Gen. Math. Notes, Vol. 25, No. 2, December 2014, pp. 95-109
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Quadruple Fixed Point Theorem for Four Mappings

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(Received: 3-8-14 / Accepted: 12-9-14)

Abstract

In this paper we have proved a unique common quadruple fixed point theorem for four mappings satisfying w-compatible in partially ordered metric space with two altering distance functions. An example has been given to validate the result.

Keywords: *Quadruple fixed point, Compatible mapping, Partially ordered set, Complete metric space.*

1 Introduction

Fixed point theory has fascinated hundreds of researchers since 1922 with the celebrated Banach's fixed point theorem. This theorem provides a technique for solving a variety of problems in mathematical sciences and engineering. This study is a very active field of research at present.

T.G. Bhashkar et al. [13] introduced the concept of a coupled fixed point and proved theorems in partially ordered complete metric spaces.

V. Lakshmikantham et al. [14] proved coupled coincidence and coupled common fixed point theorems for nonlinear mappings in partially ordered complete metric spaces. Later, many results on coupled fixed point have been obtained [2, 3, 4, 10, 11, 12].

V. Berinde et al. [15] introduced the concept of a tripled fixed point.

B. Samet et al. [1] introduced fixed point of order $N \geq 3$ for the first time. Very recently, E. Karapınar [5] used the notion of quadruple fixed point and obtained some quadruple fixed point theorems in partially ordered metric spaces. Many researchers [6-9] were motivated and proved theorems on quadruple fixed points with monotone property whereas in the present paper a unique common quadruple fixed point theorem for four mappings without using the monotone property and satisfying w-compatible condition in pairs has been proved.

2 Preliminaries

2.1 Quadruple Fixed Point

Let $F: X\times X\times X\times X \to X$. An element (x, y, z, w) is called a quadruple fixed point of F if $F(x, y, z, w) = x$, $F(y, z, w, x) = y$, $F(z, w, x, y) = z$, $F(w, x, y, z) = w$.

2.2 Quadruple Coincidence Point

Let $F: X\times X\times X\times X \to X$ and $g: X \to X$. An element (x, y, z, w) is called a quadruple coincidence point of F and g if $F(x, y, z, w) = gx$, $F(y, z, w, x) = gy$, $F(z, w, x, y) = gz$, $F(w, x, y, z) = gw$.

2.3 Quadruple Common Fixed Point

Let $F: X\times X\times X\times X \to X$ and $g: X \to X$. An element (x, y, z, w) is called a quadruple common fixed point of F and g if $F(x, y, z, w) = gx = x$, $F(y, z, w, x) = gy = y$, $F(z, w, x, y) = gz = z$, $F(w, x, y, z) = gw = w$.

2.4 W-Compatible Mapping

$F: X\times X\times X\times X \to X$ and $g: X \to X$ are called w-compatible if $F(gx, gy, gz, gw) = g(F(x, y, z, w))$ whenever $F(x, y, z, w) = gx$, $F(y, z, w, x) = gy$, $F(z, w, x, y) = gz$, $F(w, x, y, z) = gw$.

2.5 Alternating Distance Function

Let Φ denote all the functions $\xi \in \Phi$ such that $\xi : [0,\infty) \to [0,\infty)$ which satisfy

(i) $\xi(t) = 0$ if and only if $t = 0$,

(ii) ξ(t) is continuous and non-decreasing.

3 Main Theorem

Theorem 3.1: *Let (X,$\leq$) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space .Suppose* $F,G : X^4 \to X$ *and* $g,f : X \to X$ *are such that F, G are continuous.* ξ , $\phi \in \Phi$ *and* $L \geq 0$ *such that*

(i) $\xi(d(F(x,y,z,w),G(p,q,r,s)$

$$\leq \xi\left(\max\{d(fx,gp),d(fy,gq),d(fz,gr),d(fw,gs))\}\right)$$
$$-\phi\left(\max\{d(fx,gp),d(fy,gq),d(fz,gr),d(fw,gs))\}\right)$$
$$+L\min\left(d(fx,G(p,q,r,s)),d(gp,F(x,y,z,w),d(gp,G(p,q,r,s))\right)...........(i)$$

for all $x,y,z,w,p,q,r,s \in X$.

(ii) $F(X^4) \subset g(X), G(X^4) \subset f(X)$,

(iii) the pairs (F, f) and (G, g) are W - compatible .

Then F, G, f and g have a unique common coupled fixed point in X^4 *and also they have a unique common fixed point in X.*

Proof: Let $x_0, y_0, z_0, w_0 \in X$.

$\because F(X^4) \subset g(X), G(X^4) \subset f(X)$.

$\therefore$ we may find x_1, y_1, z_1, w_1 so that $F(x_0,y_0,z_0,w_0) = g(x_1), F(y_0,z_0,w_0,x_0) = g(y_1)$,
$F(z_0,w_0,x_0,y_0) = g(z_1), F(w_0,x_0,y_0,z_0) = g(w_1)$.

Similarly we may find x_2, y_2, z_2, w_2 so that $G(x_1,y_1,z_1,w_1) = f(x_2), G(y_1,z_1,w_1,x_1) = f(y_2)$,
$G(z_1,w_1,x_1,y_1) = f(z_2), G(w_1,x_1,y_1,z_1) = f(w_2)$.

