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Common Fixed Point for Intimate Mappings in Complex Valued Metric Spaces

Gopal Meena

Department of Applied Mathematics
Jabalpur Engineering College Jabalpur (M.P.)-482011, India
E-mail: gopal.g1981@rediffmail.com

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Abstract

The aim of this paper is to introduce the concept of intimate mappings in Complex valued metric space and prove a lemma and a common fixed point theorem for four mappings and provide examples in support of main theorem.

Keywords: *Intimate mappings, Common fixed point, Complex valued metric space.*

1 Introduction

In 2001 Sahu et al. [7] introduced the concept of Intimate mappings, in fact Intimate mappings are the generalization of the Compatible mappings of type (A), introduced by Kang and Kim[4]. The interesting feature of Compatible mappings of type (A), weakly Compatible mappings and weakly Compatible mappings of type (A) is that all the above stated mappings commute at coincidence point whereas Intimate mappings do not necessarily commute at coincidence point. Recently Chugh and Aggarwal [3] introduce the concept of Intimate mappings in Uniform spaces. In 2011 Azam et al. [2] introduced new spaces called Complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. Subsequently Rouzkard and Imdad [6] established some common fixed point theorems satisfying certain rational expressions in Complex valued metric spaces also can refer related results in [8]. Recently Ahmad et.al [1] prove some common fixed results for the mappings satisfying rational expressions on a closed ball in Complex valued metric spaces.

In this paper, we continue the study of fixed point theorems in Complex valued metric spaces. The obtained result for four mappings (with intimate condition) is the generalization of recent results for pair of mappings proved by Nashine et al. [5]. Some illustrative examples are also furnished to support the usability of our results, and a lemma has been proved whose metric version is available in [7].

2 Preliminaries

In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)\,,\; Im(z_1) \leq \mathrm{Im}(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)\,,\; Im(z_1) < Im(z_2)$,
(ii) $\mathrm{Re}(z_1) < Re(z_2)\,,\; Im(z_1) = \mathrm{Im}(z_2)$,
(iii) $\mathrm{Re}(z_1) < Re(z_2)\,,\; Im(z_1) < Im(z_2)$,
(iv) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)\,,\; Im(z_1) = \mathrm{Im}(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \precsim z_2\,, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

The following definition is recently introduced by Azam et.al [2].

Definition 2.1: *Let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping*: $X \times X \to \mathbb{C}$ *, satisfies the following conditions:*

(I) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(II) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(III) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y \in X$.

Then d is called a complex valued metric on X , and $(X,\, d)$ is called a complex valued metric spaces.

Definition 2.2: *Let (X, d) be complex valued metric space and* $\{x_n\}_{n\geq 1}$ *be a sequence in X and* $x \in X$.

We say that

(i) The sequence $\{x_n\}_{n\geq 1}$ converges to x if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_n x_n = x$, or $x_n \to \infty$, as $n \to \infty$.

(ii) The sequence $\{x_n\}_{n\geq 1}$ is Cauchy sequence if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$.

Definition 2.3: *A metric space (X, d) is a complete complex valued metric space if every Cauchy sequence is convergent.*

In [2], Azam et al. established the following two lemmas.

Lemma 2.1: *Let (X, d) be a complex valued metric space and let* $\{x_n\}$ *be a sequence in X. Then* $\{x_n\}$ *converges to x if and only if* $|d(x_n, x)| \to 0$, *as* $n \to \infty$.

Lemma 2.2: *Let (X, d) be a complex valued metric space and let* $\{x_n\}$ *be a sequence in X. Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $|d(x_n, x_{n+m})| \to 0$, *as* $n \to \infty$.

3 Main Results

First we introduce the concept of intimate mappings in complex valued metric spaces and prove a useful lemma whose metric version is available in [7].

