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Distance Two Labeling of Some Total Graphs

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Abstract

An $L(2,1)$-labeling (or distance two labeling) of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$. The $L(2,1)$-labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)$-labeling with $max\{f(v) : v \in V(G)\} = k$. In this paper we completely determine the λ-number for total graphs of path P_n, cycle C_n, star $K_{1,n}$ and friendship graph F_n.

Keywords: $L(2,1)$-*labeling,* λ-*number, Total Graph.*

1 Introduction

The channel assignment problem is the problem to assign a channel (non negative integer) to each TV or radio transmitters located at various places such that communication do not interfere. This problem was first formulated as a graph coloring problem by Hale[5] who introduced the notion of T-coloring of a graph.

In a graph model of this problem, the transmitters are represented by the vertices of a graph; two vertices are *very close* if they are adjacent in the graph and *close* if they are at distance two apart in the graph.

In a private communication with Griggs during 1988 Roberts proposed a variation of the channel assignment problem in which *close* transmitters must receive different channels and *very close* transmitters must receive channels that are at least two apart. Motivated by this problem Griggs and Yeh[4] introduced $L(2,1)$-labeling which is defined as follows.

Definition 1.1 *An $L(2,1)$-labeling (or distance two labeling) of a graph $G = (V(G), E(G))$ is a function f from vertex set $V(G)$ to the set of all nonnegative integers such that the following conditions are satisfied:*
(1) $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$
(2) $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$

A k-$L(2,1)$-labeling is an $L(2,1)$-labeling such that no labels is greater than k. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$ or λ, is the smallest number k such that G has a k-$L(2,1)$-labeling. In the discussion of $L(2,1)$-labeling we take $[0,k] = \{0,1,2,...,k\}$. The $L(2,1)$-labeling has been extensively studied in recent past by many researchers like Yeh[10], Sakai[7], Georges, Mauro and Whittlesey[3], Georges and Mauro[2] and Vaidya et al.[8].

The common trend in most of the research papers is either to determine the λ-number or to suggest bounds for particular graph structures. In the present work we completely determine the λ-number of total graph of various graphs.

Through out this work, we consider the finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For all other standard terminology and notations we refer to West[9]. We will give brief summary of definitions and information which are prerequisites for the present work.

Definition 1.2 *The total graph of a graph G is the graph whose vertex set is $V(G) \bigcup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G. The total graph of G is denoted by $T(G)$.*

Proposition 1.3 *[4] The λ-number of a star $K_{1,\Delta}$ is $\Delta + 1$, where Δ is the maximum degree.*

Proposition 1.4 *[4] The λ-number of a complete graph K_n is $2n - 2$.*

Proposition 1.5 *[1] $\lambda(H) \leq \lambda(G)$, for any subgraph H of a graph G.*

Proposition 1.6 *[6] Let G be a graph with maximum degree $\Delta \geq 2$. If G contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$.*

2 Main Results

Theorem 2.1 *For the total graph $T(P_n)$ of path P_n,*

$$\lambda(T(P_n)) = \begin{cases} 4; & if \quad n = 2 \\ 5; & if \quad n = 3 \\ 6; & if \quad n \geq 4 \end{cases}$$

Proof: Let v_0, v_1, ..., v_{n-1} and $e_0 = (v_0, v_1)$, $e_1 = (v_1, v_2)$, ..., $e_{n-2} = (v_{n-2}, v_{n-1})$ are the vertices and edges of path P_n then $V(T(P_n)) = \{v_0, v_1, ..., v_{n-1}, e_0, e_1, ..., e_{n-2}\}$. For $n = 2$, the graph $T(P_2)$ is a graph K_3 and hence by Proposition 1.4, $\lambda(T(P_2)) = 2(3) - 2 = 4$. For $n \geq 3$, the graph $K_{1,4}$ is a subgraph of $T(P_n)$ and hence by Proposition 1.3 and 1.5, $\lambda(T(P_n)) \geq 5$. For $n \geq 4$, in the graph $T(P_n)$, the close neighborhood of each e_i where $i = 1, ..., n-3$ contains three vertices with degree Δ. Hence by Proposition 1.6, $\lambda(T(P_n)) \geq 6$. Now define $f : V(T(P_n)) \rightarrow \{0, 1, ..., 6\}$ as follows.

