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Iterative Scheme for Solution to the Falkner-Skan Boundary Layer Wedge Flow Problem

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Abstract

We study Falkner-Skan boundary layer wedge flow problem of an incompressible viscous fluid and develop an iterative scheme, the generalized approximation method (GAM), to obtain a solution to the problem. By using similarity transformation, the nonlinear partial differential equation for the velocity field is transformed to a third order nonlinear ordinary differential equation. We provide estimates for the exact solution of the nonlinear problem. These estimates determine the region of existence of solution of the problem. Based on these estimates, we develop the generalized approximation method GAM for the solution of the problem. The GAM is a monotone iterative scheme which generates a bounded monotone sequence of solutions of linear problems. The sequence converges monotonically and rapidly to a solution of the original nonlinear problem. We study the effect of the Falkner-Skan power-law parameter on the velocity field and the shear stress. For the numerical simulations, we use Mathematica.

Keywords: *Falkner-Skan equation; upper and lower solutions; approximation method.*

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1. Introduction

The well-known Falkner-Skan equation is one of the most important equations in laminar boundary layer theory and is used to describe the steady two-dimensional flow of a viscous incompressible fluid past wedge shaped bodies of angles $\lambda\pi$, where $\lambda \in \mathbb{R}$ is a parameter involved in the equation. In two-dimensional, the equation of continuity and the laminar boundary layer equations for the steady flow of an incompressible viscous fluid over a wedge are given by

$$\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} = 0, \tag{1.1}$$

$$v\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial y} = U\frac{dU}{dx} + \mu\frac{\partial^2 v}{\partial y^2}, \tag{1.2}$$

where v, w are components of velocity in x and y direction of the fluid flow, μ is the viscosity, $U(x)$ is the velocity at the edge of the boundary layer. We consider a general case of a power law free stream velocity, that is, $U(x) = U_\infty(x/L)^m$, where U_∞ is uniform free stream velocity, L is the length of the wedge, x is measured from the tip of the wedge and m is the Falkner-Skan power-law parameter. The boundary conditions are given by

$$v(x,0) = w(x,0) = 0, v(x,y) \to U(x) \text{ as } y \to \infty. \tag{1.3}$$

The continuity equation (1.1) is automatically satisfied by the stream function $\psi(x,y)$ such that

$$v = \frac{\partial \psi}{\partial y},\ w = -\frac{\partial \psi}{\partial x},$$

and the momentum equation (1.2) takes the form

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U\frac{dU}{dx} + \mu\frac{\partial^3 \psi}{\partial y^3}. \tag{1.4}$$

By the similarity transformation

$$f(\eta) = \Big[\frac{1+m}{2}\frac{L^m}{\mu U_\infty}\frac{1}{x^{1+m}}\Big]^{\frac{1}{2}}\psi,\ \eta = \Big[\frac{1+m}{2}\frac{U_\infty}{\mu L^m}\frac{1}{x^{1-m}}\Big]^{\frac{1}{2}}y,$$

where f is a dimensionless stream function and η is a dimensionless distance from the edge, called similarity variable, the partial differential equation (1.4) reduces to third order nonlinear ordinary differential equation

$$f'''(\eta) + f(\eta)f''(\eta) + \lambda[1 - (f'(\eta))^2] = 0,\ \eta \in (0,\infty), \tag{1.5}$$

where $\lambda\pi$ is the wedge angle and is related to the Falkner- Skan power-law parameter m through the expression $\lambda = \frac{2m}{1+m}$ and prime $'$ denotes differentiation with respect to η. The equation (1.5) is well known Falkner-Skan boundary layer equation [4] in which $f'(\eta)$ defines the dimensionless velocity component in η-direction and $f''(\eta)$ defines the dimensionless shear stress in the boundary layer.

The boundary conditions (1.3) can be written as

$$f(0) = f'(0) = 0, \text{ and } f'(\eta) \to 1 \text{ as } \eta \to \infty. \tag{1.6}$$

Approximate solutions of the nonlinear third order boundary value problem (1.5), (1.6) are obtained by many researchers, see for example, [1, 2, 4, 9, 10, 11, 12] and the references therein.