Continuing in the same way we may form sequences

$\{x_n\},\{y_n\},\{z_n\},\{w_n\},\{a_n\},\{b_n\},\{c_n\},\{d_n\}$ in X such that

$$a_{2n} = g(x_{2n+1}) = F(x_{2n}, y_{2n}, z_{2n}, w_{2n}), b_{2n} = g(y_{2n+1}) = F(y_{2n}, z_{2n}, w_{2n}, x_{2n})$$
$$c_{2n} = g(z_{2n+1}) = F(z_{2n}, w_{2n}, x_{2n}, y_{2n}), d_{2n} = g(w_{2n+1}) = F(w_{2n}, x_{2n}, y_{2n}, z_{2n})$$

and

$$a_{2n+1} = f(x_{2n+2}) = G\left(x_{2n+1}, y_{2n+1}, z_{2n+1}, w_{2n+1}\right),$$
$$b_{2n+1} = f(y_{2n+2}) = G\left(y_{2n+1}, z_{2n+1}, w_{2n+1}, x_{2n+1}\right)$$
$$c_{2n+1} = f(z_{2n+2}) = G\left(z_{2n+1}, w_{2n+1}, x_{2n+1}, y_{2n+1}\right),$$
$$d_{2n+1} = f(w_{2n+2}) = G\left(w_{2n+1}, x_{2n+1}, y_{2n+1}, z_{2n+1}\right)$$

Putting $x = x_{2n}$, $y = y_{2n}$. $z = z_{2n}$, $w = w_{2n}$, $p = x_{2n+1}$, $q = y_{2n+1}$, $r = z_{2n+1}$, $s = w_{2n+1}$ in

(i) we get

$$\begin{aligned}\xi(d(a_{2n}, a_{2n+1}) &\le \xi\left(\max\left\{d(a_{2n-1}, a_{2n}), d(b_{2n-1}, b_{2n}), d(c_{2n-1}, c_{2n}), d(d_{2n-1}, d_{2n})\right\}\right)\\ &- \phi\left(\max\left\{d(a_{2n-1}, a_{2n}), d(b_{2n-1}, b_{2n}), d(c_{2n-1}, c_{2n}), d(d_{2n-1}, d_{2n})\right\}\right)\\ &+ L\min\left(d(a_{2n-1}, a_{2n+1}), d(a_{2n}, a_{2n}), d(a_{2n}, a_{2n+1})\right)\end{aligned}$$

$$\begin{aligned}\therefore \xi(d(a_{2n}, a_{2n+1}) &\le \xi\left(\max\left\{d(a_{2n-1}, a_{2n}), d(b_{2n-1}, b_{2n}), d(c_{2n-1}, c_{2n}), d(d_{2n-1}, d_{2n})\right\}\right)\\ &- \phi\left(\max\left\{d(a_{2n-1}, a_{2n}), d(b_{2n-1}, b_{2n}), d(c_{2n-1}, c_{2n}), d(d_{2n-1}, d_{2n})\right\}\right) \text{....(ii)}\end{aligned}$$

Similarly if we consider,

$$\begin{aligned}\xi(d(F(y,z,w,x), G(q,r,s,p) &\le \xi\left(\max\left\{d(fy,gq), d(fz,gr), d(fw,gs), d(fx,gp))\right\}\right)\\ &- \phi\left(\max\left\{d(fy,gq), d(fz,gr), d(fw,gs), d(fx,gp)\right\}\right)\\ &+ L\min\left(d(fy, G(q,r,s,p)), d(gq, F(y,z,w,x)), d(gq, G(q,r,s,p))\right)\end{aligned}$$

We may prove that

$$\begin{aligned}\xi(d(b_{2n}, b_{2n+1}) &\le \xi\left(\max\left\{d(b_{2n-1}, b_{2n}), d(c_{2n-1}, c_{2n}), d(d_{2n-1}, d_{2n}), d(a_{2n-1}, a_{2n})\right\}\right)\\ &- \phi\left(\max\left\{d(b_{2n-1}, b_{2n}), d(c_{2n-1}, c_{2n}), d(d_{2n-1}, d_{2n}), d(a_{2n-1}, a_{2n})\right\}\right) \text{..........(iii)}\end{aligned}$$

Similarly we may prove that

$$\begin{aligned}\xi(d(c_{2n}, c_{2n+1}) &\le \xi\left(\max\left\{d(c_{2n-1}, c_{2n}), d(d_{2n-1}, d_{2n}), d(a_{2n-1}, a_{2n}), d(b_{2n-1}, b_{2n})\right\}\right)\\ &- \phi\left(\max\left\{d(c_{2n-1}, c_{2n}), d(d_{2n-1}, d_{2n}), d(a_{2n-1}, a_{2n}), d(b_{2n-1}, b_{2n})\right\}\right) \text{........(iv)}\end{aligned}$$