$$\begin{array}{l} f(v_i) = 4,\ f(e_i) = 2 \text{ if } i \equiv 0 \pmod 7 \\ f(v_i) = 0,\ f(e_i) = 5 \text{ if } i \equiv 1 \pmod 7 \\ f(v_i) = 3,\ f(e_i) = 1 \text{ if } i \equiv 2 \pmod 7 \\ f(v_i) = 6,\ f(e_i) = 4 \text{ if } i \equiv 3 \pmod 7 \\ f(v_i) = 2,\ f(e_i) = 0 \text{ if } i \equiv 4 \pmod 7 \\ f(v_i) = 5,\ f(e_i) = 3 \text{ if } i \equiv 5 \pmod 7 \\ f(v_i) = 1,\ f(e_i) = 6 \text{ if } i \equiv 6 \pmod 7 \end{array}$$

The above defined function provides $L(2,1)$-labeling for $T(P_n)$ and from the definition of f it is clear that $\lambda(T(P_3)) \leq 5$ and for $n \geq 4$, $\lambda(T(P_n)) \leq 6$.
Thus, we have

$$\lambda(T(P_n)) = \begin{cases} 4; & \text{if} \quad n = 2 \\ 5; & \text{if} \quad n = 3 \\ 6; & \text{if} \quad n \geq 4 \end{cases}$$

Example 2.2 *In* Figure-1*, the $L(2,1)$-labeling of graph $T(P_8)$ is shown where $\lambda(T(P_8))$=6.*

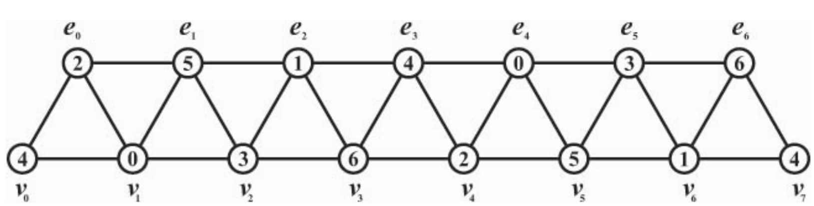

Figure-1

Theorem 2.3 *For the total graph $T(C_n)$ of cycle C_n, $\lambda(T(C_n)) \in [6, 8]$.*

Proof: Let v_0, v_1, ..., v_{n-1} and $e_0 = (v_0, v_1)$,$e_1 = (v_1, v_2)$,...,$e_{n-1} = (v_{n-1}, v_0)$ are the vertices and edges of cycle C_n then $V(T(C_n)) = \{v_0, v_1, ..., v_{n-1}, e_0, e_1, ..., e_{n-1}\}$. The graph $K_{1,4}$ is a subgraph of $T(C_n)$ and hence by Proposition 1.3 and 1.5, $\lambda(T(C_n)) \geq 5$. In the graph $T(C_n)$, the close neighborhood of each e_i where $i = 0, 1, ..., n-1$ contains three vertices with degree Δ. Hence by Proposition 1.6, $\lambda(T(C_n)) \geq 6$. Now define labeling as follows.
Case 1: $n = 7k$

$$\begin{array}{l} f(v_i) = 4,\ f(e_i) = 2 \text{ if } i \equiv 0 \pmod 7 \\ f(v_i) = 0,\ f(e_i) = 5 \text{ if } i \equiv 1 \pmod 7 \\ f(v_i) = 3,\ f(e_i) = 1 \text{ if } i \equiv 2 \pmod 7 \\ f(v_i) = 6,\ f(e_i) = 4 \text{ if } i \equiv 3 \pmod 7 \\ f(v_i) = 2,\ f(e_i) = 0 \text{ if } i \equiv 4 \pmod 7 \\ f(v_i) = 5,\ f(e_i) = 3 \text{ if } i \equiv 5 \pmod 7 \\ f(v_i) = 1,\ f(e_i) = 6 \text{ if } i \equiv 6 \pmod 7 \end{array}$$

Case 2: $n \neq 7k$
(1) $n \equiv 0 \pmod 3$

$$\begin{array}{l} f(v_i) = 0 \text{ if } i \equiv 0 \pmod 3 \\ f(v_i) = 3 \text{ if } i \equiv 1 \pmod 3 \\ f(v_i) = 6 \text{ if } i \equiv 2 \pmod 3 \\ f(e_i) = 5 \text{ if } i \equiv 0 \pmod 3 \\ f(e_i) = 8 \text{ if } i \equiv 1 \pmod 3 \\ f(e_i) = 2 \text{ if } i \equiv 2 \pmod 3 \end{array}$$

(2) $n \equiv 1$ or $2 \pmod 3$ then redefine the above $f(v_{n-1})$ and $f(e_{n-1})$ as

$$\begin{array}{l} f(v_i) = 4 \text{ if } i = n-1 \\ f(e_i) = 7 \text{ if } i = n-1 \end{array}$$

The above defined function provides $L(2, 1)$-labeling for $T(C_n)$ and hence $\lambda(T(C_n)) \leq 8$.
Thus, we have $\lambda(T(C_n)) \in [6, 8]$.

