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Involute Curve of the Biharmonic Curve in the Special Three-Dimensional Kenmotsu Manifold K with η-Parallel Ricci Tensor

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Abstract

In this paper, we study involute curve of the biharmonic curve in the special three-dimensional Kenmotsu manifold K *with* η *-parallel ricci tensor. We characterize involute curve by means of biharmonic curves in the special three-dimensional Kenmotsu manifold* K *with* η*-parallel ricci tensor.*

***Keywords**: biharmonic curve, involute curve, Kenmotsu manifold.*

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1 Introduction

The idea of a string involute is due to C. Huygens (1658), who is also known for his work in optics. He discovered involutes while trying to build a more accurate clock (see [2]). The involute of a given curve is a well-known concept in Euclidean-3 space $\mathbb{E}^3$.

An evolute and its involute, are defined in mutual pairs. The evolute of any curve is defined as the locus of the centers of curvature of the curve. The original curve is then defined as the involute of the evolute. The simplest case is that of a circle, which has only one center of curvature (its center), which is a degenerate evolute. The circle itself is the involute of this point.

In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to Lorentz manifolds. For instance, in [16], the authors extended and studied spacelike involute-evolute curves in Minkowski space-time.

A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left| \mathcal{T}(\phi) \right|^2 dv_h,$$

where $\mathcal{T}(\phi) := \operatorname{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler--Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \operatorname{tr}R(\mathcal{T}(\phi), d\phi)d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study involute curve of the biharmonic curve in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor. We characterize involute curve by means of biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor.

2 Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold with 1-form η, the associated vector field ξ, $(1,1)$-tensor field ϕ and the associated Riemannian metric g. It is well known that [1]

$$\phi\xi = 0, \eta(\xi) = 1, \eta(\phi X) = 0, \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$g(X,\xi)=\eta(X), \tag{2.3}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X, Y on M. Moreover,

$$(\nabla_X\phi)Y=-\eta(Y)\phi(X)-g(X,\phi Y)\xi, \quad X,Y\in\chi(M), \tag{2.5}$$

$$\nabla_X\xi=X-\eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g, then (M,ϕ,ξ,η,g) is called an almost Kenmotsu manifold [1].

In Kenmotsu manifolds the following relations hold [1]:

$$(\nabla_X\eta)Y=g(\phi X,\phi Y),$$
$$\eta(R(X,Y)Z)=\eta(Y)g(X,Z)-\eta(X)g(Y,Z),$$
$$R(\xi,X)Y=\eta(Y)X-g(X,Y)\xi,$$
$$R(\xi,X)\xi=X-\eta(X)\xi,$$
$$S(\phi X,\phi Y)=S(X,Y)+2n\eta(X)\eta(Y),$$
$$(\nabla_X R)(X,Y)\xi=g(Z,X)Y-g(Z,Y)X-R(X,Y)Z,$$

where R is the Riemannian curvature tensor and S is the Ricci tensor. In a Riemannian manifold we also have

$$g(R(W,X)Y,Z)+g(R(W,X)Z,Y)=0,$$

for every vector fields X,Y,Z.

3 Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$ with η-Parallel Ricci Tensor

Definition 3.1 The Ricci tensor S of a Kenmotsu manifold is called η-parallel if it satisfies

$$(\nabla_X S)(\phi Y,\phi Z)=0.$$

We consider the three-dimensional manifold

$$\mathbb{K}=\{(x^1,x^2,x^3)\in\mathbb{R}^3:(x^1,x^2,x^3)\neq(0,0,0)\},$$

where (x^1,x^2,x^3) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$\mathbf{e}_1=x^3\frac{\partial}{\partial x^1},\ \mathbf{e}_2=x^3\frac{\partial}{\partial x^2},\ \mathbf{e}_3=-x^3\frac{\partial}{\partial x^3} \tag{3.1}$$

are linearly independent at each point of $\mathbb{K}$. Let g be the Riemannian metric defined by

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1, \tag{3.2}$$
$$g(\mathbf{e}_1,\mathbf{e}_2)=g(\mathbf{e}_2,\mathbf{e}_3)=g(\mathbf{e}_1,\mathbf{e}_3)=0.$$

