\leq | \, \varphi(\hat{x}(s_{l*}(k,m^*), s_{l*}(k,m^*)) \, | + 3\delta.
\end{aligned} \tag{5.10}$$

Thus, we arrive at:

$$\left(\varphi_{0k}^2 + \frac{\varphi_{1k}^2}{(\pi k)^2} \right)^{1/2} \leq \frac{1}{\gamma_*} (\|\varphi(\hat{x}(\cdot),\cdot)\|_{L^\infty(0,T)} + 3\delta). \tag{5.11}$$

Recall now that (5.11) was derived uniformly with respect to the choice of φ. Hence, replacing φ by $\alpha\varphi$, $\alpha \in R$ yields with $\alpha \to \infty$:

$$(5.12) \qquad \left(\varphi_{0k}^2 + \frac{\varphi_{1k}^2}{(\pi k)^2}\right)^{1/2} \leq \frac{1}{\gamma_*} \|\varphi(\hat{x}(\cdot), \cdot)\|_{L^\infty(0,T)}, \quad k = 1, \dots .$$

The last estimate implies the assertion 1. in Theorem 3.2. The proof of the assertion 2. is analogous. In particular, instead of (5.12) one can obtain for the observation (3.4):

$$(5.13) \qquad ((\pi k \varphi_{0k})^2 + \varphi_{1k}^2)^{1/2} \leq \frac{1}{\gamma_*} \|\varphi_x(\hat{x}(\cdot), \cdot)\|_{L^\infty(0,T)}, \quad k = 1, \dots .$$

$L^\infty(0,T)$-norm in (5.12) and (5.13) can equally be replaced by the space $C((0,T)\backslash\{t_k^i\}_{i=1,k=1}^{2,\infty})$-one. This completes the proof of Theorem 3.2. ∎

Remark 5.1. The straightforward extension of the above scheme to the case of the observation (3.4) admits the situation when a part of the skeleton for an observation curve lies on the boundary of $\Omega = (0,1)$. This is due to the fact that $\cos \pi k x$, $k = 1, \dots$ do not vanish at $x = 0, 1$. However, it is readily seen that all such points (if they exist) can be replaced by strictly internal ones close enough to preserve (5.13) (with, maybe, different γ_*). The same comment can be made in the multidimensional case (see Appendix A, Remark A.1).

Proof of Corollary 3.1. Set $t_k^1 = T$, $t_k^2 = t^* \in (0,T)$, $k = 1, \dots,$ where t^* is such that $t_a = T - t^*$ is an algebraic number of degree 2. Observe that (4.4) yields the existence of $\varepsilon_k, k = 1, \dots$ such that

$$|\sin(\pi k t_a + \alpha_k)| = |\sin(\varepsilon_k + \alpha_k)|, \qquad \frac{\text{const}}{k} \leq |\varepsilon_k| \leq \frac{\pi}{2},$$

$$\varepsilon_k = \pi \times \min\{k t_a - [k t_a], k t_a + 1 - [k t_a]\}.$$

Instead of (5.4), we obtain,

$$\max_{i=1,2}\{|\sin(\pi k(T - t_k^i) + \alpha_k)|\} = \max\{|\sin \alpha_k|, |\sin(\pi k t_a + \alpha_k)|\} \geq \frac{\text{const}}{k},$$

$$\forall \alpha_k \in [-\frac{\pi}{2}, \frac{\pi}{2}], \quad k = 1, \dots .$$

The rest of the proof follows Steps 1-5 in the above. ∎

6. Proofs of Theorems 2.1 and 2.2

We begin by studying the regularity properties of (1.1). Denote (see [15], p. 230)

$$H_D^{s-1}(Q) = \{f \mid f \in H^{s-1}(Q), \ f|_{\partial\Omega\times[0,T]} = \dots = \Delta^{[s/2]-1} f|_{\partial\Omega\times[0,T]} = 0\},$$

$$s > 1, \quad H_D^0(Q) = L^2(Q), \quad H_D^{-s}(Q) = [H_D^s(Q)]'.$$

Theorem 6.1. *Let* $\partial\Omega$ *be of class* $C^{[n/2]+1}$ *in the case of point control (1.2) and of class* $C^{[n/2]+2}$ *in the case of point controls (1.3)/(1.4). Let* $\hat{x}(\cdot)$ *be an arbitrary measurable function such that* $\hat{x}(t) \in \Omega$ *a.e. on* $(0,T)$. *Then with* $V = L^2(0,T)$ *or* $[L^\infty(0,T)]'$ *for (1.2), (1.4) and* $V = L^2(0,T;R^n)$ *or* $[L^\infty(0,T;R^n)]'$ *for (1.3):*

(i) *the problem (1.1), (1.2) admits a unique solution in the space* $H_D^{-[n/2]}(Q)$ *and the mapping*

$$v \to \{y, \; y\,|_{t=T}, \; y_t\,|_{t=T}\}$$

is linear continuous from V *into* $H_D^{-[n/2]}(Q) \times H_D^{-[n/2]}(\Omega)$ $\times H_D^{-[n/2]-1}(\Omega)$.