&

$$\xi(d(d_{2n},d_{2n+1})$$
$$\leq \xi\left(\max\{d(d_{2n-1},d_{2n}),d(a_{2n-1},a_{2n}),d(b_{2n-1},b_{2n}),d(c_{2n-1},c_{2n})\}\right)$$
$$-\phi\left(\max\{d(d_{2n-1},d_{2n}),d(a_{2n-1},a_{2n}),d(b_{2n-1},b_{2n}),d(c_{2n-1},c_{2n})\}\right)........(v)$$

Combining (ii), (iii), (iv), (v) we get

$$\xi(\max\{d(a_{2n},a_{2n+1}),d(b_{2n},b_{2n+1}),d(c_{2n},c_{2n+1}),d(d_{2n},d_{2n+1})\}$$
$$\leq \xi\left(\max\{d(a_{2n-1},a_{2n}),d(b_{2n-1},b_{2n}),d(c_{2n-1},c_{2n}),d(d_{2n-1},d_{2n})\}\right)$$
$$-\phi\left(\max\{d(a_{2n-1},a_{2n}),d(b_{2n-1},b_{2n}),d(c_{2n-1},c_{2n}),d(d_{2n-1},d_{2n})\}\right)...(vi)$$

$$\therefore \xi(\max\{d(a_{2n},a_{2n+1}),d(b_{2n},b_{2n+1}),d(c_{2n},c_{2n+1}),d(d_{2n},d_{2n+1})\}$$
$$\leq \xi\left(\max\{d(a_{2n-1},a_{2n}),d(b_{2n-1},b_{2n}),d(c_{2n-1},c_{2n}),d(d_{2n-1},d_{2n})\}\right)$$

$\because \xi(t)$ is an a non-decreasing sequence

$$\therefore \max\{d(a_{2n},a_{2n+1}),d(b_{2n},b_{2n+1}),d(c_{2n},c_{2n+1}),d(d_{2n},d_{2n+1})\}$$
$$\leq \max\{d(a_{2n-1},a_{2n}),d(b_{2n-1},b_{2n}),d(c_{2n-1},c_{2n}),d(d_{2n-1},d_{2n})\}$$

$\therefore \{d(a_{2n},a_{2n+1}),d(b_{2n},b_{2n+1}),d(c_{2n},c_{2n+1}),d(d_{2n},d_{2n+1})\}$ is a sequence of non-increasing positive real numbers. So, it must converge to a positive real number say $\delta > 0$.

$$\therefore \lim_{n\to\infty} \{d(a_{2n},a_{2n+1}),d(b_{2n},b_{2n+1}),d(c_{2n},c_{2n+1}),d(d_{2n},d_{2n+1})\} = \delta$$

$\therefore$ Taking $\lim_{n\to\infty}$ on (vi) we get

$$\lim_{n\to\infty} \xi(\max\{d(a_{2n},a_{2n+1}),d(b_{2n},b_{2n+1}),d(c_{2n},c_{2n+1}),d(d_{2n},d_{2n+1})\}$$
$$\leq \lim_{n\to\infty} \xi\left(\max\{d(a_{2n-1},a_{2n}),d(b_{2n-1},b_{2n}),d(c_{2n-1},c_{2n}),d(d_{2n-1},d_{2n})\}\right)$$
$$-\lim_{n\to\infty} \phi\left(\max\{d(a_{2n-1},a_{2n}),d(b_{2n-1},b_{2n}),d(c_{2n-1},c_{2n}),d(d_{2n-1},d_{2n})\}\right)$$

$$\therefore \xi(\delta) \leq \xi(\delta) - \phi(\delta) \Rightarrow \phi(\delta) = 0$$
$$\Rightarrow \delta = 0$$
$$\therefore \lim_{n\to\infty} \{d(a_{2n},a_{2n+1}),d(b_{2n},b_{2n+1}),d(c_{2n},c_{2n+1}),d(d_{2n},d_{2n+1})\} = 0$$

$\therefore$ Generalising we get,

$$\lim_{n\to\infty} \{d(a_n,a_{n+1}),d(b_n,b_{n+1}),d(c_n,c_{n+1}),d(d_n,d_{n+1})\} = 0..................(vii)$$

We will show that $\{a_n\},\{b_n\},\{c_n\},\{d_n\}$ are Cauchy sequences. Assume on the contrary, that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are not Cauchy sequences, consequently, $\lim_{n\to\infty} d(a_n,a_m) \neq 0$, $\lim_{n\to\infty} d(b_n,b_m) \neq 0$, $\lim_{n\to\infty} d(c_n,c_m) \neq 0$

and $\lim_{n\to\infty} d(d_n, d_m) \neq 0$.

Let there exists $\in > 0$ for which we can find subsequence of integers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k > k$.