The characterising properties of $\chi(\mathrm{K})$ are the following commutation relations:

$$[\mathbf{e}_1,\mathbf{e}_2]=0,\ [\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_1,\ [\mathbf{e}_2,\mathbf{e}_3]=\mathbf{e}_2. \tag{3.3}$$

Let η be the 1-form defined by

$$\eta(Z)=g(Z,\mathbf{e}_3)\text{for any } Z\in\chi(M)$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1)=-\mathbf{e}_2,\phi(\mathbf{e}_2)=\mathbf{e}_1,\phi(\mathbf{e}_3)=0.$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3)=1, \tag{3.4}$$

$$\phi^2(Z)=-Z+\eta(Z)\mathbf{e}_3, \tag{3.5}$$

$$g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W), \tag{3.6}$$

for any $Z,W\in\chi(M)$. Thus for $\mathbf{e}_3=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on M.

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned}&2g(\nabla_X Y,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)\\&-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]),\end{aligned}$$

which is known as Koszul's formula.

Koszul's formula yields

$$\begin{aligned}&\nabla_{\mathbf{e}_1}\mathbf{e}_1=0,\nabla_{\mathbf{e}_1}\mathbf{e}_2=0,\nabla_{\mathbf{e}_1}\mathbf{e}_3=\mathbf{e}_1,\\&\nabla_{\mathbf{e}_2}\mathbf{e}_1=0,\ \nabla_{\mathbf{e}_2}\mathbf{e}_2=0,\nabla_{\mathbf{e}_2}\mathbf{e}_3=\mathbf{e}_2,\\&\nabla_{\mathbf{e}_3}\mathbf{e}_1=0,\nabla_{\mathbf{e}_3}\mathbf{e}_2=0,\nabla_{\mathbf{e}_3}\mathbf{e}_3=0.\end{aligned} \tag{3.7}$$

Moreover we put

$$R_{ijk}=R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k,\ R_{ijkl}=R(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_l),$$

where the indices i,j,k and l take the values $1,2$ and 3.

$$R_{121}=0,\ R_{131}=R_{232}=\mathbf{e}_3$$

and

$$R_{1212}=0,\ R_{1313}=R_{2323}=1. \tag{3.8}$$

4 Biharmonic Curves in the Special Three-Dimensional Kenmotsu Manifold K with η-Parallel Ricci Tensor

Biharmonic equation for the curve γ reduces to

$$\nabla_{\mathbf{T}}^{3}\mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0, \tag{4.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (4.1).

Let us consider biharmonicity of curves in the special three-dimensional Kenmotsu manifold K with η-parallel ricci tensor. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{4.2}$$

where $\kappa = |\mathrm{T}(\gamma)| = |\nabla_{\mathbf{T}}\mathbf{T}|$ is the curvature of γ and τ its torsion and

$$\begin{aligned} &g(\mathbf{T},\mathbf{T}) = 1,\ g(\mathbf{N},\mathbf{N}) = 1,\ g(\mathbf{B},\mathbf{B}) = 1, \\ &g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{4.3}$$

Theorem 4.1 *(see [11]) $\gamma : I \to \mathrm{K}$ is a biharmonic curve if and only if*

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\kappa^2 + \tau^2 = 1 - B_3^2, \\ &\tau' = N_3 B_3. \end{aligned} \tag{4.4}$$

Theorem 4.2 *(see [11]) Let $\gamma : I \to \mathrm{K}$ be a non-geodesic curve on the special three-dimensional Kenmotsu manifold K with η-parallel ricci tensor parametrized by arc length. If κ is constant and $N_3B_3 \neq 0$, then γ is not biharmonic.*

5 Involute Curves in the Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$ with η-Parallel Ricci Tensor

Definition 5.1 *Let unit speed curve $\gamma : I \to \mathbb{K}$ and the curve $\beta : I \to \mathbb{K}$ be given. For $\forall s \in I$, then the curve β is called the involute of the curve γ, if the tangent at the point $\gamma(s)$ to the curve γ passes through the tangent at the point $\beta(s)$ to the curve β and*

$$g\left(\mathbf{T}^*(s), \mathbf{T}(s)\right) = 0. \tag{5.1}$$

Let the Frenet-Serret frames of the curves γ and β be $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and $\{\mathbf{T}^*,\mathbf{N}^*,\mathbf{B}^*\}$, respectively.