In this paper, we revisit the problem and provide estimates for the exact solution of the problem. These estimates determine the region of existence of solution of the problem. Based on these estimates, we apply the generalized approximation method GAM, [5, 6, 7, 8], to obtain a approximate solution of the problem. We shall show that only few iterations lead to an accurate result. The GAM generates a bounded monotone sequence of solutions of linear problems that converges uniformly and rapidly to a solution of the original problem. Moreover, the solution is bracketed between the iterates and a fixed upper solution. We shall show that our results are consistent and accurately represent the actual solution of the problem for any values of the parameter. We study the effect of the fluid parameter on the velocity field. For the numerical simulations, we use the computer programme, Mathematica.

By using the transformation $f(\eta) = \int_0^\eta u(s)ds$, the boundary value problem (1.5), (1.6) can be written as integro-differential equation

$$\begin{aligned} &-u''(\eta) - u'(\eta) = (\int_0^\eta u(s)ds - 1)u'(\eta) + \lambda[1 - u^2(\eta)],\ \eta \in (0,\infty), \\ &u(0) = 0,\ u(\infty) = 1, \end{aligned} \tag{1.7}$$

By a solution of (2.1), we mean solution of the corresponding integral equation

$$u(\eta) = (1 - e^{-\eta}) + \int_0^\infty G(\eta, s)\big[(\int_0^s u(r)dr - 1)u'(s) + \lambda(1 - u^2(s))\big]ds,\ \eta \in (0,\infty), \tag{1.8}$$

where

$$G(\eta, s) = \begin{cases} 1 - e^{-\eta}, & 0 \le \eta < s \le \infty \\ (1 - e^{-\eta})e^{s-\eta}, & 0 \le s < \eta \le \infty, \end{cases}$$

is the Green's function. Clearly, $G(\eta, s) > 0$ on $(0,\infty) \times (0,\infty)$.

2. Upper and Lower Solutions

Recall the concept of lower and upper solutions corresponding to the BVP (2.1).

Definition 2.1. A function $\alpha \in C^1(I)$ is called a lower solution of the BVP (2.1) if it satisfies the following inequalities,

$$\begin{aligned} -\alpha''(\eta) - \alpha'(\eta) &\le g(\alpha(\eta), \alpha'(\eta)), \quad \eta \in (0,\infty) \\ \alpha(0) &\le 0,\ \alpha(\infty) \le 1, \end{aligned}$$

where $g(u(\eta), u'(\eta)) = (\int_0^\eta u(s)ds - 1)u'(\eta) + \lambda[1 - u^2(\eta)]$. An upper solution $\beta \in C^1(I)$ of the BVP (2.1) is defined similarly by reversing the inequalities.

For example, $\alpha = 0$ and $\beta = 1$ are lower and upper solutions of the BVP (2.1) respectively and these functions provide estimates for the exact solution of the problem.

Definition 2.2. For $T > 0$, a continuous function $\omega : (0, \infty) \to (0, \infty)$ is called a Nagumo function if

$$\int_{\nu}^{\infty} \frac{sds}{\omega(s)} = \infty,$$

where $\nu = \max\{|\alpha(0) - \beta(T)|, |\alpha(T) - \beta(0)|\}$. We say that $g \in C[\mathbb{R} \times \mathbb{R}]$ satisfies a Nagumo condition on $[0, T]$ relative to α, β if for $x \in [\min \alpha, \max \beta]$, there exists a Nagumo function ω such that $|g(u, u')| \leq \omega(|u'|)$.

The following result is known [3] (Theorem 1.7.1, Page 31).

