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An Omitting Types Theorem for Finite Schematizable Algebraic Logic

Tarek Sayed Ahmed

Department of Mathematics, Cairo University
rutahmed@gmail.com

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Abstract

We prove an Omitting Types Theorem for the extension of first order logic studied by Németi, Sain and others as a solution to the so-called Finitization Problem in Algebraic Logic. A new omitting types theorem for first order logic is obtained.

Keywords: *Algebraic logic, finitizability problem, polyadic algebras, omitting types*

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1 Introduction

We follow the terminology of [11]. Let α be a countably infinite ordinal, and $G \subseteq {}^{\alpha}\alpha$. Then the class of G polyadic set algebras, or $\mathbf{GPSA}_\alpha$ for short, is defined in [11, 1.1]. The class of abstract G polyadic algebras, or $\mathbf{GPA}_\alpha$ for short, is defined (by a finite schema of equations) in [11, 2.2]. The notion of rich and strongly rich semigroups is defined in [11, 1.4]. One of the the main Theorems in [9], cf. Theorem 1.8 therein, is that when G is a rich semigroup then the abstract and concrete G algebras coincide, i.e. $\mathbf{GPSA}_\alpha = \mathbf{GPA}_\alpha$. When further G happens to be finitely presented, then $\mathbf{GPSA}_\alpha$ is (term equivalent to a variety that is) finitely axiomatizable [11]. This provides a solution to the so- called *Finitization Problem* (FP) in algebraic logic. The research direction represented by the FP is underlined by a striving of making all syntactical entities involved in the axiomatization of the validities of the logic in question (a variant of first order logic) thoroughly finite.

The FP, a central classic problem in algebraic logic, is extensively discussed in the introduction of [9] and elaborated upon in the reference [14]. It is worth noting that the dissertation [10] is devoted to this problem alone.

The main result in [11], cf. Theorem 1.5 therein, is that when G happens to be a *strongly* rich semigroup then G polyadic (set) algebras have the super amalgamation property. The results of [9] and [11] combined provide a finitizable extension of first order logic without equality that is complete and enjoys the definability properties of Beth and Craig. Here we show - using a Baire Category approach - that this logic further enjoys an Omitting Types Theorem. Our result applies to ordinary first order logic, giving a new omitting types theorem.

2 The Main Result

We use standard notation adopted in [11]. In what follows $\prod$ and $\sum$ denote infimum and supremum respectively.

Definition 2.1 *Let A be an algebra with a boolean reduct. $X \subseteq A$ is non-principal if $\prod X = 0$.*

We shall need the following form of the Baire Category Theorem for compact Hausdorff spaces [2]. Let $covK$ be the least cardinal such that the real line can be covered by $covK$ many nowhere dense sets. Then $\omega < covK \leq 2^{Alepho_0}$.

Lemma 2.2 *Let $\kappa < covK$. Let X be a compact Hausdorff space and $\{X_i : i \in \kappa\}$ be a family of nowhere dense sets in X. Then $X \setminus \bigcup_{i\in\kappa} X_i$ is dense in X.*

We formulate and prove our main Theorem:

Theorem 2.3 *Let G be a countable strongly rich sub-semigroup of ${}^{\omega}\omega$. Let $A \in \mathbf{GPA}_\omega$ be countable. Let $\kappa < covK$. Let $\{X_i : i \in \kappa\}$ be a family of non-principal types of A. Then for all non-zero $a \in A$, there exists a countable $C \in \mathbf{GPSA}_\omega$, and a homomorphism $f : A \to C$ such that $f(a) \neq 0$, and C omits the given non-principal types in the sense that $\bigcap f(X_i) = \emptyset$ for all $i \in \kappa$.*

Proof Let $A \in \mathbf{GPA}_\omega$ be countable. Then the following hold cf. [9, 2.15] or [11, 2.9, 2.10]: There exist $B \in \bar{G}PA_{\omega+\omega}$ and an embedding of G algebras $e : A \to Nr_\omega B$. Here $\bar{G}$ denotes the subsemigroup of ${}^{\omega+\omega}\omega + \omega$ generated by the set $\{\bar{\tau} : \tau \in G\}$, where $\bar{\tau}$ is the transformation that agrees with τ on ω and otherwise is the identity, together with all transpositions and replacements on $\omega + \omega$. $Nr_\omega B$, on the other hand, is the algebra whose universe is the set of ω dimensional elements of B, i.e those elements for which $\mathsf{c}_i x = x$ for all