$\{d(a_{2m_k}, a_{2n_k}), d(b_{2m_k}, b_{2n_k}), d(c_{2m_k}, c_{2n_k}), d(d_{2m_k}, d_{2n_k})\} \geq \in$ and

$\{d(a_{2m_k}, a_{2n_k-1}), d(b_{2m_k}, b_{2n_k-1}), d(c_{2m_k}, c_{2n_k-1}), d(d_{2m_k}, d_{2n_k-1})\} < \in$

By the triangle inequality, we have

$$\in \leq \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k}) = \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k-1}) + \lim_{k\to\infty} d(a_{2n_k-1}, a_{2n_k})$$

$$\in \leq \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k}) < \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k-1}) \leq \in$$

$$\therefore \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k}) = \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k-1}) = \in$$

Similarly we may that

$$\lim_{k\to\infty} d(b_{2m_k}, b_{2n_k}) = \lim_{k\to\infty} d(b_{2m_k}, b_{2n_k-1}) = \in$$

$$\lim_{k\to\infty} d(c_{2m_k}, c_{2n_k}) = \lim_{k\to\infty} d(c_{2m_k}, c_{2n_k-1}) = \in$$

$$\lim_{k\to\infty} d(d_{2m_k}, d_{2n_k}) = \lim_{k\to\infty} d(d_{2m_k}, d_{2n_k-1}) = \in$$

$$d(a_{2m_k}, a_{2n_k+1}) \leq d(a_{2m_k}, a_{2n_k}) + d(a_{2m_k}, a_{2n_k+1})$$

$$\lim_{k\to\infty} d(a_{2m_k}, a_{2n_k+1}) \leq \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k}) + \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k+1})$$

$$\therefore \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k+1}) \leq \in$$

$$d(a_{2m_k-1}, a_{2n_k}) \leq d(a_{2m_k-1}, a_{2m_k}) + d(a_{2m_k}, a_{2n_k})$$

$$\lim_{k\to\infty} d(a_{2m_k-1}, a_{2n_k}) \leq \lim_{k\to\infty} d(a_{2m_k-1}, a_{2m_k}) + \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k})$$

$$\therefore \lim_{k\to\infty} d(a_{2m_k-1}, a_{2n_k}) \leq \in$$

$$d(a_{2m_k-1}, a_{2n_k+1}) \leq d(a_{2m_k-1}, a_{2m_k}) + d(a_{2m_k}, a_{2n_k}) + d(a_{2n_k}, a_{2n_k+1})$$

$$\lim_{k\to\infty} d(a_{2m_k-1}, a_{2n_k+1}) \leq \lim_{k\to\infty} d(a_{2m_k-1}, a_{2m_k}) + \lim_{k\to\infty} d(a_{2m_k}, a_{2n_k}) + \lim_{k\to\infty} d(a_{2n_k}, a_{2n_k+1})$$

$$\therefore \lim_{k\to\infty} d(a_{2m_k-1}, a_{2n_k+1}) \leq \in$$

Similarly we may prove that

$$\lim_{k\to\infty} d(b_{2m_k}, b_{2n_k+1}) \le \in, \lim_{k\to\infty} d(b_{2m_k-1}, b_{2n_k}) \le \in, \lim_{k\to\infty} d(b_{2m_k-1}, b_{2n_k+1}) \le \in$$

$$\lim_{k\to\infty} d(c_{2m_k}, c_{2n_k+1}) \le \in, \lim_{k\to\infty} d(c_{2m_k-1}, c_{2n_k}) \le \in, \lim_{k\to\infty} d(c_{2m_k-1}, c_{2n_k+1}) \le \in$$

$$\lim_{k\to\infty} d(d_{2m_k}, d_{2n_k+1}) \le \in, \lim_{k\to\infty} d(d_{2m_k-1}, d_{2n_k}) \le \in, \lim_{k\to\infty} d(d_{2m_k-1}, d_{2n_k+1}) \le \in$$

Putting $x = x_{2m_k}, y = y_{2m_k}, z = z_{2m_k}, w = w_{2m_k}, p = x_{2n_k+1}, q = y_{2n_k+1},$
$r = z_{2n_k+1}, s = w_{2n_k+1}$ in (i), we get

$$\xi(d(F(x_{2m_k}, y_{2m_k}, z_{2m_k}, w_{2m_k}), G(x_{2n_k+1}, y_{2n_k+1}, z_{2n_k+1}, w_{2n_k+1})$$

$$\le \xi\left(\max\begin{Bmatrix} d(fx_{2m_k}, gx_{2n_k+1}), d(fy_{2m_k}, gy_{2n_k+1}), d(fz_{2m_k}, gz_{2n_k+1}), \\ d(fw_{2m_k}, gw_{2n_k+1})) \end{Bmatrix}\right)$$

$$-\phi\left(\max\begin{Bmatrix} d(fx_{2m_k}, gx_{2n_k+1}), d(fy_{2m_k}, gy_{2n_k+1}), d(fz_{2m_k}, gz_{2n_k+1}), \\ d(fw_{2m_k}, gw_{2n_k+1})) \end{Bmatrix}\right)$$

$$+ L\min\begin{pmatrix} d(fx_{2m_k}, G(x_{2n_k+1}, y_{2n_k+1}, z_{2n_k+1}, w_{2n_k+1})), \\ d(gx_{2n_k+1}, F(x_{2m_k}, y_{2m_k}, z_{2m_k}, w_{2m_k}), \\ d(gx_{2n_k+1}, G(x_{2n_k+1}, y_{2n_k+1}, z_{2n_k+1}, w_{2n_k+1})) \end{pmatrix}$$

