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Neighborhoods of Certain Class of Analytic Functions of Complex Order with Negative Coefficients

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Abstract

In this paper we prove several inclusion relations associated with (n,δ) *-neighborhood of certain subclasses of analytic functions of complex order with negative coefficients by making use of the familiar concept of neighborhoods of analytic functions.. Special cases of some of these inclusion relations are shown to yield known results.*

Keywords: *univalent functions, starlike functions, convex functions,* (n,δ) *-neighborhood and linear operator.*

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1 Introduction

Let $A(n)$ denote the class of functions f of the form :

(1.1) $$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad , \qquad \left(a_k \geq 0; \quad n \in N : \{1,2,3,...\} \right),$$

which are analytic in the unit disk $U = \{z :: z \in C \, and |z| < 1 \}$.

Following the works of Goodman [9] and Ruscheweyh [11], we define the (n,δ)-neighborhood of a function $f(z) \in A(n)$ by (see also [2], [3],[4], and [13])

(1.2) $$N_{n,\delta}(f) = \left\{ g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \right\}.$$

In particular, for the identity function

$$e(z) = z,$$

we immediately have

(1.4) $$N_{n,\delta}(e) = \left\{ g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=n+1}^{\infty} k|b_k| \leq \delta \right\}.$$

The above concept of (n,δ)-neighborhoods was extended and applied recently to families of analytically multivalent functions by Altintas et al. [6].The main object of the present paper is to investigate the (n,δ)-neighborhoods of several subclasses $A(n)$ of normalized analytic functions in U with negative and missing coefficients, which are introduced below by making use of the Ruscheweyh derivatives .

A function $f \in A(n)$ is said be starlike of complex order γ $(\gamma \in C \setminus \{0\})$,that is $f \in S_n^*(\gamma)$, if it satisfies the inequality:

(1.5) $$\operatorname{Re}\left\{ 1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \qquad \left(z \in U; \ \gamma \in C - \{0\} \right) .$$

Furthermore, a function $f \in A(n)$ is said be convex of complex order γ $(\gamma \in C \setminus \{0\})$, that is , $f \in C_n(\gamma)$, if it also satisfies the following inequality

(1.6) $$\operatorname{Re}\left\{ 1 + \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)} \right) \right\} > 0$$

$\left(z \in U; \ \gamma \in C \setminus \{0\} \right)$.

The classes $S_n^*(\gamma)$ and $C_n(\gamma)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr and Aou] (see also[5, 6]) .

Next, for the functions $f_j \, (j = 1,2)$ given by

(1.7) $$f(z) = z + \sum_{k=n+1}^{\infty} a_{k,j} z^k \qquad (j = 1,2),$$

Let $f_1 * f_2$ denote the Hadamard product (or convolution) of f_1 and f_2 , defined by

(1.8) $$(f_1 * f_2)(z) := z + \sum_{k=n+1}^{\infty} a_{k,1}\, a_{k,2} z^k =: (f_2 * f_1)(z) \ .$$

Now we define the function $\phi(a,c;z)$ by

(1.9) $$\phi(a,c;z) = z + \sum_{k=n+1}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \ .$$

For $c \neq 0,-1,-2,..., a \neq -1; z \in \Delta$ where $(\lambda)_n$ is the Pochhammer symbol defined by

(1.10) $$(\lambda)_n = \frac{\Gamma(n+\lambda)}{(\Gamma)} = \left\{ \begin{array}{ll} 1; & n = 0 \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+n-1)\ , & n \in N = \{1,2,...\} \end{array} \right\}$$

Carson and Shaffer [1] introduced a linear operator $L(a,c)$, by

$$L(a.c) f(z) = \phi(a,c;z) * f(z)$$

(1.11) $$= z + \sum_{k=n-1}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_k z^k \ , \ z \in \Delta.$$

Where * stands for the Hadamard product or convolution product of two power series

$$\varphi(z) = \sum_{k=n-1}^{\infty} \varphi_n z^n \quad \text{and} \quad \psi(z) = \sum_{k=n-1}^{\infty} \psi_n z^n$$

Defined by

$$(\varphi * \psi)(z) = \varphi(z) * \psi(z) = \sum_{k=n-1}^{\infty} \varphi_n \psi_n z^n$$

We note that $L(a,a) f(z) = f(z)$, $L(2,1) f(z) = z\, f'(z)$.