Theorem 2.3. *Assume that for each $T > 0$, $g(u, u')$ satisfies Nagumo's condition on $[0, T]$ relative to the pair $\alpha, \beta \in C^1[[0, \infty), \mathbb{R}]$ with $\alpha \leq \beta$ on $[0, \infty)$. Suppose also that α, β are lower and upper solutions of (2.1) on $[0, \infty)$, respectively. Then the BVP (2.1) has a solution $u \in C^2[[0, \infty), \mathbb{R}]$ such that $\alpha \leq u \leq \beta$ on $[0, \infty)$.*

Since $\alpha = 0$, $\beta = 1$, therefore for each $T > 0$, $\eta \in [0, T]$ and $u(\eta) \in [0, 1]$, we have

$$|g(u, u')| = |(\int_0^{\eta} u(s)ds - 1)u'(\eta) + \lambda[1 - u^2(\eta)]| \leq (T + 1)|u'(\eta)| + |\lambda| = \omega(|u'|).$$

Hence, for each $T > 0$, g satisfies a Nagomo condition with $\omega(s) = (T+1)s + |\lambda|$ as a Nagumo function and $\nu = 1$. By Theorem 2.4, the the BVP (2.1) has a solution u such that $\alpha \leq u \leq \beta$ on $[0, \infty)$, that is $0 \leq u(\eta) \leq 1$, $\eta \in [0, \infty)$.

By using the transformation $f(\eta) = \int_0^{\eta} u(s)ds$, the boundary value problem (1.5), (1.6) can be written as integro-differential equation

$$\begin{aligned} &-u''(\eta) - u'(\eta) = (\int_0^{\eta} u(s)ds - 1)u'(\eta) + \lambda[1 - u^2(\eta)],\ \eta \in (0, \infty), \\ &u(0) = 0,\ u(\infty) = 1, \end{aligned} \tag{2.1}$$

By a solution of (2.1), we mean solution of the corresponding integral equation

$$u(\eta) = (1 - e^{-\eta}) + \int_0^{\infty} G(\eta, s)\big[(\int_0^{s} u(r)dr - 1)u'(s) + \lambda(1 - u^2(s))\big]ds,\ \eta \in (0, \infty), \tag{2.2}$$

where

$$G(\eta, s) = \begin{cases} 1 - e^{-\eta}, & 0 \leq \eta < s \leq \infty \\ (1 - e^{-\eta})e^{s-\eta}, & 0 \leq s < \eta \leq \infty, \end{cases}$$

is the Green's function. Clearly, $G(\eta, s) > 0$ on $(0, \infty) \times (0, \infty)$. Recall the concept of lower and upper solutions corresponding to the BVP (2.1).

Definition 2.1. A function $\alpha \in C^1(I)$ is called a lower solution of the BVP (2.1) if it satisfies the following inequalities,

$$\begin{aligned} -\alpha''(\eta) - \alpha'(\eta) &\leq g(\alpha(\eta), \alpha'(\eta)), \quad \eta \in (0, \infty) \\ \alpha(0) &\leq 0,\ \alpha(\infty) \leq 1, \end{aligned}$$

where $g(u(\eta), u'(\eta)) = (\int_0^\eta u(s)ds - 1)u'(\eta) + \lambda[1 - u^2(\eta)]$. An upper solution $\beta \in C^1(I)$ of the BVP (2.1) is defined similarly by reversing the inequalities.

For example, $\alpha = 0$ and $\beta = 1$ are lower and upper solutions of the BVP (2.1) respectively and these functions provide estimates for the exact solution of the problem.

Definition 2.2. For $T > 0$, a continuous function $\omega : (0, \infty) \to (0, \infty)$ is called a Nagumo function if

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where $\nu = \max\{|\alpha(0) - \beta(T)|, |\alpha(T) - \beta(0)|\}$. We say that $g \in C[\mathbb{R} \times \mathbb{R}]$ satisfies a Nagumo condition on $[0, T]$ relative to α, β if for $x \in [\min \alpha, \max \beta]$, there exists a Nagumo function ω such that $|g(u, u')| \leq \omega(|u'|)$.

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Since $\alpha = 0$, $\beta = 1$, therefore for each $T > 0$, $\eta \in [0, T]$ and $u(\eta) \in [0, 1]$, we have

$$|g(u, u')| = |(\int_0^\eta u(s)ds - 1)u'(\eta) + \lambda[1 - u^2(\eta)]| \leq (T + 1)|u'(\eta)| + |\lambda| = \omega(|u'|).$$

Hence, for each $T > 0$, g satisfies a Nagomo condition with $\omega(s) = (T + 1)s + |\lambda|$ as a Nagumo function and $\nu = 1$. By Theorem 2.4, the the BVP (2.1) has a solution u such that $\alpha \leq u \leq \beta$ on $[0, \infty)$, that is $0 \leq u(\eta) \leq 1$, $\eta \in [0, \infty)$.