$$\xi(d(a_{2m_k}, a_{2n_k+1}) \le \xi\left(\max\begin{Bmatrix} d(a_{2m_k-1}, a_{2n_k}), d(b_{2m_k-1}, b_{2n_k}), d(c_{2m_k-1}, c_{2n_k}), \\ d(d_{2m_k-1}, d_{2n_k}) \end{Bmatrix}\right)$$

$$-\phi\left(\max\left\{d(a_{2m_k-1}, a_{2n_k}), d(b_{2m_k-1}, b_{2n_k}), d(c_{2m_k-1}, c_{2n_k}), d(d_{2m_k-1}, d_{2n_k})\right\}\right)$$

$$+ L\min\left(d(a_{2m_k-1}, a_{2n_k+1}), d(a_{2n_k}, a_{2m_k}), d(a_{2n_k}, a_{2n_k+1})\right)$$

Taking $\lim_{n\to\infty}$, we get

$\xi(\in) \le \xi(\in) - \phi(\in)$

$\therefore \phi(\in) = 0$

$\therefore \in = 0$, which is not possible

$\{a_n\}$ is a cauchy sequence.

Similarly we may prove that $\{b_n\}, \{c_n\}, \{d_n\}$ are cauchy sequences.

As (X, d) is a complete metric space. So,
$a_{2n+1} = f(x_{2n+2}), b_{2n+1} = f(y_{2n+2}), c_{2n+1} = f(z_{2n+2}), d_{2n+1} = f(w_{2n+2})$
converge to some $\alpha, \beta, \gamma, \theta$ in X.

Hence there exists x, y, z, w in X, such that $\alpha = fx$, $\beta = fy$, $\gamma = fz$, $\theta = fw$.
Also the subsequences $\{a_{2n}\},\{b_{2n}\},\{c_{2n}\},\{d_{2n}\}$ converge to $\alpha.\beta,\gamma,\theta$

$$\begin{aligned}
&\xi(d(F(x,y,z,w),G(x_{2n+1},y_{2n+1},z_{2n+1},w_{2n+1}))\\
&\qquad \le \xi\left(\max\left\{d(fx,gx_{2n+1}),d(fy,gy_{2n+1}),d(fz,gz_{2n+1}),d(fw,gw_{2n+1}))\right\}\right)\\
&\qquad -\phi\left(\max\left\{d(fx,gx_{2n+1}),d(fy,gy_{2n+1}),d(fz,gz_{2n+1}),d(fw,gw_{2n+1}))\right\}\right)\\
&\qquad +L\min\begin{pmatrix} d(fx,G(x_{2n+1},y_{2n+1},z_{2n+1},w_{2n+1})),d(gx_{2n+1},F(x,y,z,w)),\\ d(gx_{2n+1},G(x_{2n+1},y_{2n+1},z_{2n+1},w_{2n+1})) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&\xi(d(F(x,y,z,w),a_{2n+1})\\
&\qquad \le \xi\left(\max\left\{d(\alpha,a_{2n}),d(\beta,b_{2n}),d(\gamma,c_{2n}),d(\theta,d_{2n})\right\}\right)\\
&\qquad -\phi\left(\max\left\{d(\alpha,a_{2n}),d(\beta,b_{2n}),d(\gamma,c_{2n}),d(\theta,d_{2n})\right\}\right)\\
&\qquad +L\min\left(d(\alpha,a_{2n+1}),d(a_{2n},F(x,y,z,w)),d(a_{2n},a_{2n+1})\right)
\end{aligned}$$

Taking $\lim_{n\to\infty}$ to both sides, we get

$\therefore \xi(d(F(x,y,z,w),\alpha) \le 0$
$\therefore F(x,y,z,w) = \alpha$
$\therefore F(x,y,z,w) = fx = \alpha$
$\because (F,f)$ are w-compatible.
$\therefore f\alpha = f(F(x,y,z,w)) = F(fx,fy,fz,fw) = F(\alpha,\beta,\gamma,\theta)$

Similarly we may prove that

$f\beta = F(\beta,\gamma,\theta,\alpha)$
$f\gamma = F(\gamma,\theta,\alpha,\beta)$
$f\theta = F(\theta,\alpha,\beta,\gamma)$

Putting $x = \alpha, y = \beta, z = \gamma, w = \theta$ we get

$$\begin{aligned}
&\xi(d(F(\alpha,\beta,\gamma,\theta),G(x_{2n+1},y_{2n+1},z_{2n+1},w_{2n+1}))\\
&\qquad \le \xi\left(\max\left\{d(f\alpha,gx_{2n+1}),d(f\beta,gy_{2n+1}),d(f\gamma,gz_{2n+1}),d(f\theta,gw_{2n+1}))\right\}\right)\\
&\qquad -\phi\left(\max\left\{d(f\alpha,gx_{2n+1}),d(f\beta,gy_{2n+1}),d(f\gamma,gz_{2n+1}),d(f\theta,gw_{2n+1}))\right\}\right)\\
&\qquad +L\min\begin{pmatrix} d(f\alpha,G(x_{2n+1},y_{2n+1},z_{2n+1},w_{2n+1})),d(gx_{2n+1},F(\alpha,\beta,\gamma,\theta)),\\ d(gx_{2n+1},G(x_{2n+1},y_{2n+1},z_{2n+1},w_{2n+1})) \end{pmatrix}
\end{aligned}$$