$i \in \omega + \omega \sim \omega$. The booleans and cylindrifications in $Nr_\omega B$ are those induced by B. For $\tau \in G$ and $x \in Nr_\omega B$, $\mathsf{s}_\tau^{Nr_\omega B} x$ is defined by $\mathsf{s}_{\bar{\tau}}^B x$. Assume further that G is strongly rich. For an algebra D and $X \subseteq D$, let $Sg^D X$ denote the subalgebra of D generated by X. Then in [11, 2.10] it is shown that B and e can be chosen to satisfy

(a) For all $X \subseteq A$, $e(Sg^A X) = Nr\omega Sg^B(e(X).)$

(b) B is *dimension-complemented*, in the sense that

$$(\forall x \in B)(|\Delta x \sim \omega| < \omega.)$$

(c) For all $X \subseteq A$, if $\prod^A X = 0$, and $\sigma \in \bar{G}$, then $\prod \mathsf{s}_\sigma^B e(X) = 0$.

Let $\beta = \omega + \omega$. Upon identifying e with the identity function let $B \in \bar{G}PA_\beta$ be as in (a), (b) and (c) above, i.e $A = Nr_\omega B$, A is a generating set for B and $\prod \mathsf{s}_\sigma^B X_i = 0$ for each $i < \kappa$ and $\sigma \in \bar{G}$. Recall that $\bar{G}$ contains all replacements. For $i, j \in \beta$, we write, following the convention of [5], s_j^i for the substitution corresponding to the replacement $[i|j]$. Since B is dimension complemented, it follows from [5, 1.11.6] that

$$(1) \quad (\forall j < \beta)(\forall x \in B)(\mathsf{c}_j x = \sum_{i \in \beta \sim \Delta x} \mathsf{s}_i^j x.)$$

Let V be the generalized ω-dimensional weak space $\bigcup_{\tau \in G} {}^\omega\beta^{(\tau)}$. Here ${}^\omega\beta^{(\tau)} = \{s \in {}^\omega\beta : |\{i \in \omega : \mathsf{s}_i \neq \tau_i\}| < \omega\}$. For each $\tau \in V$ and for each $i \in \kappa$, let

$$X_{i,\tau} = \{\mathsf{s}_{\bar{\tau}}^B x : x \in X_i\}.$$

Here we are using that for any $\tau \in V$, $\bar{\tau} \in \bar{G}$. This is straightforward, and is proved in [9, 3.19]. It follows that

$$(2) \ (\forall \tau \in V)(\forall i \in \kappa) \prod{}^B X_{i,\tau} = 0.$$

Let S denote the Stone space of the boolean part of B. For $a \in B$, let N_a be the clopen set of S consisting of all ultrafilters of the boolean part of B containing a. Then form (1) and (2) it follows that for $x \in B$, $j < \beta$, $i < \kappa$ and $\tau \in V$, the sets

$$\mathbf{G}_{j,x} = N_{\mathsf{c}_j x} \sim \bigcup_{i \notin \Delta x} N_{\mathsf{s}_i^j x} and \mathbf{H}_{i,\tau} = \bigcap_{x \in X_i} N_{\mathsf{s}_{\bar{\tau}} x}$$

are closed nowhere dense sets in S. Indeed, the set $\mathbf{G}_{j,x}$ is closed since it is the diference of a closed and an open set. Now, suppose seeking a contradiction, that $\mathbf{G}_{j,x}$ is not nowhere dense. Then it necessarily contains an open set N_a say, for some non-zero element a. Since $N_a \subseteq N_{\mathsf{c}_j x}$ we have $a \leq \mathsf{c}_j x$ thus $\mathsf{c}_j x - a < \mathsf{c}_j x$. On the other hand

$$N_a \subseteq N_{\mathsf{c}_j x} \setminus N_{\mathsf{s}_i^j x}$$

i.e

$$N_{\mathsf{s}_i^j x} \subseteq N_{\mathsf{c}_j x} \setminus N_a = N_{\mathsf{c}_j x - a}$$