Finally, in term of the Carlson and Shaffer[1]defined by (1.11), let $S_n(\gamma,\lambda,\beta)$ denote the subclass of $A(n)$ consisting of functions f which satisfy the following inequality :

(1.12) $$\left| \frac{1}{\gamma}\left(\frac{z(L(a,c) f(z)}{L(a,c) f(z)} - 1 \right) \right| < \beta$$

$$(z \in U;\ \gamma \in C \setminus \{0\}\ ;\ 0 < \beta \leq 1).$$

Also let $R_n(\gamma,\lambda,\beta;\mu)$ denote the subclass of $A(n)$ consisting of functions f which satisfy the following inequality :

(1.13) $$\left| \frac{1}{\gamma}\left((1-\mu) \frac{L(a,c) f(z)}{z} + \mu(L(a,c)\ \ f(z))' - 1 \right) \right| < \beta$$

$$(z \in U;\ \gamma \in C \setminus \{0\}\ ;\quad 0 < \beta \leq 1\ ;\ 0 \leq \mu \leq 1).$$

Various further subclasses $S_n(\gamma,a,c,\beta)$ and $R_n(\gamma,a,c,\beta;\mu)$ with $\gamma = 1$ were studied in many earlier works (cf., e.g.,[7]); see also the references cited in these earlier works) . Clearly , in the case of (for example) the class $S_n(\gamma,a,c,\beta)$, we have

(1.14) $$S_n(\gamma,a,c,\beta)\subset S_n^*(\gamma) \quad\text{and}\quad S_n(\gamma,a,c,\beta)\subset C_n(\gamma)$$
$$(n\in N;\gamma\in C\setminus\{0\}).$$

2 Inclusion Relations Involving $N_{n,\delta}(e)$.

In our investigation of the inclusion relations involving $N_{n,\delta}(e)$, we shall require Lemma 1 and Lemma 2 below.

Lemma 1 Let the function $f\in A(n)$ be defined by (1.1). Then f is in the class $S_n(\gamma,a,c,\beta)$ if and only if

(2.1) $$\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(\beta|\gamma|+k-1))\ a_k\le\beta|\gamma|\ .$$

Proof. We first suppose that $f\in S_n(\gamma,a,c,\beta)$, then using condition (1.13) we get

(2.2) $$\operatorname{Re}\left(\frac{z\big(L(a,c)f(z)\big)'}{L(a,c)f(z)}-1\right)>-\beta|\gamma| \qquad (z\in U)$$

or, equivalently ,

(2.3) $$\operatorname{Re}\left(\frac{-\displaystyle\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(k-1)^m a_k z^k}{z-\displaystyle\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^k}\right)>-\beta|\gamma| \qquad (z\in U).$$

Where we have made use of (1.10) and the definition (1.1) .We now choose values of z on the real axis and let $z\to 1$ through the real values . Than the inequality (2.3) immediately yields the desired condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z|=1$, we find that

$$\left|\frac{z(L(a,c)f(z))'}{L(a,c)\ \ f(z)}-1\right|=\left|\frac{\displaystyle\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(k-1)a_k z^k}{z-\displaystyle\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^k}\right|$$

$$\leq \frac{\beta|\gamma|\left(1-\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}a_k\right)}{1-\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}a_k}$$

$$\leq \beta|\gamma|. \tag{2.4}$$

Hence, by the maximum modulus theorem, we have $f \in S_n(a,c,\beta)$, which completes the proof of Lemma 1.

Similarly, we can prove the following Lemma.

Lemma 2 Let the function $f \in A(n)$ be defined by (1.1). Then f is in the class $R_n(a,c,\beta;\mu)$ if and only if

$$\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}[\mu(k-1)]a_k \leq \beta|\gamma| . \tag{2.5}$$

Remark 1 A special case of Lemma 1 when

$$n=1,\ \gamma=1,\ \text{and}\ \beta=1-\alpha\ \ (0\leq\alpha<1)$$

was given earlier by Ahuja [1] .

Our first inclusion relation involving $N_{n,\delta}(e)$ is given by Theorem 1 below.