(ii) *[7]: the problems (1.1), (1.3) or (1.4) admit unique solutions from* $H_D^{-[n/2]-1}(Q)$ *and the mapping*

$$v \to \{y, \; y\,|_{t=T}, \; y_t\,|_{t=T}\}$$

is linear continuous from V *into* $H_D^{-[n/2]-1}(Q) \times H_D^{-[n/2]-1}(\Omega)$ $\times H_D^{-[n/2]-2}(\Omega)$.

(iii) *All the above mappings* $y \to \{y\,|_{t=T}, \; y_t\,|_{t=T}\}$ *are injective.*

Corollary 6.1. *Let* $\hat{x}(\cdot)$ *be continuous everywhere on* $(0,T)$ *except a countable number of isolated points* $\{t_i\}_{i=1}^\infty$. *Then the results of Theorem 6.1 hold accordingly for* $V = [C((0,T)\backslash\{t_i\}_{i=1}^\infty)]', \; [C((0,T)\backslash\{t_i\}_{i=1}^\infty; R^n)]'$.

The assertion 6.1(ii) was proven by transposition in [7]. The assertion 6.1(i) and Corollary 6.1 can be established in a similar way. The terminal conditions in Theorem 6.1 and Corollary 6.1 satisfy the following identity:

$$(6.1) \quad <\varphi_1, y\,|_{t=T}>_{\Phi_0} \; - \; <\varphi_0, y_t\,|_{t=T}>_{\Phi_1} = < -v, \mathbf{G}(\hat{x}(\cdot))\varphi\} >_V, \; v \in V,$$

which is verified for any solution φ to (3.1) with

$$(6.2) \qquad \{\varphi_0, \varphi_1\} \in H_D^{[n/2]+1}(\Omega) \times H_D^{[n/2]}(\Omega)$$

for the system (1.1), (1.2) and with

$$(6.3) \qquad \{\varphi_0, \varphi_1\} \in H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega)$$

for the systems (1.1), (1.3)/(1.4). In the above $< \cdot, \cdot >_B$ indicates the duality associated with the Banach space B; Φ_0, Φ_1 are the Hilbert spaces for the terminal pair $\{y\,|_{t=T}, y_t\,|_{t=T}\}$, as they are specified in Theorem 6.1.

Remark 6.1. The injectivity between the solution of (1.1) and its terminal pair, defined by (6.1), is treated in Theorem 6.1 in the following sense: the solution of equation (1.1) evolving in backward time from this terminal pair coincides (as an element of $H_D^{-[n/2]}(Q)$ or $H_D^{-[n/2]-1}(Q)$) with the solution of the direct problem (1.1). In particular, the latter can be continuous in time in some other functional space. A detailed study of the regularity of (1.1) requires a separate investigation. The following lemma and Example 6.1 expose the problem arising here.

Lemma 6.1. *Let* $\Omega = (0,1)$, $V = L^2(0,T)$ *and* $T > 0$ *be given. Then the solution of (1.1), (1.2) lies in* $C([0,T]; L^2(0,1) \times H^{-1}(0,1))$.

Proof. The solution of (1.1), (1.2) can be represented as follows:

$$y(x,t) = \sum_{k=1}^{\infty} \frac{2}{\pi k} \int_0^t v(\tau) \sin \pi k(t-\tau)\ \sin \pi k \hat{x}(\tau)\, d\tau\ \sin \pi k x, \tag{6.4}$$

$$y_t(x,t) = 2 \sum_{k=1}^{\infty} \int_0^t v(\tau) \cos \pi k(t-\tau) \sin \pi k \hat{x}(\tau)\ d\tau\ \sin \pi k x.$$

It is readily seen that the first series converges in $C([0,T]; L^2(0,1))$, and the second (see (4.7)) in $C([0,T]; H^{-1}(0,1))$. ■

As it was shown in [18], [20], the solution of (1.1), (1.2), with $\Omega = (0,1)$, $V = L^2(0,T)$, $\hat{x}(\cdot) \equiv \bar{x}$ lies in $C([0,T]; H_0^1(0,1) \times L^2(0,1))$. However, the following example shows that Lemma 6.1 cannot be embedded in this result.