hence

$$N_{\mathsf{s}_i^j x} \leq N_{\mathsf{c}_j x - a}.$$

It thus follows that

$$\mathsf{s}_i^j x \leq \mathsf{c}_j x - a$$

for every j, which is a contradiction. Similarly it can be proved that each $\mathbf{H}_{i,\tau}$ is closed and nowhere dense. Let

$$\mathbf{G} = \bigcup_{j\in\beta}\bigcup_{x\in B}\mathbf{G}_{j,x}, \quad \mathbf{H} = \bigcup_{i\in\kappa}\bigcup_{\tau\in V}\mathbf{H}_{i,\tau}.$$

Then $X = S \sim \mathbf{H} \cup \mathbf{G}$ is dense in S. Accordingly, let F be an ultrafilter in $N_a \cap X$. Then, by the very choice of F, we have the following

(3) For $j < \beta$ and $x \in B$, if $\mathsf{c}_j x \in F$ then there exists $j \notin \Delta x$ such that $\mathsf{s}_j^i x \in F$ and

(4) for each $i < \kappa$ and $\tau \in V$, there exists $x \in X_i$ such that $\mathsf{s}_{\bar{\tau}} x \notin F$.

Let $\wp(V)$ be the full boolean set algebra with unit V. Let f be the function with domain A such that

$$f(a) = \{\tau \in V : \mathsf{s}_{\bar{\tau}}^B a \in F\}.$$

Then, following the notation of [11, 2.21], f is the desired homorphism from A into the set algebra

$$\langle \wp(V), \mathsf{c}_i, \mathsf{s}_\tau \rangle_{i\in\omega,\tau\in G}.$$

Indeed, that h is a homomorphism follows from (3), cf. [9] Claim 3.22 p.536, that $f(a) \neq 0$ follows from that Id, the identity function on β, is in $f(a)$ since $\mathsf{s}_{Id} a = a \in F$, and finally, that C omits the non-principal X_i's all $i < covK$, follows directly from (4).

Bearing in mind that theories are represented by abstract algebras, models by set algebras, and satisfiability by homomorphisms, we arrive at the following metalogical reading of our main Theorem 4. In the non-degenerate case, the algebra A corresponds to a countable consistent theory – in the algebraizable extensions of first order logic without equality studied in [9, §4] – the X_i's correspond to non-principal types over this theory, and the set algebra C, having a countable base, corresponds to a countable model uniformly omitting these types. Furthermore, in this model the formula corresponding to the non-zero element a is satisfiable.

An important difference from first order logic is that, in our present context, we do not assume an upper bound on the number of (free) variables occuring in the types omitted, i.e. these types need not be finitary, they can have infinitely many free variables. But there is a price we pay for this improvement. The model omitting these types is not a standard model, since it corresponds to a a union of weak cartesian spaces, i.e sets of the form ${}^{\omega}U^{(p)}$ and not a set algebra whose unit is a union of cartesian square, i.e. sets of the form ${}^{\omega}U$. The classical proof (by forcing) of omitting types for first order logic breaks down when types consisting of formulas having infinitely many free variables are considered because there are uncountably many assignments to free variables, but only countably many stages of the forcing construction to consider them in. When we consider only those assignments that are eventually constant (which we do), this problem of cardinality disappeas. But let us now concentrate on finitary types.

Let A be a G algebra. Then $Nr_nA = \{x \in A : \mathsf{c}_i x = x, \forall i \in \omega \sim n\}$. A finitary type is a set $\Gamma \subseteq Nr_nA$ for some $n \in \omega$. Γ is isolated or principal if there exists $a \in A$ such that $a \leq x$ for all $x \in \Gamma$. Γ is omitted by A if there exists $B \in \mathbf{GPSA}_\omega$ and an isomorphism $f : A \to B$ such that $\bigcap f(\Gamma) = \emptyset$. It seems likely that if we consider only $< covK$ many *finitary* non-principal types then we can square the unit in our Theorem, obtaining a standard model. But in any case if a finitary type Γ is non-principal then it can be omitted by a non-standard model as shown in our theorem. It is easy to construct examples where *principal* types are omitted. We now give a necessary and sufficient condition for when a single finitary type can be omitted.

