*Gen. Math. Notes, Vol. 2, No. 1, January 2011, pp. 221-231*
*ISSN 2219-7184; Copyright © ICSRS Publication, 2011*
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# An Analytic Approach for Some Equations Arising in Heat Transfer in a Quiescent Medium with Exponential Nonlinearities

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(Received 25.10.2010, Accepted 15.12.2010)

**Abstract**

*In this article, the homotopy analysis method (HAM) has been applied to obtain the analytical approach for obtaining approximate series solutions to some initial value problems arising in heat transfer in a quiescent medium. This method does not need linearization, weak nonlinearity assumptions or perturbation theory. The validity of this method has successfully been accomplished by applying it to some examples. The results show that the method is very effective and convenient for solving such equations.*

**Keywords**: *Heat transfer; Series solution; Homotopy analysis method; Symbolic computation; Analytical solution.*

# 1 Introduction

Partial differential equations arise in engineering, applied mathematics and several branches of physics, and have attracted much attention. However, it is usually difficult to obtain closed-form solutions for such equations, especially for nonlinear ones. In most cases, only approximate solutions (either numerical solutions or analytical solutions) can be expected. Some numerical methods such as finite difference method [1], finite element method [2] have been developed for obtaining approximate solutions to partial differential equations. Perturbation method [3] is one of the well-known analytical methods for solving such nonlinear problems. However, it strongly depends on the existence of small/large parameters. Traditional non-perturbation methods such as Adomian's decomposition method [4] and Homotopy perturbation method [5] have been developed for solving nonlinear differential equations. However, these methods cannot provide a mechanism to adjust and control the convergence region and rate of the series solutions.

The rest of present contribution is organized, as follows.

In Section 2 of this paper, based on the Homotopy analysis method [6-8] (HAM) we propose an analytical approach for solving the following type of PDEs

$$\frac{\partial w}{\partial t} = a \frac{\partial}{\partial x}\left( e^{\lambda w} \frac{\partial w}{\partial x} \right) + f(w),\ (x,t) \in \Omega. \tag{1}$$

subject to the initial condition

$$w(x,0) = g(x)$$

where $w(x,0) = g(x)$ is a known functions.

This equation is often encountered in nonlinear problems of unsteady heat transfer in a quiescent medium in the case where the thermal diffusivity is exponentially dependent on temperature [9].

To demonstrate its effectiveness, the approach is applied to solve two cases of $f(w)$ in Section 3. It is shown that the series solution obtained via this approach is good agreement with exact ones. The success of this approach lies in the fact that the HAM provides a convenient way to adjust and control the convergence region and rate of the series solutions obtained. Finally in Section 4, some concluding remarks are given.

## 2 The HAM-based Approach

In order to obtain a convergent series solution to the nonlinear problem (1), we first construct the zeroth order deformation equation

$$(1-p)L[\omega(x,t;p)-w_0(x,t;p)]=p\hbar L[\omega(x,t;p)], \tag{2}$$

where $p\in[0,1]$ is an embedding parameter, $\hbar\neq 0$ is a convergence-control parameter, and $\omega(x,t;p)$ is an unknown function, respectively.

According to (1) the nonlinear operator $N$ is given by

$$N[\omega(x,t;p)]=\frac{\partial\omega}{\partial t}-a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial\omega}{\partial x}\right)-f(\omega).$$

The initial guess $w_0(x,t)$ of the solution $w(x,t)$ can be determined by the rule of solution expression as follows.

We now focus on how to obtain higher order approximations to the problem (1). From (2), when $p=0$ and $p=1$,

$$\omega(x,t;0)=w_0(x,t)=w(x,0)\ \ and\ \ \omega(x,t;1)=w(x,t) \tag{3}$$

both hold. Therefore, as $p$ increases from 0 to 1, the solution $\omega(x,t;p)$ varies from the initial guess $w_0(x,t)$ to the solution $w(x,t)$. Expanding $\omega(x,t;p)$ in Taylor series with respect to $p$, one has

$$\omega(x,t;p)=w_0(x,t)+\sum_{m=1}^{\infty}w_m(x,t)p^m. \tag{4}$$

where

$$w_k(x,t)=\frac{1}{m!}\frac{\partial^m\omega(x,t;p)}{\partial p^m}\bigg|_{p=0}.$$

Now the convergence of the series (4) depends on the parameter $\hbar$.