Definition 3.1: *Let S and T be self maps of complex valued metric space (X, d). Then the pair {S, T} is said to be T-intimate if and only if* $\alpha d(TSx_n, Tx_n) \precsim \alpha d(SSx_n, Sx_n)$. *Where* $\alpha = lim\ sup$ *or* $lim\ inf$ $\{x_n\}$ *is a sequence in X such that* $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = t$ *for some t in X.*

Lemma 3.2: *Let S and T be self maps of complex valued metric space (X, d). If the pair {S, T} is T-intimate and* $St = Tt = p \in X$ *for some t in X, then* $d(Tp, p) \precsim d(Sp, p)$.

Proof: Suppose $x_n = t$ for all $n \geq 1$, so $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = St = Tt = p \in X$.

Since the pair $\{S, T\}$ is T-intimate, then

$$d(TSt, Tt) = \lim_{n\to\infty} d(TSx_n, Tx_n) \precsim \lim_{n\to\infty} d(SSx_n, Sx_n) = d(SSt, St),$$

Implies $d(Tp, p) \precsim d(Sp, p)$. This completes the proof.

Our main result runs as follows:

Theorem 3.3: *Let A, B, S and T be the four mappings from a complex valued metric space (X, d) into itself, such that*

(3.3.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$;

(3.3.2) $d(Ax, By) \precsim \alpha d(Sx, Ty) + \frac{\beta d(Ax,Sx).d(By,Ty)}{d(Ax,Ty)+d(Sx,By)+d(Sx,Ty)}$, for all $x, y \in X$ and

$$d(Ax, Ty) + d(Sx, By) + d(Sx, Ty) \neq 0,$$

Where α , β are non-negative real numbers with $\alpha + \beta < 1$;

(3.3.3) (A, S) is S-intimate and (B, T) is T-intimate;

(3.3.4) *S(X)* is complete.

Then *A*, *B*, *S* and *T* have a unique Common fixed point in *X*.

Proof: Let x_0 be an arbitrary point in X, by (3.3.1) there exists a point $x_1 \in X$, such that $Ax_0 = Tx_1$ and for $x_1 \in X$ we can choose a point $x_2 \in X$, such that $Bx_1 = Sx_2$ and so on .

Inductively we can define a sequence $\{y_n\}$ in, such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

Consider

$$d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1})$$

$$\precsim \alpha d(Sx_{2n}, Tx_{2n+1}) + \frac{\beta d(Ax_{2n},Sx_{2n}).d(Bx_{2n+1},Tx_{2n+1})}{d(Ax_{2n},Tx_{2n+1})+d(Sx_{2n},Bx_{2n+1})+d(Sx_{2n},Tx_{2n+1})}$$

$$\precsim \alpha d(y_{2n-1}, y_{2n}) + \frac{\beta d(y_{2n}, y_{2n-1}).d(y_{2n+1}, y_{2n})}{d(y_{2n},y_{2n})+d(y_{2n-1},y_{2n+1})+d(y_{2n-1}, y_{2n})}$$

$$|d(y_{2n}, y_{2n+1})| \leq \alpha|d(y_{2n-1}, y_{2n})| + \frac{\beta|d(y_{2n}, y_{2n-1})|.|d(y_{2n+1}, y_{2n})|}{|d(y_{2n-1},y_{2n+1})+d(y_{2n-1}, y_{2n})|}$$

Since $|d(y_{2n+1}, y_{2n})| \leq |d(y_{2n-1}, y_{2n+1}) + d(y_{2n-1}, y_{2n})|$, therefore

$$|d(y_{2n}, y_{2n+1})| \leq (\alpha + \beta)|d(y_{2n-1}, y_{2n})|$$

$$|d(y_{2n}, y_{2n+1})| \leq \gamma|d(y_{2n-1}, y_{2n})| \quad \text{, where } \gamma = \alpha + \beta < 1.$$