Notation: ${}^{<\omega}\omega$ stands for the set of finite sequences of ω into ω. For $A \in \mathbf{GPA}_\omega$ and $n \in \omega$, $\mathsf{c}_{(n)}$ abbreviates $\mathsf{c}_0\mathsf{c}_1..\mathsf{c}_{n-1}$.

Definition 2.4 *Let $A \in \mathbf{GPA}_\omega$ and $X \subseteq Nr_nA$. Fix $\langle x_i : i \in \omega\rangle$ an enumeration of X. By a principal tree for X in A we understand a mapping $a : {}^{<\omega}\omega \to A$ such that:*

(i) $\mathsf{c}_{(n)}a_\emptyset = 1$

(ii) $-\mathsf{c}_{(n)}a_{s^\wedge<i>} \wedge a_s \leq x_i \forall s \in {}^{<\omega}\omega, i \in \omega.$

(iii) $\forall f \in {}^{\omega}\omega \exists n \in \omega,\ a_{f|n} \leq x$

Theorem 2.5 *Let $A \in \mathbf{GPA}_\omega$ be countable. Let $X \subseteq Nr_nA$. Then X is omitted if and only if there exists no principal tree for X (relative to any fixed enumeration of X) in A*

Let everything be as in the hypothesis. If we can find a principal tree of X, then it is easy to see following the branches of the tree that X cannot be omitted. Condition (iii) entails that this procedure come to an end. Conversely assume that X cannot be omitted. We construct a principal tree for X in A. Let F be a filter in A. As usual, a filter is the equivalence class of 1 under a congruence relation on A. Write $a \leq b(modF)$ if $a \vee -b \in F$.

For $Y \subseteq A$, write $a \leq Y(modF)$ if $a \leq y(modF)$ for every $y \in Y$.

Now set:

$$I(F,X) = \{a \in A : a \leq X(modF)\}$$

and

$$F[X] = F \cup \{-\mathsf{c}_{(n)}a : a \in A,\ a \leq X(modF)\} =$$

$$\{-\mathsf{c}_{(n)}a : a \in A,\ a \in I(F,X)\}.$$

We now define an increasing sequence of filters $\{F_\alpha\}_{\alpha \in Ord}$ as follows:

$$F_0 = \{1\}$$

$$F_{\alpha+1} = F_\alpha[X],$$

and if α is a limit ordinal

$$F_\alpha = \bigcup_{\beta<\alpha} F_\beta$$

By definition it is clear that

$$I(F_\beta, X) \subseteq I(F_\alpha, X), \beta \leq \alpha$$

and

$$F_\alpha = \bigcup \{-\mathsf{c}_{(n)}a : a \in \bigcup_{\beta<\alpha} I(F_\beta, X)\}.$$

Since the sequence of filters is increasing it follows that it eventually stops, so for some α we have

$$F_\alpha = F_{\alpha+1} = F_\alpha[X].$$

It follows that X is not isolated in F_α, that is

$$\prod(X/F_\alpha) = \prod\{x/F_\alpha : x \in X\} = 0$$

in A/F_α. By the our omitting types Theorem-upon noting that A/F_α is a G algebra - and the assumption that X is not omitted by A, hence it it is not omitted by A/F_α, it follows that F_α is inconsistent i.e.

$$0 \in F_{\alpha+1},$$

so that $F_{\alpha+1} = A$. Then there is some

$$\beta_\emptyset < \alpha$$

and

$$a_\emptyset \in I(F_{\beta_\emptyset}, X)\mathsf{c}_{(n)}a_\emptyset = 1$$

Now

$$a_\emptyset \in I(F_{\beta_\emptyset}, X)$$

implies that

$$a_\emptyset \le X(modF_{\beta_\emptyset})$$

where

$$F_{\beta_\emptyset} = \bigcup\{-\mathsf{c}_{(n)}a : a \in \bigcup_{\mu<\beta_\emptyset} I(F_\mu, X)\}.$$

and thus for any $i \in \omega$ there is some $\beta_{<i>} < \beta_\emptyset$ and $a_{<i>} \in I(F_{\beta_{<i>}}, X)$ such that