Assuming that $\hbar$ is chosen so properly that the series (4) is convergent at $p=1$, we have the solution series

$$w(x,t)=\omega(x,t,1)=w_0(x,t)+\sum_{k=1}^{\infty}w_k(x,t).$$

which must be one of the solutions of the original problem (1), as proved by Liao in [7].

Our next goal is to determine the higher order terms $w_m(x,t)(m\geq 1)$. Define the vector

$$\vec{w}_s(x,t)=\{w_0(x,t),\ldots,w_s(x,t)\}.$$

Differentiating the zeroth order deformation equation (2) $m$ times with respect to $p$, then setting $p=0$, finally dividing them by $m!$, we obtain the $m$ th order deformation equation

$$L[w_m(x,t)-\chi_m w_{m-1}(x,t)]=\hbar R_m(\vec{w}_{m-1}(x,\xi)), \tag{5}$$

and its initial conditions

$$w_m(x,0)=0,$$

where

$$R_m(\vec{w}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathrm{N}[\omega(x,t,p)]}{\partial p^{m-1}}\bigg|_{p=0},$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases}$$

Note that the $m$ th order deformation equation (5) becomes

$$w_m(x,t)=\chi_m w_{m-1}(x,t)+\hbar\int_0^t R_m(\vec{w}_{m-1}(x,\xi))d\xi+c_m.$$

In view of the $w_m(x,0)=0,$ the coefficients $c_m$ can be determined.

In this way, we can obtain $w_k(x,t)$'s recursively.

The $m$ th order approximation to the problem (1) can be generally expressed by

$$w(x,t)=\sum_{k=0}^{m}w_k(x,t).$$

Such equation is a family of solutions to the problem (1) expressed in terms of the parameter $\hbar$.

To obtain an accurate approximation to the problem (1), a proper value of $\hbar$ must be found. First, the valid region of $\hbar$ can be obtained via the $\hbar$-curve as follows.

Let $c_0=(x_0,t_0)\in\Omega$. Then $u(c_0,\hbar)$ is a function of $\hbar$, and the curve $u(c_0,\hbar)$ versus $\hbar$ contains a horizontal line segment which corresponds to the valid region of $h$. The reason is that all convergent series given by different values of $\hbar$ converge to its exact value. So, if the solution is unique, then all of these series converge to the same value and therefore there exists a horizontal line segment in the curve. We call such kind of curve the $\hbar$-curve.

Although the solution series given by different values in the valid region of converge to the exact solution, the convergence rates of these solution series are usually different. A more accurate approximation can be obtained by assigning $\hbar$ a proper value. By substituting the $m$ th order approximation into the original governing equation (1) and then integrating the square residual error over the whole domain of problem, one gets a function of $\hbar$, denoted $F(\hbar)$. Minimizing $F(\hbar)$ gives the best value of $\hbar$ which corresponds to the best approximation. However, it is usually difficult to minimize $F(\hbar)$. Alternatively, one can obtain a proper value of $\hbar$ just by observation.

# 3 Applications

In this section, the approach proposed in Section 2 is applied to following two cases of $f(w)$ in (1), which are taken from [9].

Example 1. If we take $f(w)=0$ in Eq. (1). Thus the equation becomes

$$\frac{\partial w}{\partial t}=a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right). \tag{6}$$

Exact solution of this equation with the initial condition

$$w(x,0)=\frac{1}{\lambda}\ln\left(\lambda\mu x^2+Ax+B\right).$$

is

$$w(x,t)=\frac{1}{\lambda}\ln\left(\frac{\lambda\mu x^2+Ax+B}{1-2a\lambda\mu t}\right). \tag{7}$$

where $A, B, C$ and $\mu$ are arbitrary constants.