3. GENERALIZED APPROXIMATION METHOD (GAM)

Differentiating g with respect to u, u', we obtain

$$g_u = -2\lambda u,\ g_{u'} = f(\eta) - 1,\ g_{uu} = -2\lambda,\ g_{uu'} = 0,\ g_{u'u'} = 0.$$

Define $\phi(u) = \lambda u^2$ and $F(u, u') = g(u, u') + \phi(u)$. Then,

$$F_u(u, u') = 0,\ F_{u'}(u, u') = f(\eta) - 1,\ F_{uu}(u, u') = 0,\ F_{uu'}(u, u') = 0,\ F_{u'u'}(u, u') = 0.$$

Hence, the quadratic form

(3.1)
$$v^T H(F) v = (u - z)^2 F_{uu}(z, z') + 2(u - z)(u' - z')F_{uu'}(z, z') + (u' - z')^2 F_{u'u'}(z, z') = 0,$$

where $H(F) = \begin{pmatrix} F_{uu} & F_{uu'} \\ F_{uu'} & F_{u'u'} \end{pmatrix}$ is the Hessian matrix and $v = \begin{pmatrix} u - z \\ u' - z' \end{pmatrix}$.

Using (3.1), we obtain

$$F(u, u') = F(z, z') + F_u(z, z')(u - z) + F_{u'}(z, z')(u' - z'),\ z,\ z' \in \mathbb{R},$$

which implies that

$$g(u,u') = g(z,z') + F_{u'}(z,z')(u'-z') - (\phi(u)-\phi(z)),\ z,\ z' \in \mathbb{R}. \tag{3.2}$$

Using the mean value theorem, we have

$$\begin{aligned}\phi(u)-\phi(z) &= \phi_u(\xi)(u-z),\ z \le \xi \le u\\ &\le 2\lambda(u-z) \text{ for } u \ge z.\end{aligned} \tag{3.3}$$

Substituting (3.3) in (3.2), we obtain

$$\begin{aligned}g(u,u') \ge& g(z,z') + F_{u'}(z,z')(u'-z') - 2\lambda(u-z),\ \text{for } u \ge z\\ &= B(z,z') + (\int_0^\eta z(s)ds - 1)u' - 2\lambda u,\end{aligned} \tag{3.4}$$

where $B(z,z') = \lambda(1 - z'^2(\eta) + 2z(\eta))$.

Define $h : \mathbb{R}^4 \to \mathbb{R}$ by

$$h(u,u';z,z') = B(z,z') + (\int_0^\eta z(s)ds - 1)u' - 2\lambda u.$$

Clearly, h is continuous and satisfies the following relations

$$\begin{cases} g(u,u') \ge h(u,u';z,z'),\ u \ge z\\ g(u,u') = h(u,u';u,u'). \end{cases} \tag{3.5}$$

Now, consider the following linear BVP

$$\begin{aligned}-u''(\eta) - u'(\eta) &= h(u(\eta),u'(\eta);z(\eta),z'(\eta))\\ u(0) = 0,&\quad u(\infty) = 1.\end{aligned} \tag{3.6}$$

As an initial approximation, choose $w_0 = \alpha = 0$ and consider the following linear BVP

$$\begin{aligned}u''(\eta) - u'(\eta) &= h(u(\eta),u'(\eta);w_0'(\eta),w_0'(\eta)),\\ u(0) = 0,&\quad u(\infty) = 1.\end{aligned} \tag{3.7}$$