$$\therefore \xi(d(f\alpha, a_{2n+1}) \le \xi\left(\max\{d(f\alpha, a_{2n}), d(f\beta, b_{2n}), d(f\gamma, c_{2n}), d(f\theta, d_{2n})\}\right)$$
$$-\phi\left(\max\{d(f\alpha, a_{2n}), d(f\beta, b_{2n}), d(f\gamma, c_{2n}), d(f\theta, d_{2n})\}\right)$$
$$+ L\min\left(d(f\alpha, a_{2n+1}), d(a_{2n}, F(\alpha,\beta,\gamma,\theta)), d(a_{2n}, a_{2n+1})\right)$$

$$\xi(d(f\alpha,\alpha)) \le \xi\left(\max\{d(f\alpha,\alpha), d(f\beta,\beta), d(f\gamma,\gamma), d(f\theta,\theta)\}\right) -$$
$$\phi\left(\max\{d(f\alpha,\alpha), d(f\beta,\beta), d(f\gamma,\gamma), d(f\theta,\theta)\}\right)$$

$$\xi(d(f\beta,\beta)) \le \xi\left(\max\{d(f\beta,\beta), d(f\gamma,\gamma), d(f\theta,\theta), d(f\alpha,\alpha)\}\right) -$$
$$\phi\left(\max\{d(f\beta,\beta), d(f\gamma,\gamma), d(f\theta,\theta), d(f\alpha,\alpha)\}\right)$$

$$\xi(d(f\theta,\theta)) \le \xi\left(\max\{d(f\theta,\theta), d(f\alpha,\alpha), d(f\beta,\beta), d(f\gamma,\gamma)\}\right) -$$
$$\phi\left(\max\{d(f\theta,\theta), d(f\alpha,\alpha), d(f\beta,\beta), d(f\gamma,\gamma)\}\right)$$

$$\xi(d(f\gamma,\gamma)) \le \xi\left(\max\{d(f\gamma,\gamma), d(f\theta,\theta), d(f\alpha,\alpha), d(f\beta,\beta)\}\right) -$$
$$\phi\left(\max\{d(f\gamma,\gamma), d(f\theta,\theta), d(f\alpha,\alpha), d(f\beta,\beta)\}\right)$$

$$\therefore \xi(\max\{d(f\alpha,\alpha), d(f\beta,\beta), d(f\gamma,\gamma), d(f\theta,\theta)\}) \le$$
$$\xi\left(\max\{d(f\alpha,\alpha), d(f\beta,\beta), d(f\gamma,\gamma), d(f\theta,\theta)\}\right)$$
$$-\phi\left(\max\{d(f\alpha,\alpha), d(f\beta,\beta), d(f\gamma,\gamma), d(f\theta,\theta)\}\right)$$

$$\therefore \phi\left(\max\{d(f\alpha,\alpha), d(f\beta,\beta), d(f\gamma,\gamma), d(f\theta,\theta)\}\right) = 0$$

$$\max\{d(f\alpha,\alpha), d(f\beta,\beta), d(f\gamma,\gamma), d(f\theta,\theta)\} = 0$$

$$\Rightarrow d(f\alpha,\alpha) = 0, d(f\beta,\beta) = 0, d(f\gamma,\gamma) = 0 \; \& \; d(f\theta,\theta) = 0$$

$$\therefore f\alpha = \alpha, \; f\beta = \beta, \; f\gamma = \gamma, \; f\theta = \theta$$

$$\therefore F(\alpha,\beta,\gamma,\theta) = f\alpha = \alpha \qquad F(\beta,\gamma,\theta,\alpha,) = f\beta = \beta$$

$$F(\gamma,\theta,\alpha,\beta) = f\gamma = \gamma \qquad F(\theta,\alpha,\beta,\gamma) = f\theta = \theta$$

$\because F(X^4) \subset g(X)$, there exists $\upsilon, \nu, \sigma, \tau$ such that $g(\upsilon) = F(\alpha,\beta,\gamma,\theta) = f\alpha = \alpha$

$g(\nu) = F(\beta,\gamma,\theta,\alpha,) = f\beta = \beta \quad g(\sigma) = F(\gamma,\theta,\alpha,\beta) = f\gamma = \gamma \quad g(\tau) = F(\theta,\alpha,\beta,\gamma) = f\theta = \theta$

$$\therefore \xi(d(g(\upsilon), G(\upsilon,\nu,\sigma,\tau)) = \xi(d(F(\alpha,\beta,\gamma,\theta), G(\upsilon,\nu,\sigma,\tau))$$
$$\le \xi\left(\max\{d(f\alpha, g\upsilon), d(f\beta, g\nu), d(f\gamma, g\sigma), d(f\theta, g\tau))\}\right)$$
$$-\phi\left(\max\{d(f\alpha, g\upsilon), d(f\beta, g\nu), d(f\gamma, g\sigma), d(f\theta, g\tau))\}\right)$$
$$+ L\min\left(d(f\alpha, G(\upsilon,\nu,\sigma,\tau)), d(g\upsilon, F(\alpha,\beta,\gamma,\theta), d(g\upsilon, G(\upsilon,\nu,\sigma,\tau))\right)$$