$$\mathsf{c}_{(n)}a_{<i>} \vee -a_\emptyset \vee x_i$$

holds in A. We abbreviate the above by

$$A \models -\mathsf{c}_{(n)}a_{<i>} \wedge a_\emptyset \le x_i$$

For any $i \in \omega$, applying again that

$$a_{<i>} \in I(F_{\beta_{<i>}}, X)$$

we can find $\beta_{<i,j>} < \beta_{<i>}$ and $a_{<i,j>} \in I(F_{\beta_{(i,j)}}, X)$ such that

$$A \models -\mathsf{c}_{(n)}a_{<i,j>} \wedge a_{<i>} \le x_j.$$

Repeating this procedure we find

$$a_s, s \in {}^{<\omega}\omega.$$

Now the sequence of ordinals

$$\beta_\emptyset > \beta_{<i>} > \beta_{<i,j>} \cdots$$

cannot go on for ever. So for any $f \in {}^{\omega}\omega$ there is some $n \in \omega$ such that $a_{f|n} = \emptyset$. Define $a : {}^{<\omega}\omega \to A$ as follows:

$$s \mapsto a_s.$$

Then it is easy to see that f defines a principal tree of X in A.

It is easy to see that our characterization first for (ordinary) first order logic, by replacing G algebras by (countable) locally finite cylindric algebras. By the same token it is easy to characterize omitting $< covK$ many types for first order logic. We write $\phi \to \Gamma$ if $\phi \to \psi$ for all $\psi \in \Gamma$. Let $P = \{\Gamma_\alpha(\bar{x_\alpha}) : i < \alpha\}$ be a family of types, where for each $i < \lambda$, $\Gamma_\alpha(\bar{x}_\alpha) = \{\sigma_i^\alpha(\bar{x}_\alpha) : i \in \omega\}$. A family of formulas ϕ_s $s \in {}^{<\omega}\omega$ is a principal tree of P in T iff there are natural numbers n_s $(s \in {}^{<\omega}\omega)$ and finite sequences of ordinals $\bar{\alpha_s}$ $s \in {}^{<\omega}\omega$ such that for every $s \in {}^{<\omega}\omega$ $\bar{\alpha_s} = (\alpha_s^1, \ldots \alpha_s^{n_s})$ with $\alpha_s^i < \lambda$, $\phi_s = (\phi_s^1, \ldots \phi_s^{n_s})$ with $\phi_s^i = \phi_s^i(\bar{x_{\alpha_s^i}})$ and the following conditions (where $\bar{y}_s = (\bar{x}_{\alpha_s^1}, \ldots \bar{x}_{\alpha_s{n_s}}))$ are satisfied.

(i) $T \models \exists \bar{y}_\emptyset \phi_\emptyset(\bar{y}_\emptyset)$

(ii) $T \models \forall \bar{y}_{s^\wedge<j>} \neg \phi_{s^\wedge<j>} \wedge \phi^i_s(\bar{x}_{\alpha^i_s}) \to \sigma_j^{\alpha^i_s}(\bar{x}_{\alpha^i_s})$. for all $s \in {}^{<\omega}\omega$ $j \in \omega$ and every i ($1 \leq i \leq n_s$.)

(iii) For all $f \in {}^\omega\omega$ there is some $m \in \omega$ such that for every i ($1 \leq i \leq n_{f|m}$), $T \models \phi^i_{f|m}(\bar{x}_{\alpha^i_{f|m}}) \to \Gamma_{\alpha_{f|m^i}}(\bar{x}_{\alpha_{f|m^i}})$

The proof of the following (new) Omitting Types Theorem for first order logic can be destilled from our proofs so far [4].

Theorem 2.6 *Let T be a countable first order theory. Let $P = \{\Gamma_\alpha(\bar{x}_\alpha) : \alpha < \lambda\}$ and $\lambda < covK$. There is a model omitting P iff there is no principal tree of P in T.*

3 Conclusion

The main results of the paper gives a new omitting types theorem for certain extensions of first order logic, that are important in algebraic logic. The study of these logics was initiated by Tarski, Henkin and Monk, and later was further investigated by Andréka, Németi and Sain among others. Our result applied to the classical case of first order logic gives the omitting types theorem proved in [4]. Our results in this paper contrast negative results on omitting types (for finite variable fragments of first order logic) proved in [1] and [13].

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