Using (3.5) and the definition of lower and upper solutions, we obtain

$$\begin{aligned}h(w_0(\eta),w_0'(\eta);\ w_0(\eta),w_0'(\eta);w_0'(\eta)) &= g(w_0(\eta),w_0'(\eta)) \ge -w_0''(\eta) - w_0'(\eta),\ \eta \in (0,\infty)\\ h(\beta(\eta),\beta'(\eta);\ w_0(\eta),w_0'(\eta)) &\le g(\beta(\eta),\beta'(\eta)) \le -\beta''(\eta) - \beta'(\eta),\ \eta \in (0,\infty),\end{aligned}$$

which imply that w_0 and β are lower and upper solutions of (3.7). Hence, by Theorem 2.4, solution w_1 of (3.7) satisfies $w_0 \le w_1 \le \beta$ on $[0,\infty)$. Moreover, in view of (3.5) and the fact that w_1 is a solution of (3.7), we obtain

$$-w_1''(\eta) - w_1'(\eta) = h(w_1(\eta),w_1'(\eta);\ w_0(\eta);w_0'(\eta)) \le g(w_1(\eta),w_1'(\eta)),\ \eta \in (0,\infty) \tag{3.8}$$

which implies that w_1 is a lower solution of (2.1).

Similarly, we can show that w_1 and β are lower and upper solutions of the linear BVP

$$\begin{aligned}-u''(\eta) - u'(\eta) &= h(u(\eta),u'(\eta);w_1'(\eta),w_1'(\eta)),\\ u(0) = 0,&\quad u(\infty) = 1.\end{aligned} \tag{3.9}$$

By Theorem 2.4, there exists a solution w_2 of (3.9) such that $w_1 \leq w_2 \leq \beta$ on $(0,\infty)$.

Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions of linear problems satisfying

$$\alpha = w_0 \leq w_1 \leq w_2 \leq w_3 \leq ... \leq w_{n-1} \leq w_n \leq \beta \text{ on } (0,\infty),$$

where the element w_n is a solution of the following linear problem

$$\begin{aligned} -u''(\eta) - u'(\eta) &= g(u(\eta), u'(\eta); w_{n-1}(\eta), w'_{n-1}(\eta)),\\ u(0) = 0, &\quad u(\infty) = 1, \end{aligned}$$

and is given by

(3.10)

$$w_n(y) = (1-e^{-\eta}) + \int_0^\infty G(\eta,s)h(w_n(s), w'_n(s); w_{n-1}(s), w'_{n-1}(s))ds, \ \eta \in (0,\infty).$$

The sequence of functions w_n is uniformly bounded and equicontinuous. The monotonicity and uniform boundedness of the sequence $\{w_n\}$ implies the existence of a pointwise limit ω on $(0,\infty)$. From the boundary conditions, we have

$$0 = w_n(0) \to \omega(0) \text{ and } 1 = w_n(\infty) \to \omega(\infty).$$

Hence ω satisfy the boundary conditions. Moreover, by the dominated convergence theorem, for any $\eta \in (0,\infty)$, we have

$$\int_0^\infty G(\eta,s)h(w_n(s), w'_n(s); w_{n-1}(s), w'_{n-1}(s))ds \to \int_0^\infty G(\eta,s)g(\omega(s),\omega'(s))ds.$$

Passing to the limit as $n \to \infty$, (3.10) yields

$$\omega(\eta) = (1-e^{-\eta}) + \int_0^\infty G(\eta,s)g(\omega(s),\omega'(s))ds, \ \eta \in (0,\infty),$$

which implies that ω is a solution of (2.1). Hence, the sequence of approximants $\{w_n\}$ converges to a solution of the nonlinear BVP (2.1).

4. NUMERICAL RESULTS AND DISCUSSION

Results via GAM for different values of the parameter λ are obtained. Numerical simulation shows that only few iterations generated by the GAM lead to the exact solution of the problem independent of the choice of the parameter and the convergence is very fast. For example, see figures 1 and 2 for $f'(\eta)$ [or $u(\eta)$] corresponding to $\lambda = 0$, 0.2, 0.4 and 1. The case $\lambda = 0$ corresponds to the flat plane and $\lambda = 1$ corresponds to the plane stagnation point. Figures 3 and 4 represent the velocity profile corresponding to $\lambda = 0$, 0.2, 0.4, 1. Asymptotic behavior of $f'(\eta)$ is observed at η close to 4 and this property is attained much before 4 as the value of the parameter increases. The dimensionless shear stress $f''(\eta)$ [or $u'(\eta)$] corresponding to $\lambda = 0$, 0.2, 0.4, 1 are shown in figures 5 and 6. Finally, we study the effect of the parameter λ on the velocity field u. We observe that the dimensionless velocity $f'(\eta)$ of the fluid in the x-direction increases as the value of the parameter λ increases from 0 to 1 see for example, fig. 6.