Theorem 1 If

$$\delta=\frac{(n+1)\,\beta|\gamma|}{(\beta|\gamma|+n)\dfrac{(a)_n}{(c)_n}} \qquad (|\gamma|<1), \tag{2.6}$$

then

$$S_n(\gamma,a,c,\beta)\subset N_{n,\delta}(e). \tag{2.7}$$

Proof. For a function $f \in S_n(\gamma,a,c,\beta)$ of the form (1.1), Lemma 1 immediately yields

$$(\beta|\gamma|+n)\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty}a_k \leq \beta|\gamma|,$$

so that

(2.8) $$\sum_{k=n+1}^{\infty} a_k \le \frac{\beta|\gamma|}{(\beta|\gamma|+n)\dfrac{(a)_n}{(c)_n}}.$$

On the other hand, we also find from (2.1) and (2.8) that

$$\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty} ka_k \le \beta|\gamma| + (1-\beta|\gamma|)\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty} a_k$$

$$\le \beta|\gamma| + (1-\beta|\gamma|)\frac{(a)_n}{(c)_n}\,\frac{\beta|\gamma|}{(\beta|\gamma|+n)\dfrac{(a)_n}{(c)_n}}$$

$$\le \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)} \qquad (|\gamma|<1),$$

that is,

(2.9) $$\sum_{k=n+1}^{\infty} ka_k \le \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)\dfrac{(a)_n}{(c)_n}} := \delta \qquad |\gamma|<1.$$

Which, in view of definition (1.4), proves Theorem 1.
Similarly apply Lemma 2 instead of Lemma 1, we can prove the following Theorem.

Theorem 2 If

(2.10) $$\delta := \frac{(n+1)\beta|\gamma|}{(\mu n+1)\dfrac{(a)_n}{(c)_n}}$$

then

(2.11) $$R_n(\gamma,a,c,\beta;\mu) \subset N_{n,\delta}(e).$$

Proof . Suppose that a function $f \in R(\gamma,a,c,\beta;\mu)$ is of the form (1.1) . Then we find from the assertion (2.5) of lemma 2 that

$$\frac{(a)_n}{(c)_n}(\mu k+1)\sum_{k=n+1}^{\infty} a_k \le \beta|\gamma|,$$

Which yield the following coefficient inequality ;

(2.12) $$\sum_{k=n+1}^{\infty} a_k \le \frac{\beta|\gamma|}{(\mu k+1)\dfrac{(a)_n}{(c)_n}}$$

Making use of (2.5) in conjunction with (2.12), we also have

$$\mu\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty}ka_k \le \beta|\gamma|+(\mu-1)\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty}a_k$$

(2.13)
$$\le \beta|\gamma|+(\mu-1)\frac{(a)_n}{(c)_n}\,\frac{\beta|\gamma|}{(\mu n+1)\frac{(a)_n}{(c)_n}}$$

that is

$$\sum_{k=n+1}^{\infty}ka_k \le \frac{(n+1)\beta|\gamma|}{(\mu n+1)\frac{(a)_n}{(c)_n}} =: \delta$$

which, in light of the definition (1.4), completes the proof of Theorem 2 .

Remark 2 By suitably specializing the various parameters involved in Theorem 1 and Theorem 2, we can derive the corresponding inclusion relations for many relatively more familiar function classes (see also Equation (1.15) and Remark 1 above) .

3 Neighborhoods for the class $S_n^{(\alpha)}(\gamma,a,c,\beta)$ and $R_n^{(\alpha)}(\gamma,a,c,\beta)$.

In this section, we determine the Neighborhoods for each of the class $S_n^{\alpha}(\gamma,a,c,\beta)$ and $R_n^{\alpha}(\gamma,a,c,\beta,)$,
which we define as follows. A function $f\in A(n)$ is said to be in the class $S_n^{\alpha}(\gamma,a,c,\beta)$ if there exist a function $g\in S_n(\gamma,a,c,\beta)$ such that

(3.1)
$$\left|\frac{f(z)}{g(z)}-1\right|<1-\alpha \qquad (z\in U; 0\le\alpha<1).$$

Analogously, a function $f\in A(n)$ is said to be in the class $R_n^{\alpha}(\gamma,a,c,\beta,;\mu)$ if there exist a function $g\in R_n(\gamma,a,c,\beta)$ such that the inequality (3.1)holds true.

Theorem 3 If $g\in S_n(\gamma,a,c,\beta)$ and

(3.2)
$$\alpha=1-\frac{\delta(\beta|\gamma|+n)\frac{(a)_n}{(c)_n}}{(n+1)[(\beta|\gamma|+n)\frac{(a)_n}{(c)_n}-\beta|\gamma|]},$$

then

(3.3) $$N_{n,\delta}(g) \subset S_n^{(\alpha)}(\gamma, a, c, \beta).$$

Proof. Suppose that $f \in N_{n,\delta}(g)$. We then find from the definition (1.2) that

(3.4) $$\sum_{k=n+1}^{\infty} k|a_k - b_k| \le \delta,$$

which readily implies the coefficient inequality:

(3.5) $$\sum_{k=n+1}^{\infty} |a_k - b_k| \le \frac{\delta}{n+1} \qquad (n \in N).$$

Next, since $g \in S_n(\gamma, a, c, \beta)$, we have by [cf.Equation(2.8)]

(3.6) $$\sum_{k=n+1}^{\infty} b_k \le \frac{\beta|\lambda|}{(\beta|\gamma| + n)\frac{(a)_n}{(c)_n}},$$

So that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k}$$

$$\le \frac{\delta}{n+1} \cdot \frac{(\beta|\gamma| + n)\frac{(a)_n}{(c)_n}}{(\beta|\gamma| + n)\frac{(a)_n}{(c)_n} - \beta|\gamma|}$$

(3.7) $$= 1 - \alpha,$$

provided that α is given by (3.2). thus, by definition, $f \in S_n^{(\alpha)}(\gamma, a, c, \beta)$ for α given by (3.2). This evidently completes our proof of Theorem 3.

Theorem 4. If $g \in R_n(\gamma, a, c, \beta)$ and

(3.8) $$\rho = 1 - \frac{\delta(\mu n + 1)\frac{(a)_n}{(c)_n}}{(n+1)[(\mu n + 1)\frac{(a)_n}{(c)_n} - \beta|\gamma|]},$$

then

(3.9) $$N_{n,\delta}(g) \subset R_n^{(\alpha)}(\gamma, a, c, \beta).$$

Remark 5. By suitable specializing the various parameters in Theorem 3 and Theorem 4, we can derive the corresponding neighborhood results for many relative more familiar function classes as in [8].

3. Subordination Theorem

Before stating and proving our subordination Theorem for the class $S_n^{(\alpha)}(\alpha,a,c,\beta)$
we shall make use of the following definition results .

Definition 4.1 For two functions f and g analytic functions in U **,** we say that the function f is subordination to g in U , denoted by $(f \prec g)$ if there exists a function
$W(z)$**,** analytic in U with $w(0)=0$ and $|w(z)| \le |z| < 1$ $(z \in U)$, such that
$f(z) = g(w(z))$.

Definition 4.2 A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called subordination factor sequence if whenever $f(z)$ is analytic univalent and convex in U **,** then

$$\sum_{k=n+1}^{\infty} b_n a_n z^n \prec f(z) \quad , \ (z \in U, a_i = 1) \ . \tag{4.1}$$

Lemma 4.1 . (cf.Wilf [8]) the sequence $\{b_n\}_{n=1}^{\infty}$ is subordination factor sequence if and only if

$$\operatorname{Re}\left\{1 + 2\sum_{k=n+1}^{\infty} b_n z^k\right\} > 0 \qquad z \in U \ . \tag{4.2}$$

Theorem 4.1 let $f(z)$ of the form (1.1) satisfy the coefficient inequality (2.5) ,then

$$\left\{\frac{(a)_n(\beta|\gamma|+n)}{2[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]}(f*g)(z) \prec g(z)\right\} \tag{4.3}$$

$(-1<\alpha<1, \beta>0, z \in U, \forall k \in N \setminus (\{1\}))$,
for every function $g(z)$ in the class of convex functions . In particular

$$\operatorname{Re}\{f(z)\} \ge \frac{-\left[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|\right]}{(a)_n(\beta|\gamma|+n)} \qquad (z \in U) \ , \tag{4.4}$$

the constant factor

$$\frac{(a)_n(\beta|\gamma|+n)}{2[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]} \tag{4.5}$$

in the subordination result (4.3) can not be replace by any larger one .

Proof . let $f(z)$ be defined by (1.1) , the coefficient inequality (4.3) of our Theorem will hold true if the sequence

(4.6) $$\left\{\frac{(a)_n(\beta|\gamma|+n)}{2[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]}a_k\right\}_{k=1}^{\infty} \qquad ,a_1=1.$$

Is a subordination factor sequence which by virtue Lemma (4.1) is equivalent to the inequality

(4.7) $$\operatorname{Re}\left\{1+2\sum_{k=n+1}^{\infty}\frac{(a)_n(\beta|\gamma|+n)}{2[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]}a_k z^k\right\}>0 \qquad ,z\in U$$