Example 2.4 *In* Figure-2, *the $L(2, 1)$-labeling of graphs $T(C_3)$, $T(C_6)$ and $T(C_7)$ is shown where $\lambda(T(C_n)) \in [6, 8]$.*

Theorem 2.5 *The λ-number of $T(K_{1,n})$ is $2n + 1$.*

Figure-2

Proof: Let v_0, v_1,...,v_n and $e_1 = (v_0, v_1)$,...,$e_n = (v_0, v_n)$ are the vertices and edges of star $K_{1,n}$ then $V(T(K_{1,n})) = \{v_0,...,v_n,e_1,...,e_n\}$. The star $K_{1,2n}$ is a subgraph of $T(K_{1,n})$ and hence by Proposition 1.3 and 1.5, $\lambda(T(K_{1,n})) \geq 2n+1$. Now define $f:V(T(K_{1,n})) \rightarrow \{0, 1, ..., 2n + 1\}$ as follows.

$f(v_0) = 0$

$f(e_i) = 2i,\ i = 1, 2, ..., n$

$f(v_i) = 2i - 3, i = 3, 4, ..., n$

$f(v_1) = 2n - 1$

$f(v_2) = 2n + 1$

The above defined function provides $L(2, 1)$-labeling for $T(K_{1,n})$ and hence $\lambda(T(K_{1,n})) \leq 2n + 1$.

Thus, we have $\lambda(T(K_{1,n}) = 2n + 1$.

Example 2.6 *In* Figure-3*, the $L(2, 1)$-labeling of graph $T(K_{1,3})$ is shown where $\lambda(T(K_{1,3}))$=7.*

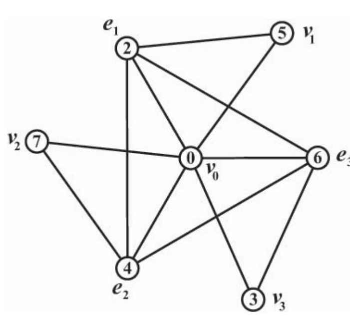

Figure-3

Theorem 2.7 $\lambda(T(F_n)) = 4n+1$ *where F_n is a friendship graph (A friendship graph is a one point union of n copies of cycle C_3).*

Proof: Let F_n be a friendship graph form by n triangles C_1,...,C_n. Let v_0, v_1,...,v_{2n} and $e_1 = (v_0, v_1)$,...,$e_{2n} = (v_0, v_{2n})$, $e_1^{'} = (v_1, v_2)$, $e_2^{'} = (v_3, v_4)$,...,$e_n^{'}$ $= (v_{2n-1}, v_{2n})$ are the vertices and edges of F_n then $V(T(F_n)) = \{v_0,...,v_{2n}$, $e_1,...,e_{2n}$, $e_1^{'},...,e_n^{'}\}$. The star $K_{1,4n}$ is a subgraph of $T(F_n)$ and hence by Proposition 1.3 and 1.5, $\lambda(T(F_n)) \geq 4n+1$. Now define f:$V(T(F_n)) \rightarrow$\{0,1,...,4n+1\} as follows.

$f(v_0) = 0$
$f(e_i) = 2i$, $i = 1, 2, ..., n$
$f(v_{2i+1}) = 4i - 1$, $i = 1, 2, ..., (n - 1)$
$f(v_{2i}) = 4i - 3$, $i = 2, 3, ..., n$
$f(e_i^{'}) = 4i - 3$, $i = 1, 2, ..., n$
$f(v_1) = 4n - 1$
$f(v_2) = 4n + 1$

The above defined function provides $L(2, 1)$-labeling for $T(F_n)$ and hence $\lambda(T(F_n)) \leq 4n + 1$.
Thus, we have $\lambda(T(F_n) = 4n + 1$.