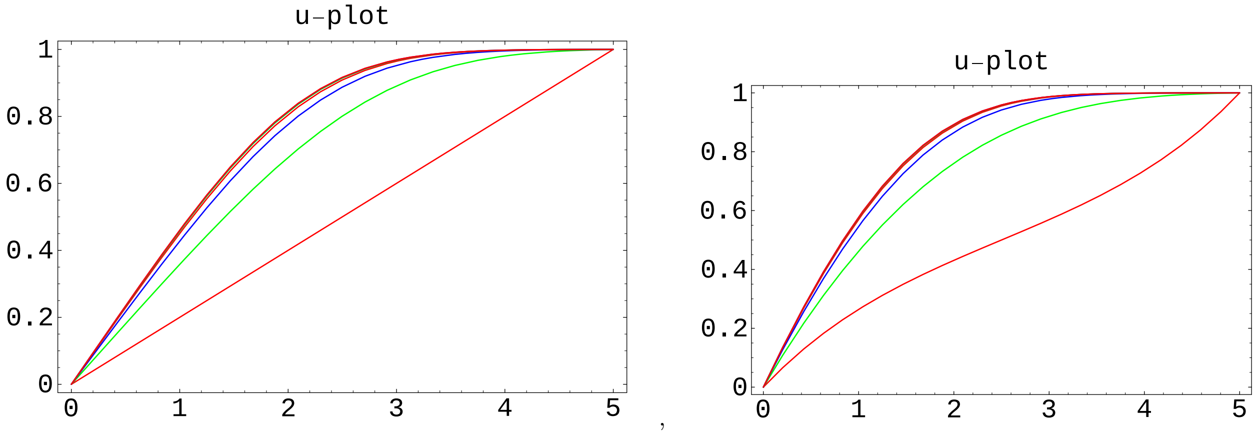

Fig.1, GAM iterations for $\lambda = 0$ (left graph) and $\lambda = 0.2$ (right graph)

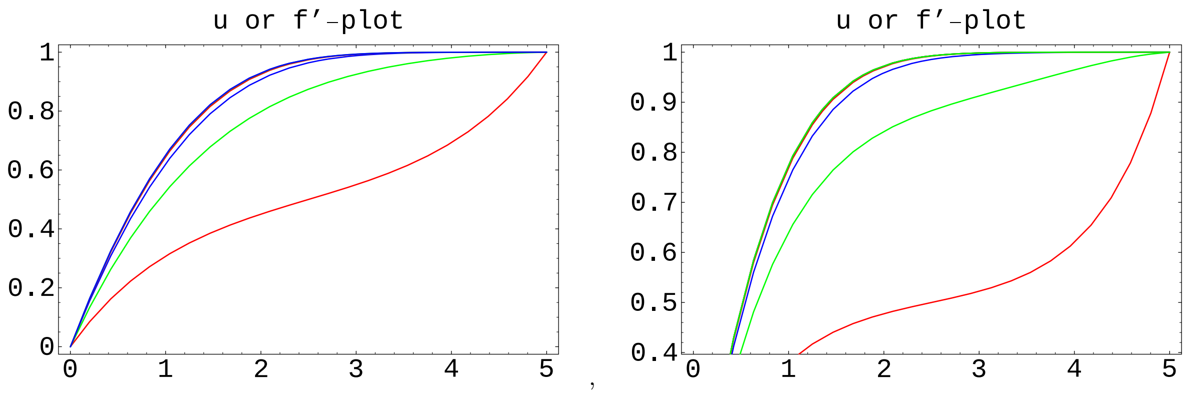

Fig.2, GAM iterations for $\lambda = 0.4$ (left graph) and $\lambda = 1$ (right graph)