Example 6.1. Let $\Omega = (0,1)$, $T = 1$, $\hat{x}(t) = (1-t)$, $v(t) = 1$, $t \in (0,1)$. Then,

$$y(x,1) = 2 \sum_{k=1}^{\infty} \frac{1}{\pi k} \int_0^1 \sin^2 \pi k(1-\tau)\, d\tau\ \sin \pi k x\ = \sum_{k=1}^{\infty} \frac{1}{\pi k}\ \sin \pi k x.$$

Hence, $y(\cdot,1) \notin H_0^1(0,1)$. ■

Proofs of Theorems 2.1, 2.2. Those assertions of Theorems 2.1, 2.2, and Corollary 2.1 dealing with approximate controllability follow straightforward from (6.1), Theorems 3.1-3.3, and Corollary 3.1 by applying the standard Hilbert space duality argument, and those dealing with exact controllability follow by a *direct duality method* - see, e.g., [3], pp. 194-195, [19] - applied in the form discussed in detail in [7]. This method is related to establishing a bound from below for the norm of the operator dual to the solution one which, in turn, is equivalent to exact controllability. The scheme of our proofs employs a suitable $L^\infty(0,T)$- or $C(((0,T)\backslash\{t_i\}_{i=1}^{\infty})$-exact observability estimate for the corresponding dual system (3.1), (3.2) with respect to the norm dual of the norm in question (see Theorems 2.1, 2.2) on a narrower space consistent with the well-posedness of the observations (see Theorems 3.1-3.3) which, in turn, is dense in the space dual of the controllability space of interest. To complete the proof, we show then, by making use of the regularity results discussed in the above, that the operator dual (via (6.1)) of the *final state→output* mapping (via (3.2)) coincides with the solution operator of system (1.1). ■

7. Appendix A: Proof of Theorem 3.3

The sketch of the proof below is given for the observation (3.3) and follows Step 1-5 of Section 5, while emphasizing the difference between the one dimensional and the multidimensional cases.

Recall that the problem (3.1), (6.2) admits a unique solution from the space $H^{[n/2]+1}(Q)$ and the following estimate holds (see, e.g., [15], pp. 307-308 for details):

$$(A.1)\qquad \sum_{p=0}^{[n/2]+1} \left\| \frac{\partial^p \varphi}{\partial t^p} \right\|^2_{H^{[n/2]+1-p}(\Omega)} \le \text{const}(\|\varphi_0\|^2_{H^{[n/2]+1}(\Omega)} + \|\varphi_1\|^2_{H^{[n/2]}(\Omega)} + \|f\|^2_{H^{[n/2]}(Q)}), \quad \forall t \in [0,T].$$

The mixed problem (3.1), (6.3) in its turn admits a unique solution from the space $H^{[n/2]+2}(Q)$ and the following estimate holds (see, e.g., [15], pp. 307-308 for details):

$$\sum_{p=0}^{[n/2]+2} \left\| \frac{\partial^p \varphi}{\partial t^p} \right\|^2_{H^{[n/2]+2-p}(\Omega)} \le \text{const}(\|\varphi_0\|^2_{H^{[n/2]+2}(\Omega)} + \|\varphi_1\|^2_{H^{[n/2]+1}(\Omega)} + \|f\|^2_{H^{[n/2]+1}(Q)}), \quad \forall t \in [0,T].$$

The principal difference between the one-dimensional and the general cases is that that the latter admits multiple eigenvalues. Let $\{\beta_k\}_{k=1}^\infty$ denote the sequence of all the distinct eigenvalues. Denote their multiplicities and the respective eigenfunctions accordingly by J_k and $\{\omega_{kj}\}_{j=1,k=1}^{J_k,\infty}$.