Theorem 5.2 *Let the curve β be involute of the the curve γ and let ρ be a constant real number. Then, the parametric equation of involute curve β are*

$$\begin{aligned}
&x_\beta^1(s) = C_2 - \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s}(\cos[s+C] - \cos\varphi \sin[s+C]) \\
&+ (\rho - s) C_1 e^{-\cos\varphi s} \sin\varphi \sin[s+C], \\
&x_\beta^2(s) = C_3 + \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s}(\cos\varphi \cos[s+C] - \sin[s+C]) \\
&+ (\rho - s) C_1 e^{-\cos\varphi s} \sin\varphi \cos[s+C], \\
&x^3(s) = C_1 e^{-\cos\varphi s} - (\rho - s) C_1 e^{-\cos\varphi s} \cos\varphi,
\end{aligned} \tag{5.2}$$

where C, C_1, C_2, C_3 are constants of integration and $= \sqrt{-\cos^2\varphi + \dfrac{\kappa^2}{\sin^2\varphi}}$.

Proof. The curve $\beta(s)$ may be given as

$$\beta(s) = \gamma(s) + u(s)\mathbf{T}(s). \tag{5.3}$$

On the other hand, since γ is biharmonic, γ is a helix. So, without loss of generality, we take the axis of γ is parallel to the vector $\mathbf{e}_3$. Then,

$$g(\mathbf{T}, \mathbf{e}_3) = T_3 = \cos\varphi, \tag{5.4}$$

where φ is constant angle.

If we take the derivative (5.3), then we have

$$\beta'(s) = \left(1 + u'(s)\right)\mathbf{T}(s) + u(s)\kappa(s)\mathbf{N}(s).$$

Since the curve β is involute of the curve γ, $g(\mathbf{T}^*(s), \mathbf{T}(s)) = 0$. Then, we get

$$1 + u'(s) = 0 \text{or}\ u(s) = \rho - s, \tag{5.5}$$

where ρ is constant of integration.

Substituting (5.5) into (5.3), we get

$$\beta(s) = \gamma(s) + (\rho - s)\mathbf{T}(s). \tag{5.6}$$

The tangent vector can be written in the following form

$$\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3. \tag{5.7}$$

On the other hand the tangent vector $\mathbf{T}$ is a unit vector, so the following condition is satisfied

$$T_1^2 + T_2^2 = 1 - \cos^2\varphi. \tag{5.8}$$

Noting that $\cos^2\varphi + \sin^2\varphi = 1$, we have

$$T_1^2 + T_2^2 = \sin^2\varphi. \tag{5.9}$$

The general solution of (5.9) can be written in the following form

$$T_1 = \sin\varphi \sin\mu, \tag{5.10}$$

$$T_2 = \sin\varphi \cos\mu,$$

where μ is an arbitrary function of s.

So, substituting the components T_1, T_2 and T_3 in the equation (5.3), we have the following equation

$$\mathbf{T} = \sin\varphi \sin\mu\mathbf{e}_1 + \sin\varphi \cos\mu\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \tag{5.11}$$

Since $|\nabla_{\mathbf{T}}\mathbf{T}| = \kappa$, we obtain

$$\mu = \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C, \tag{5.12}$$

where $C \in \mathbb{R}$.

Thus (5.11) and (5.12), imply

$$\mathbf{T} = \sin\varphi \sin[s + C]\mathbf{e}_1 + \sin\varphi \cos[s + C]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3, \tag{5.13}$$

where $= \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}$.

Using (3.1) in (5.13), we obtain

$$\mathbf{T} = (x^3 \sin\varphi \sin[s + C], x^3 \sin\varphi \cos[s + C], -x^3 \cos\varphi). \tag{5.14}$$

From third component of $\mathbf{T}$, we have

$$x^3(s) = C_1 e^{-\cos\varphi s}, \tag{5.15}$$

where C_1 is constant of integration.