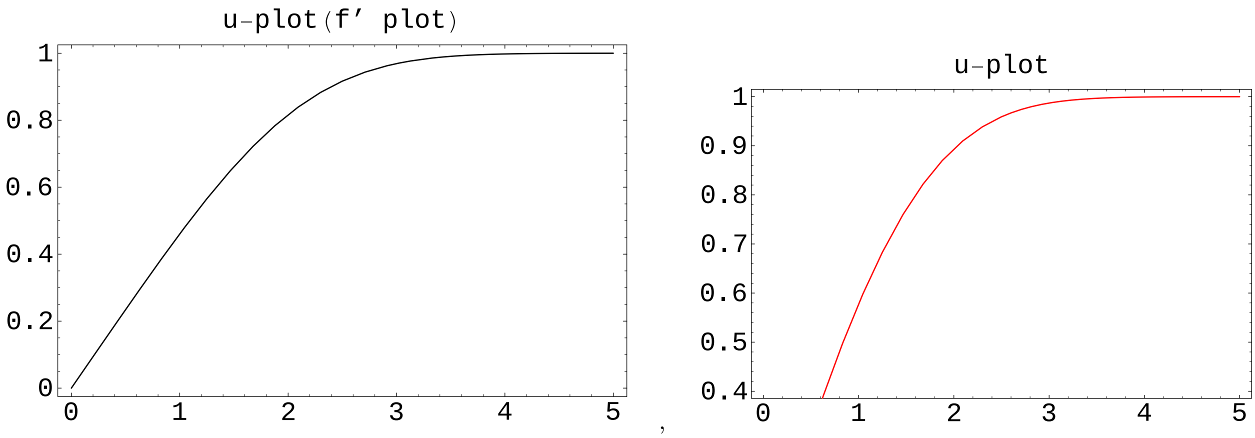

Fig.3, $u(\eta)$ or $f'(\eta)$ for $\lambda = 0$, (left graph) and $\lambda = 0.2$ (right graph)

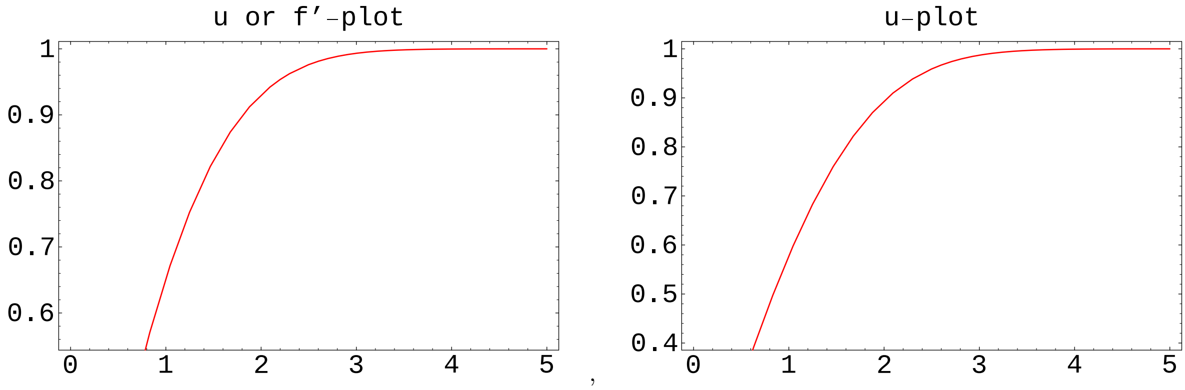

Fig.4, $u(\eta)$ or $f'(\eta)$ for $\lambda = 0.4$, (left graph), $\lambda = 1$ (right graph)