For simplicity, let us take $A=B=0, \lambda=\mu=1.$ and $a=-1$ in (7). Thus, one has

$$w(x,t)=\ln\left(\frac{x^2}{1+2t}\right). \tag{8}$$

Now Eq. (6) is solved with the consideration of the initial condition

$$w_0(x,t)=w(x,0)=\ln\left(x^2\right).$$

In this example, we consider the auxiliary linear operator $L$ as

$$L[\omega(x,t;p)]=\frac{\partial\omega}{\partial t}+\omega,$$

and the nonlinear operator $N$ given by

$$N[\omega(x,t;p)]=\frac{\partial\omega}{\partial t}+\left(e^{\omega}(\frac{\partial\omega}{\partial x})^2+e^{\omega}\frac{\partial^2\omega}{\partial x^2}\right).$$

In view of such linear operator, the solution $w(x,t)$ can be expressed by a set of base functions

$$\{t^n \,|\, n=0,1,2,\ldots\}$$

in the form

$$w(x,t)=\sum_{n=0}^{+\infty} d_n(x)t^n.$$

where $d_m(m=0,1,2,\ldots)$ are functions in $x$ to be determined later. This provides us with the rule of solution expression.

To obtain higher order terms $w_m(x,t)$, the $m$ th order deformation equation and its boundary conditions are calculated:

$$\frac{\partial}{\partial t}w_m(x,t)=\chi_m\frac{\partial}{\partial t}w_{m-1}(x,t)+\hbar R_m(\vec{w}_{m-1}(x,\xi)).$$
$$w_m(x,0)=0.$$

where

$$R_m(\vec{w}_{m-1})=\frac{\partial}{\partial t}w_{m-1}+\sum_{i=0}^{m-1}\sum_{j=0}^{i}A_i\frac{\partial w_{m-1-i-j}}{\partial x}\frac{\partial w_j}{\partial x}+\sum_{i=0}^{m-1}A_i\frac{\partial^2 w_{m-1-i}}{\partial x^2}.$$

and

$$A_k=\frac{1}{k!}\frac{\partial^k}{\partial p^k}(e^{\omega(x,t:p)})\Big|_{p=0}.$$

Here, some first few terms of $A_k$ 's have been calculated

$$A_0 = e^{w_0(x,t)}$$

$$A_1 = w_1(x,t)e^{w_0(x,t)}$$

$$A_2 = e^{w_0(x,t)}\left(w_1(x,t)+\frac{1}{2}w_1(x,t)^2\right)$$

$$A_3 = e^{w_0(x,t)}\left(w_3(x,t)+w_1(x,t)w_2(x,t)+\frac{1}{6}w_1(x,t)^3\right).$$

It should be noted that the notation of $A_k$ have used throughout. In this way, we can calculate $w_k(x,t),(k=0,1,2,\ldots)$ recursively.

The $m$ th order approximation can be expressed by

$$\tilde{w}_m(x,t) \approx \sum_{k=0}^{m} w_k(x,t).$$

To find the valid region of $\hbar$, the $\hbar$-curve given by the 5th order approximation at $(x_0,t_0)=(1,1)$ is drawn in Fig. 1, which clearly indicates that the valid region of $\hbar$ is about $-0.65<\hbar<-0.3$.

When $\hbar=-0.52$, we obtain an approximate solution which is good agreement with exact solution as shown in Fig 2, where the absolute errors of the shooting method approximation, the 5th order HAM approximations are depicted.