Step 1. Fix $T > 0$. The general solution of (3.1) admits the following representation:

$$(A.2)\quad \varphi(x,t) = \sum_{k=1}^\infty \sum_{j=1}^{J_k} (\varphi_{0kj} \cos \sqrt{\beta_k}(t-T) + \frac{\varphi_{1kj}}{\sqrt{\beta_k}} \sin \sqrt{\beta_k}(t-T))\, \omega_{kj}(x),$$

where

$$\varphi_{0kj} = \int_\Omega \varphi_0(x)\, \omega_{kj}(x)\, dx, \qquad \varphi_{1kj} = \int_\Omega \varphi_1(x)\, \omega_{kj}(x)\, dx.$$

With $\{\varphi_0, \varphi_1\} \in H_D^{[n/2]+1}(\Omega) \times H_D^{[n/2]}(\Omega)$, due to (A.1) and the corresponding embedding theorem, the series in (A.2) converges in $C(\bar\Omega \times [0,T])$. From (A.2) it follows (instead of (5.2)):

$$(A.3)\qquad \begin{aligned} \text{meas}\{\Omega\} \|\varphi(\cdot,t)\|^2_{C(\bar\Omega)} &\ge \|\varphi(\cdot,t)\|^2_{L^2(\Omega)} \\ &\ge \left(\varphi_{0kj}^2 + \frac{\varphi_{1kj}^2}{(\sqrt{\beta_k})^2} \right) \sin^2\left(\sqrt{\beta_k}(t-T) + \alpha_{kj} \right), \\ &\forall t \in [0,T],\ \forall\, k = 1, \dots,\ j = 1, \dots, J_k, \end{aligned}$$

where, similar to (5.1),

$$\alpha_{kj} \in [-\frac{\pi}{2}, +\frac{\pi}{2}], \quad |\sin \alpha_{kj}| = \frac{|\varphi_{0kj}|}{(\varphi_{0kj}^2 + \frac{\varphi_{1kj}^2}{\beta_k})^{1/2}},$$

$$|\cos \alpha_k| = \frac{\frac{|\varphi_{1kj}|}{\sqrt{\beta_k}}}{(\varphi_{0kj}^2 + \frac{\varphi_{1kj}^2}{\beta_k})^{1/2}}.$$

Step 2. Given $\varepsilon \in (0, \pi/4)$, set

$$t_k^1 = -\frac{1}{2\sqrt{\beta_k}} + T, \quad t_k^2 = -\frac{1+2\varepsilon}{2\sqrt{\beta_k}} + T, \quad k = 1, \dots .$$

Without loss of generality, as in Section 5, one can assume further that $t_k^1, t_k^2 \in (0, T)$, $k = 1, \dots$. It is readily seen that

$$\sin(\sqrt{\beta_k}(t_k^1 - T) + \alpha_{kj}) = -\cos\alpha_{kj}, \ \sin(\beta_k(t_k^2 - T) + \alpha_{kj}) = -\cos(-\varepsilon + \alpha_{kj}).$$

Hence, as in (5.4), we obtain

$$\max_{i=1,2}\{| \sin(\beta_k(T - t_k^i) + \alpha_{kj}) |\} \geq \text{(some) } \gamma_* > 0, \ k = 1, \dots , \ j = 1, \dots , j_k,$$
$$\forall\, \alpha_{jk} \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

As in Step 2 of Section 5, for any positive integers k, m select two arbitrary distinct monotone sequences $\{s_k(l,m)\}_{l=1}^\infty$ and $\{\tau_l(k,m)\}_{l=1}^\infty$.

Step 3. Denote by $S(\bar\Omega)$ the closed linear manifold in $C(\bar\Omega)$ spanned by $\{\omega_{kj}\}_{k=1,j=1}^{\infty,J_k}$. In particular, all the solutions of (3.1), (6.2) belong to $S(\bar\Omega)$, $\forall t \in [0,T]$. Fix an arbitrary $\delta > 0$. Select in its topological subset $S(\bar\Omega)$ a countable δ-net $\{p_l\}_{l=1}^\infty \subset S(\bar\Omega)$.

By taking into account the well known asymptotic properties of the (multiple) eigenvalues λ_k (namely: $\exists C_1, C_2 > 0$ and a positive integer k_0 such that (see, e.g., [15]): $C_1 k^{2/n} \leq \lambda_k \leq C_2 k^{2/n}$, $k \geq k_0$) and formula (4.7), *Steps 4-5* in the multidimensional case are as much the same as in Section 5 (though, see Remark A.1 below). The presence of multiple eigenvalues does not create any difficulties in the general case. In fact, we can use the same pair of instants $\{t_k^1, t_k^2\}$ for the evaluation of all the coefficients $\{\varphi_{0kj},\ \varphi_{1kj}\}$ associated with the multiple eigenvalue β_k. This completes the proof of Theorem 3.3. ∎