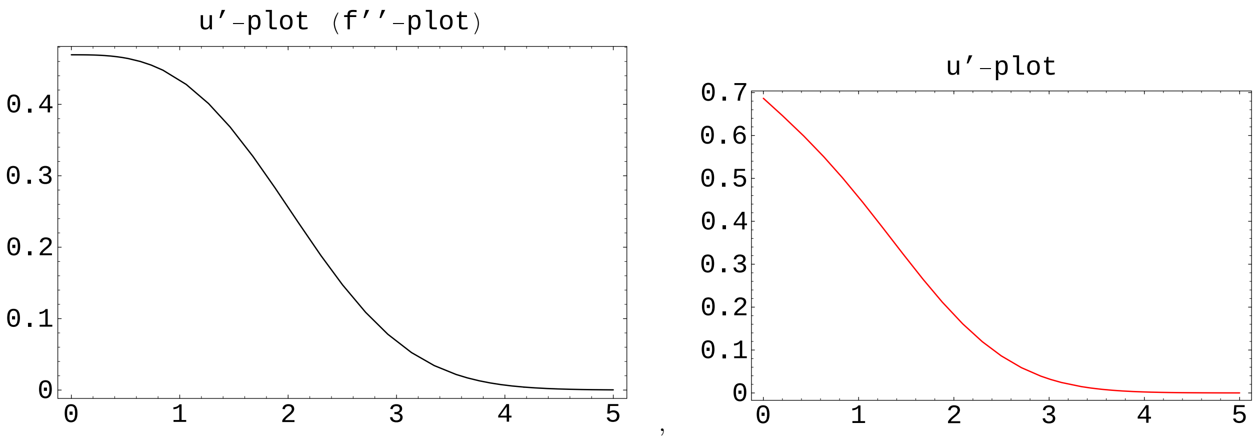

Fig.5, $u'(\eta)$ or $f''(\eta)$ for $\lambda = 0$, (left graph) and $\lambda = 0.2$ (right graph)

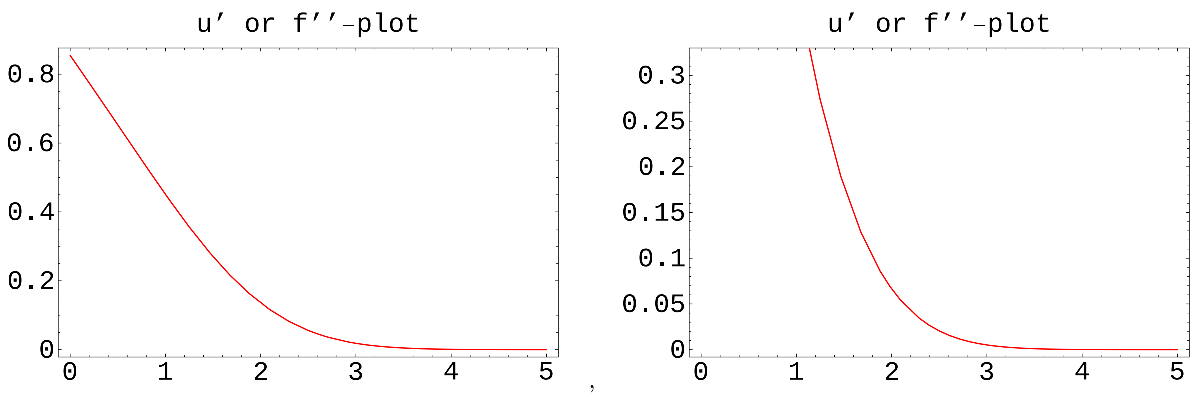

Fig.6, $u'(\eta)$ or $f''(\eta)$ for $\lambda = 0.4$ (left graph)and $\lambda = 1$ (right graph)

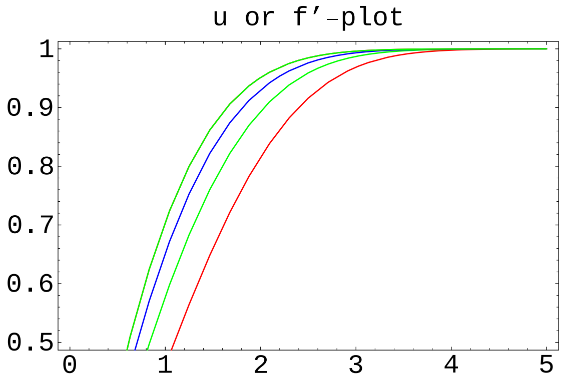

Fig.6, $u(\eta)$ or $f'(\eta)$ for $\lambda = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8$

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