$$\therefore \xi(d(g(\upsilon), G(\upsilon,\nu,\sigma,\tau)) \le 0 \Rightarrow d(g(\upsilon), G(\upsilon,\nu,\sigma,\tau) = 0$$

$$\therefore g(\upsilon) = G(\upsilon,\nu,\sigma,\tau)$$

Similarly we may prove that

$g(\nu) = G(\nu,\sigma,\tau,\upsilon), \quad g(\sigma) = G(\sigma,\tau,\upsilon,\nu), \quad g(\tau) = G(\tau,\upsilon,\nu,\sigma)$

$\because (G,g)$ are W compatible

$\therefore g\alpha = gg(\upsilon) = gG(\upsilon,\nu,\sigma,\tau) = G(g(\upsilon),g(\nu),g(\sigma),g(\tau)) = G(\alpha,\beta,\gamma,\theta)$

Similarly we may prove that

$g\beta = G(\beta,\gamma,\theta,\alpha)$, $\quad g\gamma = G(\gamma,\theta,\alpha,\beta)$, $\quad g\theta = G(\theta,\alpha,\beta,\gamma)$

Putting $x = x_{2n}$, $y = y_{2n}$. $z = z_{2n}$, $w = w_{2n}$, $p = \alpha$, $q = \beta$, $r = \gamma$, $s = \theta$ in (i) we get

$$\begin{aligned}
&\xi(d(F(x_{2n},y_{2n},z_{2n},w_{2n}),G(\alpha,\beta,\gamma,\theta)) \\
&\quad \le \xi\left(\max\left\{d(fx_{2n},g\alpha),d(fy_{2n},g\beta),d(fz_{2n},g\gamma),d(fw_{2n},g\theta))\right\}\right) \\
&\quad -\phi\left(\max\left\{d(fx_{2n},g\alpha),d(fy_{2n},g\beta),d(fz_{2n},g\gamma),d(fw_{2n},g\theta))\right\}\right) \\
&\quad + L\min\left(d(fx_{2n},G(\alpha,\beta,\gamma,\theta)),d(g\alpha,F(x_{2n},y_{2n},z_{2n},w_{2n}),d(g\alpha,G(\alpha,\beta,\gamma,\theta))\right)
\end{aligned}$$

Taking $\lim_{n\to\infty}$, we get

$$\begin{aligned}
\xi(d(\alpha,g\alpha)) &\le \xi\left(\max\left\{d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta))\right\}\right) \\
&\quad -\phi\left(\max\left\{d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta))\right\}\right) \\
&\quad + L\min\left(d(\alpha,g\alpha),d(g\alpha,\alpha),d(g\alpha,g\alpha)\right)
\end{aligned}$$

$$\begin{aligned}
\therefore \xi(d(\alpha,g\alpha)) &\le \xi\left(\max\left\{d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta))\right\}\right) \\
&\quad -\phi\left(\max\left\{d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta))\right\}\right)
\end{aligned}$$

Similarly we may prove that

$$\begin{aligned}
\xi(d(\beta,g\beta)) &\le \xi\left(\max\left\{d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta),d(\alpha,g\alpha))\right\}\right) \\
&\quad -\phi\left(\max\left\{d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta),d(\alpha,g\alpha)\right\}\right) \\
\xi(d(\gamma,g\gamma)) &\le \xi\left(\max\left\{d(\gamma,g\gamma),d(\theta,g\theta),d(\alpha,g\alpha),d(\beta,g\beta)\right\}\right) \\
&\quad -\phi\left(\max\left\{d(\gamma,g\gamma),d(\theta,g\theta),d(\alpha,g\alpha),d(\beta,g\beta)\right\}\right) \\
\xi(d(\theta,g\theta)) &\le \xi\left(\max\left\{d(\theta,g\theta),d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma)\right\}\right) \\
&\quad -\phi\left(\max\left\{d(\theta,g\theta),d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma)\right\}\right)
\end{aligned}$$

$$\begin{aligned}
&\therefore \xi\left(\max\left\{d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta))\right\}\right) \\
&\qquad \le \xi\left(\max\left\{d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta))\right\}\right) \\
&\qquad -\phi\left(\max\left\{d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta))\right\}\right)
\end{aligned}$$

$$\therefore \phi\left(\max\left\{d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta))\right\}\right) = 0$$

$$\Rightarrow \max\left\{d(\alpha,g\alpha),d(\beta,g\beta),d(\gamma,g\gamma),d(\theta,g\theta))\right\} = 0$$

$\therefore g\alpha = \alpha, \quad g\beta = \beta, \quad g\gamma = \gamma, \quad g\theta = \theta$
$g\alpha = G(\alpha,\beta,\gamma,\theta) = \alpha$