Remark A.1. In the general case the observation (3.4) has dimension n. Instead of (A.2) we have

$$(A.4) \qquad \nabla\varphi(x,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{J_k}(\varphi_{0kj}\cos\sqrt{\beta_k}(t-T) \ + \ \frac{\varphi_{1kj}}{\sqrt{\beta_k}}\sin\sqrt{\beta_k}(t-T))\ \nabla\omega_{kj}(x).$$

Noticing that

$$< \nabla\omega_k(\cdot), \nabla\omega_m(\cdot) >_{L^2(\Omega)} = \lambda_k\delta_{km}, \quad \delta_{km} = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases}$$

we obtain instead of (A.3):

$$\text{meas}\{\Omega\}\ \|\nabla\varphi(\cdot,t)\|^2_{C(\bar\Omega)} \geq \|\nabla\varphi(\cdot,t)\|^2_{L^2(\Omega)}$$
$$\geq \beta_k\left(\varphi_{0kj}^2 + \frac{\varphi_{1kj}^2}{(\sqrt{\beta_k})^2}\right)\sin^2\left(\sqrt{\beta_k}(t-T) + \alpha_{kj}\right).$$

The rest of the proof of Theorem 3.3 follows the lines of the above argument.

References

[1] A. G. Butkovski, *Structural Theory of Distributed Systems*, Ellis Horwood, Chichester, 1983.

[2] S. Dolecki, Observability for regular processes, J. Math. Anal. Appl. **58** (1977), 178-188.

[3] S. Dolecki and D.L. Russell, A general theory of observation and control, SIAM J. Control Optim. **15** (1977), 185-219.

[4] N. Dunford and J.T. Schwartz, *Linear Operators,* Interscience, New York, 1963.

[5] L. F. Ho, *Exact controllability of the one-dimensional wave equation with locally distributed control*, SIAM J. Control Optim. **28** (1990), 733-748.

[6] A. Y. Khapalov, *Observability of hyperbolic systems with interior moving sensors*, Lecture Notes in Control and Inform. Sci. **185**, (eds. R. F. Curtain, A. Bensoussan, J.-L. Lions), Springer-Verlag, New York, 1993, 489-499.

[7] A.Y. Khapalov, *Controllability of the wave equation with moving point control*, Appl. Math. Optim. **31** (1995), 155-175.

[8] A.Y. Khapalov, *Exponential decay for the one dimensional wave equation with internal pointwise damping*, Technical Report TRECE, **#95.006**, Oregon State Univ., Corvallis, Oregon, (to appear).

[9] I. M. Lasiecka, J.-L. Lions and R. Triggiani, Nonhomogeneous boundary value problems for second order hyperbolic operators, J. Math. Pures Appl. **65** (1986), 149-192.

[10] I. M. Lasiecka and R. Triggiani, *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*, Lecture Notes in Control and Inform. Sci., **164**, Springer-Verlag, New York, 1991.

[11] J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev. **30** (1988), 1-68.

[12] J.-L. Lions, *Contrôlabilité Exacte et Stabilisation de Systémes Distribués, I, Contrôlabilité Exacte*, Masson, Paris, 1988.

[13] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, **I-III**, Springer-Verlag, 1972.

[14] Y. Meyer, *Etude d'un modéle mathématique issu du contrôle des structures spatiales déformables*, In Nonlinear Partial Differential Equations and Their Applications, Collége de France Seminar, **VII**, (eds. H. Brézis and J.-L. Lions), Pitman, Boston, 1985, 234-242.

[15] V. P. Mikhailov, *Partial Differential Equations*, Mir, Moscow, 1978.

[16] D. L. Russell, *Nonharmonic Fourier series in the control theory of distributed parameter systems*, J. Math. Anal. Appl. **18** (1967), 542-560.

[17] K.B. Stolarsky, *Algebraic Numbers and Diophantine Approximation*, Marcel Dekker, New York, 1974.

[18] R. Triggiani, *Regularity of wave and plate equation with interior point control*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **2** (1991), 307-315.

[19] R. Triggiani, *Lack of exact controllability for wave and plate equation with finitely many boundary controls*, Differential Integral Equations **4** (1991), 683-705.

[20] R. Triggiani, *Interior and boundary regularity of the wave equation with interior point control*, Differential Integral Equations **6** (1993), 111-129.

[21] R. Triggiani and D. Tataru, private communication.

Department of Electrical and Computer Engineering, Oregon State University, Corvallis, OR, 97331
E-mail address: khapala@ece.orst.edu