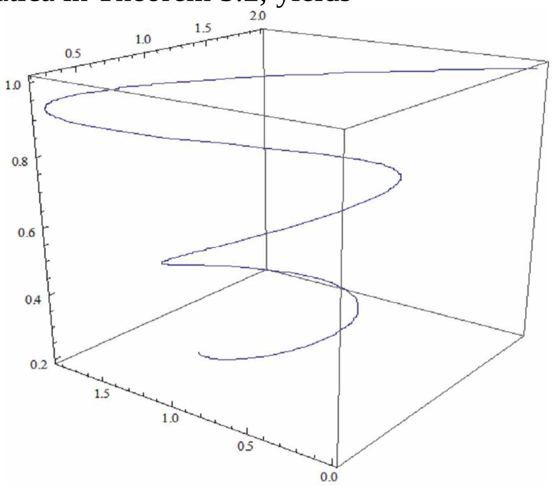
By direct calculations we have

$$x^1(s) = C_2 - \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s}(\cos[s+C] - \cos\varphi \sin[s+C]), \tag{5.16}$$

$$x^2(s) = C_3 + \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s}(\cos\varphi \cos[s+C] - \sin[s+C]). \tag{5.17}$$

Next, we substitute (5.15), (5.16), (5.17) and (5.14) into (5.6), we get (5.2). The proof is completed.

We can use Mathematica in Theorem 5.2, yields



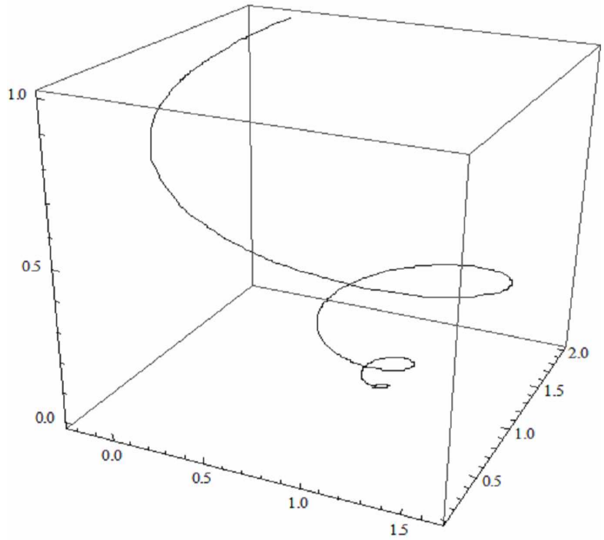
$$\cos\varphi = \sin\varphi = \frac{\sqrt{2}}{2}, \rho = C = C_1 = C_2 = C_3 = \kappa = 1.$$

Corollary 5.3 *Let $\gamma : I \to K$ be a unit speed non-geodesic curve with constant curvature. Then, the parametric equations of γ are*

$$\begin{aligned} x^1(s) &= C_2 - \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s}(\cos[s+C] - \cos\varphi \sin[s+C]), \\ x^2(s) &= C_3 - \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s}(-\cos\varphi \cos[s+C] + \sin[s+C]), \\ x^3(s) &= C_1 e^{-\cos\varphi s}, \end{aligned} \tag{5.18}$$

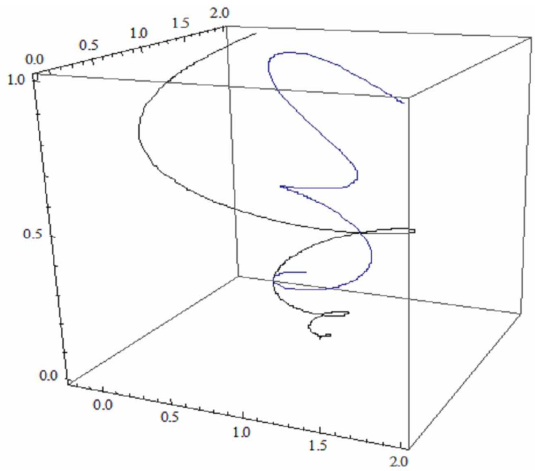
where C, C_1, C_2, C_3 are constants of integration and $= \sqrt{-\cos^2\varphi + \dfrac{\kappa^2}{\sin^2\varphi}}$.

We can use Mathematica in Corollary 5.3, yields



$$\cos\varphi = \sin\varphi = \frac{\sqrt{2}}{2}, C = C_1 = C_2 = C_3 = \kappa = 1.$$

Similarly, if we use Mathematica both involute curve and biharmonic curve, we have



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