Similarly

$$d(y_{2n+2}, y_{2n+1}) = d(Ax_{2n+2}, Bx_{2n+1})$$

$$\precsim \alpha d(Sx_{2n+2}, Tx_{2n+1}) + \frac{\beta d(Ax_{2n+2}, Sx_{2n+2}).d(Bx_{2n+1}, Tx_{2n+1})}{d(Ax_{2n+2}, Tx_{2n+1}) + d(Sx_{2n+2}, Bx_{2n+1}) + d(Sx_{2n+2}, Tx_{2n+1})}$$

$$\precsim \alpha d(y_{2n+1}, y_{2n}) + \frac{\beta d(y_{2n+2}, y_{2n+1}).d(y_{2n+1}, y_{2n})}{d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1}) + d(y_{2n+1}, y_{2n})}$$

$$|d(y_{2n+2}, y_{2n+1})| \leq \alpha |d(y_{2n+1}, y_{2n})| + \frac{\beta |d(y_{2n+2}, y_{2n+1})|.|d(y_{2n+1}, y_{2n})|}{|d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n})|}$$

Again Since $|d(y_{2n+2}, y_{2n+1})| \leq |d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n})|$, therefore

$$|d(y_{2n+2}, y_{2n+1})| \leq (\alpha + \beta)|d(y_{2n+1}, y_{2n})|$$

i.e $|d(y_{2n+2}, y_{2n+1})| \leq \gamma |d(y_{2n+1}, y_{2n})|$

$$|d(y_{2n+2}, y_{2n+1})| \leq \gamma^2 |d(y_{2n-1}, y_{2n})|$$

Thus $|d(y_{n+1}, y_{n+2})| \leq \gamma |d(y_n, y_{n+1})| \leq \cdots \leq \gamma^{n+1} |d(y_0, y_1)|$

So that for any $m > n$,

$$|d(y_n, y_m)| \leq |d(y_n, y_{n+1})| + |d(y_{n+1}, y_{n+2})| + \cdots + |d(y_{m-1}, y_m)|$$

$$\leq \gamma^n |d(y_0, y_1)| + \gamma^{n+1} |d(y_0, y_1)| + \cdots + \gamma^{m-1} |d(y_0, y_1)|$$

i.e $|d(y_n, y_m)| \leq \frac{\gamma^n}{1-\gamma} |d(y_0, y_1)| \to 0$ as $m, n \to \infty$,

which amounts to say that $\{y_n\}$ is a Cauchy sequence. i.e. $\{Sx_{2n}\}$ is a Cauchy sequence in *S(X)* , also *S(X)* is complete , then $\{y_n\}$ converges to a point $p = Su$ for some $u \in X$.

Thus $Ax_{2n}, Sx_{2n}, Bx_{2n+1}, Tx_{2n+1} \to p$

Now

$$d(Au, Bx_{2n+1}) \precsim \alpha d(Su, Tx_{2n+1}) + \frac{\beta d(Au, Su).d(Bx_{2n+1}, Tx_{2n+1})}{d(Au, Tx_{2n+1}) + d(Su, Bx_{2n+1}) + d(Su, Tx_{2n+1})}$$

$$|d(Au, Bx_{2n+1})| \leq \alpha |d(Su, Tx_{2n+1})| + \frac{\beta |d(Au, Su)|.|d(Bx_{2n+1}, Tx_{2n+1})|}{|d(Au, Tx_{2n+1}) + d(Su, Bx_{2n+1}) + d(Su, Tx_{2n+1})|}$$

Taking limit as $n \to \infty$

$$|d(Au, p)| \leq \alpha |d(Su, p)|$$

Thus $|d(Au\,,p)| = 0$, i.e. $Au = p = Su$.

Since $A(X) \subset T(X)$, there exists $v \in X$ such that $Au = Tv = p$.

Consider $d(p\,,\ Bv) = d(Au\,,Bv)$

$$\precsim \alpha d(Su\,,Tv) + \frac{\beta d(Au,Su).d(Bv,Tv)}{d(Au,Tv)+d(Su,Bv)+d(Su,Tv)}$$

$$|d(p\,,\ Bv)| \leq \alpha|d(Su\,,Tv)| + \frac{\beta|d(Au,Su)|.|d(Bv,Tv)|}{|d(Au,Tv)+d(Su,Bv)+d(Su,Tv)|}$$

Thus $|d(p\,,\ Bv)| = 0$, implies that $p = Bv = Tv = Au = Su$.