Now

$$\operatorname{Re}\left\{1+\frac{(a)_n(\beta|\gamma|+n)}{[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]}\sum_{k=n+1}^{\infty}a_k z^k\right\}$$

$$\operatorname{Re}\left\{1+\frac{(a)_n(\beta|\gamma|+n)a_1 z}{[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]}+\frac{1}{[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|\}}\sum_{k=n+1}^{\infty}(a)_n(\beta|\gamma|+n)a_k z^k\right\}$$

$$\operatorname{Re}\left\{1-\frac{(a)_n(\beta|\gamma|+n)\ r}{[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]}-\frac{(c)_n\beta|\gamma|}{[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|\}}\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}(\beta|\gamma|+(k-1)}{(c)_{k-1}\beta|\gamma|}|a_k|\,r^{k^k}\right\}$$

$$\geq 1-\left[\frac{(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|}{[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]})\right]=0$$

$$\operatorname{Re}\left[1+\frac{(a)_n(\beta|\gamma|+n)}{[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|}\right]>0$$

Hence (4.7) hold true in U ,which proves the assertion . The prove of

$$\operatorname{Re}\{f(z)\}\geq\frac{-\left[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|\right]}{(a)_n(\beta|\gamma|+n)} \qquad (z\in U)\ ,\text{for } f\in S_n^{(\alpha)}(\gamma,a,c,\beta)$$

follows by taking $g(z)=z+\sum_{k=n+1}^{\infty}z^n$. To prove the sharpness of

$$\frac{(a)_n(\beta|\gamma|+n)}{2[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]}$$

Consider the function $q(z)=z-\dfrac{(c)_n\beta|\gamma|}{(a)_n(\beta|\gamma|+n)}z^{n+1}$, which is member of the class

$S_n^{(\alpha)}(\gamma,a,c,\beta)$, and $g(z)=z+\sum_{k=n+1}^{\infty}z^n$.

Thus from relation (4.1) we obtain

$$\left\{\frac{(a)_n(\beta|\gamma|+n)}{2[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]}q(z) \prec z+\sum_{k=n+1}^{\infty}z^n\right\}$$

It can be easily shows that $\min_{|z|\leq 1} \operatorname{Re}\left\{\frac{(a)_n(\beta|\gamma|+n)}{2[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]}q(z)\right\}=-\frac{1}{2}$

This show that the constant $\frac{(a)_n(\beta|\gamma|+n)}{[(a)_n(\beta|\gamma|+n)+(c)_n\beta|\gamma|]}$ is best possible .

References

[1] O .P. Ahuja, Hadamard products of analytic functions defined by Ruscheweyh derivatives, in Current Topics in Analytic Function Theory, H,M Srivastava and S. Owa (eds.), pp. 13-28, *World Scientific Publishing Company*, Singapore, New Jersey, London and Hong Kong, (1992) .

[2] O.P. Ahuja and M, Nunokawa, Neighborhoods of analytic functions defined by Ruscheweyh derivative, *Math. Japon*., 51(2003), 487-492 .

[3]O. Altintas and S. Owa, Neighborhood of certain analytic functions with negative coefficients, *Interat. J. Math. And Sci.* , 19(1996), 797-800 .

[4] O. Altintas, O. Ozkan and H.M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, *Appl. Math. Lett.*, 13 (3)(2000), 63-67 .

[5] O.Altintas, O. Ozkan and H.M. Srivastava, Majorization by starlike functions of complex order, *Complex Variables Theory Appl*., 46(2001), 207-218 .

[6] O.Altintas and H.M. Srivastava, Some majorization problems Associated with p-valently and convex functions of complex order, *East Asian Math. J.*, 17(2001), 175-183.

[7] P.L.Duren, Univalent functions, A Series of comprehensive studies in mathematics, Vol.259.*Springer-Verlag*, New York, Berlin, Heidelberg, andTokyo,1983

[8] A.W. Goodman, Univalent functions and nonanalytic curves, *Proc. Amer. Math. soc.*, 8(1957), 598-601 .

[9] S.Ruscheweyh , New criteria for univalent functions , *Proc . Amer. Math . Soc .*, 49(1975) , 109-115 .

[10] S. Ruscheweyh , Neighborhoods of univalent functions, *Proc . Amer. Math .Soc .*, 81(1981), 521-527.

[11] H . Silverman , Univalent function with negative coefficients , *Proc . Amer .Math . Soc.*, 51(1975), 109-116

[12] M.A. Nasr and M.K. AOuf , Starlike function of complex order , *J. Natur. Sci. Math.*, 25(1985), 1-12.

[13] H. S. Wilf, Subordinating factor sequence for convex maps of the unit circle, *Proc. Amer. Math. Soc.*, 12(1961), 689-693 .

[14] H . Silverman , Neighborhoods of classes of analytic functions , *Far East J. Math. Sci .*, 3(1995), 165-169 .