Example 2.8 *In* Figure-4*, the $L(2, 1)$-labeling of graph $T(F_3)$ is shown where $\lambda(T(F_3))$=13.*

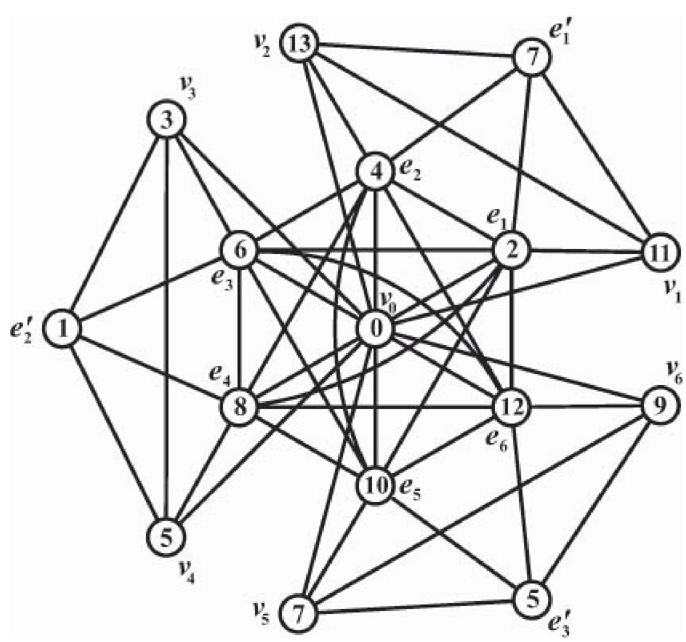

Figure-4

In the following *Table-1* the λ-numbers of some standard graphs and their total graphs are listed.

Table-1

G		$\lambda(G)$	$T(G)$	$\lambda(T(G))$
P_n	$n=2$	2	$n=2$	4
	$n=3,4$	3	$n=3$	5
	$n\geq 5$	4	$n\geq 4$	6
C_n	$n\geq 3$	4	$n\geq 3$	[6, 8]
$K_{1,n}$	$n\geq 3$	$n+1$	$n\geq 3$	$2n+1$
F_n	$n\geq 3$	$2n+1$	$n\geq 3$	$4n+1$

3 Conclusion

Assignment of channels for TV and radio broadcasting network is the problem which concern to real life situation. The rich quality of broadcasting will always provide satisfactory and enjoyable entertainment. In order to meet this demand, the study and investigations related to $L(2,1)$-labeling is the potential area of research. We investigate four new results corresponding to $L(2,1)$-labeling. The λ-number is completely determined for four new families of graphs. This work is an effort to relate total graph of a graph and $L(2,1)$-labeling. It is also possible to investigate similar results corresponding to other graph families.

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References

[1] G. Chang and D. Kuo, The L(2,1)-labeling problem on graphs, *SIAM J.Discrete Math.*, 9(2) (1996), 309-316.

[2] J. Georges and D. Mauro, Generalized vertex labelings with a condition at distance two, *Congr. Numer.*, 109(1995), 141-159.

[3] J. Georges, D. Mauro and M. Whittlesey, Ralating path covering to vertex labelings with a condition at distance two, *Discrete Math.*, 135(1994), 103-111.

[4] J. Griggs and R. Yeh, Labeling graphs with a condition at distance 2, *SIAM J.Discrete Math.*, 5(1992), 586-595.

[5] W. Hale, Frequency assignment: Theory and applications, *Proc. IEEE*, 68(1980), 1497-1514.

[6] P. Jha, Optimal $L(2,1)$-labeling of strong products of cycles, *IEEE Transactions on Circuits and System-1: Fundamenal Theory and Applications*, 48(4) (2001), 498-500.

[7] D. Sakai, Labeling chordal graphs: Distance two condition, *SIAM J.Discrete Math.*, 7(1) (1994), 133-140.

[8] S. Vaidya, P. Vihol, N. Dani and D. Bantva, $L(2,1)$-labeling in the context of some graph operations, *Journal of Mathematics Research*, 2(3) (2010), 109-119.

[9] D. West, *Introduction to Graph Theory*, Printice -Hall of India, (2001).

[10] R. Yeh, Labeling graphs with a condition at distance two, *Ph.D.Thesis*, Dept.of Math., University of South Carolina,Columbia,Sc, (1990).