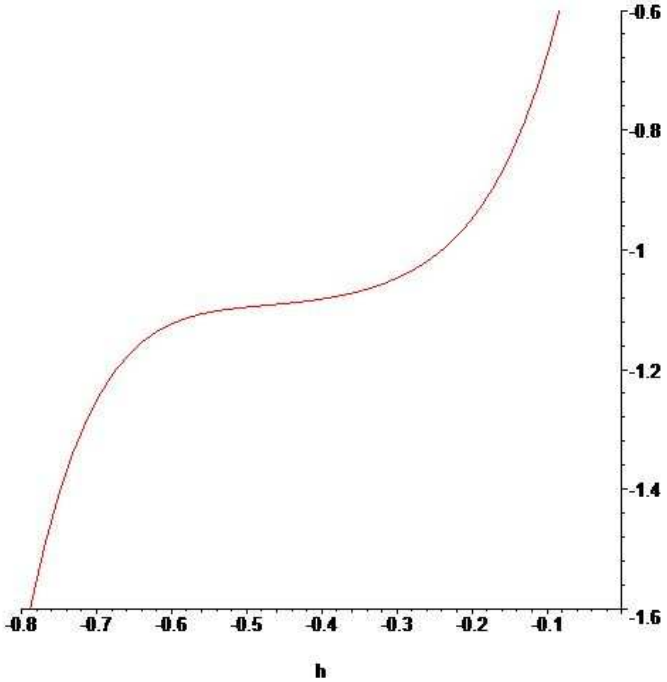


Fig. 1. $\hbar$-curve for the 5th order of HAM approximation ($\tilde{w}_5(1,1)$ versus $\hbar$)

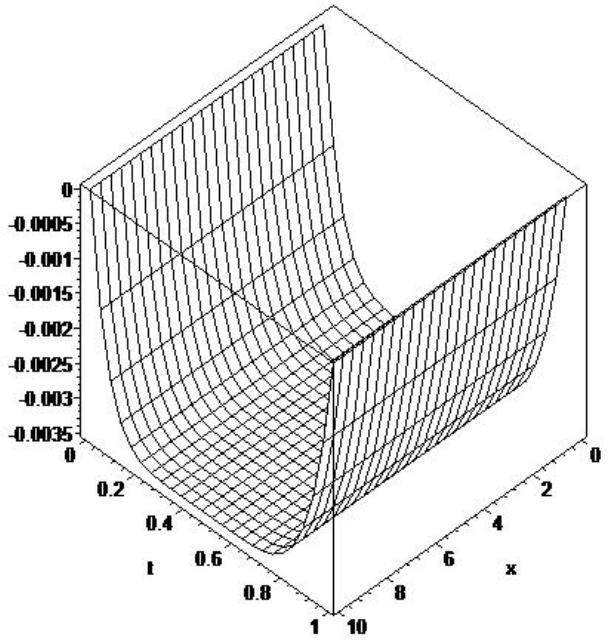


Fig. 2. Absolute errors of 5th order of HAM approximations at $\hbar = -0.52$

*Example 2.* For $f(w) = b + ce^{-\lambda w}$, (1) becomes

$$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) + b + ce^{-\lambda w}. \tag{9}$$

In [9], the authors give an exact solution to (9), as follows:

$$w(x,t) = \frac{1}{\lambda}\ln\left(c\lambda t - \frac{b\lambda}{2a}x^2 + C_1 x + C_2\right). \tag{10}$$

where $C_1$ and $C_2$ are arbitrary constants.

For the sake of simplicity, let $a = -1, b = 2, c = 1, \lambda = 1$ and $C_1 = 0, C_2 = 5$. Using such values in the solution (10), it becomes

$$w(x,t) = \ln\left(t + x^2 + 5\right). \tag{11}$$

Now (10) is solved with the consideration of the initial condition

$$w_0(x,t) = w(x,0) = \ln\left(x^2 + 5\right).$$

For the zeroth order deformation equation, following linear and nonlinear operators are respectively used

$$L[\omega(x,t;p)]=\frac{\partial\omega}{\partial t},\tag{12}$$
$$N[\omega(x,t;p)]=\frac{\partial\omega}{\partial t}+\left(e^{\omega}(\frac{\partial\omega}{\partial x})^2+e^{\omega}\frac{\partial^2\omega}{\partial x^2}\right)-2-e^{\omega}.$$

According to (12), following set of base functions is suggested

$$\{\exp(-kt)\,|\,k>0\}.$$

Therefore the solution $w(x,t)$ can be expressed in the form

$$w(x,t)=\sum_{k=0}^{+\infty}d_k(x)e^{-kt}.$$

where $d_m(m=0,1,2,\ldots)$ are functions in $x$ to be determined later.