Now, since $Au = Su = p$ and $(A\,,S)$ is S- intimate.

Then we have $|d(Sp\,,p)| \leq |d(Ap\,,p)|$

Also $d(Ap\,,p) = d(Ap\,,Bv)$

$$\precsim \alpha d(Sp\,,Tv) + \frac{\beta d(Ap,Sp).d(Bv,Tv)}{d(Ap,Tv)+d(Sp,Bv)+d(Sp,Tv)}$$

$$|d(Ap\,,p)| \leq \alpha|d(Sp\,,p)|$$

Thus $|d(Ap\,,p)| = 0$ implies that $Ap = p$ and $Sp = p$.

Similarly $Bp = Tp = p$.

Uniqueness: Let us consider p and q are common fixed points of $A\,,B\,,S$ and T such that $p \neq q$.

$$d(p\,,q) = d(Ap\,,Bq)$$

$$\precsim \alpha d(Sp\,,Tq) + \frac{\beta d(Ap,Sp).d(Bq,Tq)}{d(Ap,Tq)+d(Sp,Bq)+d(Sp,Tq)}$$

$$\precsim \alpha d(p\,,q)$$

$|d(p\,,q)| < |d(p\,,q)|$ implies that $p = q$.

The following example shows that the intimate condition for mappings is necessary in above result.

Example 3.4: *Let* $X = \{z_1\,,z_2\} \subset \mathbb{C}$ *(set of complex numbers) with* $: X \times X \to \mathbb{C}$, *defined by* $d(z_1\,,z_2) = \begin{cases} 1\,, & z_1 \neq z_2 \\ 0\,, & z_1 = z_2 \end{cases}$, *then* $(X\,,d)$ *is a complete complex valued metric space.*

Define $A = B\,,S,T : X \to X$ by

$$Az = z_1 \text{ for all } z \in X,$$

$$Sz_1 = Tz_1 = z_2 \text{ and } Sz_2 = Tz_2 = z_1.$$

All the assumptions of above theorem are satisfied except intimate condition (3.3.3).

Indeed $|d(SAz_2, Sz_2| = |d(z_2, z_1)| > 0 = |d(AAz_2\ , Az_2)|$, where $\{z_2\}$ is a constant sequence in X such that $Az_2 = Sz_2 = z_1$. Thus the pair (A, S) is not S-intimate. Therefore *A*, *S* and *T* do not have a common fixed point.

Example 3.5: *Let* $X = \mathbb{C}$ *be the set of complex numbers, defined* $d: X \times X \to X$ *by* $d(z_2, z_1) = i|z_1 - z_2|$ *where* $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$*. Then* (X, d)*, is a complete complex valued metric space. Define* $B\ , S, T : X \to X$ *as* $Az = 0\ , Bz = 0\ , Sz = z$ *and* $Tz = \frac{z}{2}$.

Clearly *A(X)* ⊂ *T(X)* and *B(X)* ⊂ *S(X)*. Now consider the sequence $\{z_n = \frac{1}{n}\ , n \in \mathbb{N}\}$ in $\mathbb{C}$, then $\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Sz_n = 0$, also we have $\lim_{n\to\infty} d(SAz_n\ , Sz_n\ ) \precsim \lim_{n\to\infty} d(AAz_n\ , Az_n\ )$. Thus the pair (A, S) is S-intimate. Also $\lim_{n\to\infty} d(TBz_n\ , Tz_n\ ) \precsim \lim_{n\to\infty} d(BBz_n\ , Bz_n\ )$. Thus the pair (B, T) is T-intimate. Moreover the mappings satisfy all the conditions of above theorem. Hence $A, B\ , S$ and T have unique common fixed point in X.

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