To obtain higher order terms $w_m(x,t)$, the $m$ th order deformation equation are calculated:

$$\frac{\partial}{\partial t}w_m(x,t)=\chi_m\frac{\partial}{\partial t}w_{m-1}(x,t)+\hbar R_m(\vec{w}_{m-1}(x,\xi)).$$
$$w_m(x,0)=0.$$

where

$$R_m(\vec{w}_{m-1})=\frac{\partial}{\partial t}w_{m-1}+\sum_{i=0}^{m-1}\sum_{j=0}^{i}A_i\frac{\partial w_{m-1-i-j}}{\partial x}\frac{\partial w_j}{\partial x}+\sum_{i=0}^{m-1}A_i\frac{\partial^2 w_{m-1-i}}{\partial x^2}+2(\chi_m-1)-A_{m-1}.$$

In this way, one can calculate $w_k(x,t)$ for $k=0,1,2,\ldots$ recursively.

To find the valid region of $\hbar$ , the $\hbar$ -curve given by the 5th order approximation at $\hbar$ is drawn in Fig. 3, which clearly indicates that the valid region of is about $-1.3<\hbar<-0.4$.

When $\hbar=-1$, we obtain an approximate series solution which agrees very well with the exact solution given (11), as shown in Table 1 where the absolute errors of the 5th order HAM approximations for $\hbar=-1$ at different points are calculated.

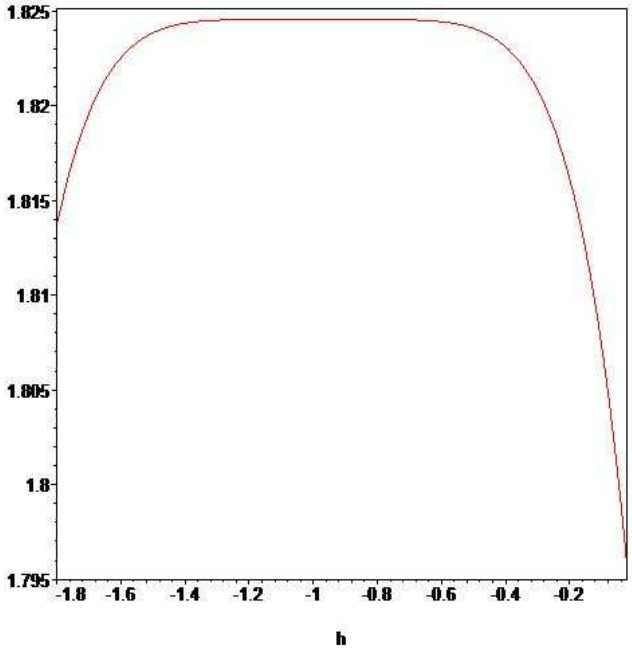


Fig. 3. $\hbar$-curve for the 5th order of HAM approximation ($\tilde{w}_5(1,1)$ versus $\hbar$)

# 4 Conclusions

In this paper, we obtained the HAM series solution of a problem in heat transfer in a quiescent medium. It can be seen that the series solution of the problem by using HAM is very close to the exact solution of the problem which is given in Examples 1 and 2. This shows us that no matter how many nonlinear terms the partial differential equations have; we can find series solution of the partial differential equation without linearization of them.

We got some tables and figures to show that the series solution converges very rapidly to the exact solution. The success of this approach lies in the fact that the HAM provides a convergence-control parameter which can be used to adjust and control the convergence region and rate of the series solutions obtained.

Table 1

The comparisons between HAM and HPM for various values of $x$ and $t$ in Example 2.

| $x$ | $t$ | HAM | Exact |
|---|---|---|---|
| 1 | 1 | 1.945909703 | 1.945910149 |
| | 3 | 2.197874312 | 2.197224578 |
| | 5 | 2.197224578 | 2.397895273 |
| | 1 | 2.302585066 | 2.302585093 |

| | | | |
|---|---|---|---|
| 2 | 3 | 2.484856037 | 2.484906650 |
| | 5 | 2.637484331 | 2.639057330 |
| 3 | 1 | 2.708050200 | 2.708050201 |
| | 3 | 2.833210848 | 2.833213344 |
| | 5 | 2.944358236 | 2.944438979 |
| 4 | 1 | 3.091042454 | 3.091042453 |
| | 3 | 3.178053676 | 3.178053830 |
| | 5 | 3.258091408 | 3.258096538 |
| 5 | 1 | 3.433987205 | 3.433987204 |
| | 3 | 3.496507549 | 3.496507561 |
| | 5 | 3.555347616 | 3.555348061 |

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