Similarly we may prove that
$g\beta = G(\beta,\gamma,\theta,\alpha) = \beta, \quad g\gamma = G(\gamma,\theta,\alpha,\beta) = \gamma, \quad g\theta = G(\theta,\alpha,\beta,\gamma) = \theta$
$\therefore F(\alpha,\beta,\gamma,\theta) = f\alpha = g\alpha = G(\alpha,\beta,\gamma,\theta) = \alpha \quad F(\beta,\gamma,\theta,\alpha,) = f\beta = g\beta = G(\beta,\gamma,\theta,\alpha) = \beta$
$F(\gamma,\theta,\alpha,\beta) = f\gamma = g\gamma = G(\gamma,\theta,\alpha,\beta) = \gamma, \quad F(\theta,\alpha,\beta,\gamma) = f\theta = g\theta = G(\theta,\alpha,\beta,\gamma) = \theta$

Thus $(\alpha,\beta,\gamma,\theta)$ is a quadruple fixed point of F,G,f,g

<u>Uniqueness</u>: Let if possible there are two fixed points say

$(\alpha,\beta,\gamma,\theta)$ and $(\alpha^*,\beta^*,\gamma^*,\theta^*)$ of F,G,f,g
$\therefore F(\alpha,\beta,\gamma,\theta) = f\alpha = g\alpha = G(\alpha,\beta,\gamma,\theta) = \alpha \quad F(\beta,\gamma,\theta,\alpha,) = f\beta = g\beta = G(\beta,\gamma,\theta,\alpha) = \beta$
$F(\gamma,\theta,\alpha,\beta) = f\gamma = g\gamma = G(\gamma,\theta,\alpha,\beta) = \gamma, \quad F(\theta,\alpha,\beta,\gamma) = f\theta = g\theta = G(\theta,\alpha,\beta,\gamma) = \theta$
and
$F(\alpha^*,\beta^*,\gamma^*,\theta^*) = f\alpha^* = g\alpha^* = G(\alpha^*,\beta^*,\gamma^*,\theta^*) = \alpha^*$
$F(\beta^*,\gamma^*,\theta^*,\alpha^*) = f\beta^* = g\beta^* = G(\beta^*,\gamma^*,\theta^*,\alpha^*) = \beta^*$
$F(\gamma^*,\theta^*,\alpha^*,\beta^*) = f\gamma^* = g\gamma^* = G(\gamma^*,\theta^*,\alpha^*,\beta^*) = \gamma^*,$
$F(\theta^*,\alpha^*,\beta^*,\gamma^*) = f\theta^* = g\theta^* = G(\theta^*,\alpha^*,\beta^*,\gamma^*) = \theta^*$

Putting $x = \alpha, y = \beta, z = \gamma, w = \theta$ and $p = \alpha^*, q = \beta^*, r = \gamma^*, s = \theta^*$

$$\begin{aligned}
&\xi(d(F(\alpha,\beta,\gamma,\theta), G(\alpha^*,\beta^*,\gamma^*,\theta^*)) \\
&\quad \le \xi\left(\max\left\{d(f\alpha,g\alpha^*),d(f\beta,g\beta^*),d(f\gamma,g\gamma^*),d(f\theta,g\theta^*)\right\}\right) \\
&\quad -\phi\left(\max\left\{d(f\alpha,g\alpha^*),d(f\beta,g\beta^*),d(f\gamma,g\gamma^*),d(f\theta,g\theta^*)\right\}\right) \\
&\quad +L\min\left(d(f\alpha,G(\alpha^*,\beta^*,\gamma^*,\theta^*)),d(g\alpha^*,F(\alpha,\beta,\gamma,\theta)),d(g\alpha^*,G(\alpha^*,\beta^*,\gamma^*,\theta^*))\right)
\end{aligned}$$

$$\begin{aligned}
\xi(d(\alpha,\alpha^*)) &\le \xi\left(\max\left\{d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*)\right\}\right) \\
&\quad -\phi\left(\max\left\{d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*)\right\}\right) \\
&\quad\quad +L\min\left(d(\alpha,\alpha^*),d(\alpha^*,\alpha),d(\alpha^*,\alpha^*)\right)
\end{aligned}$$

$$\begin{aligned}
\xi(d(\alpha,\alpha^*)) &\le \xi\left(\max\left\{d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*)\right\}\right) \\
&\quad -\phi\left(\max\left\{d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*)\right\}\right)
\end{aligned}$$

Similarly we may prove that

$$\begin{aligned}
\xi(d(\beta,\beta^*)) &\le \xi\left(\max\left\{d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*),d(\alpha,\alpha^*)\right\}\right) \\
&\quad -\phi\left(\max\left\{d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*),d(\alpha,\alpha^*)\right\}\right)
\end{aligned}$$

$$\xi(d(\gamma,\gamma^*)) \leq \xi\left(\max\left\{d(\gamma,\gamma^*),d(\theta,\theta^*),d(\alpha,\alpha^*),d(\beta,\beta^*)\right\}\right)$$
$$-\phi\left(\max\left\{d(\gamma,\gamma^*),d(\theta,\theta^*),d(\alpha,\alpha^*),d(\beta,\beta^*)\right\}\right)$$
$$\xi(d(\theta,\theta^*)) \leq \xi\left(\max\left\{d(\theta,\theta^*),d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*)\right\}\right)$$
$$-\phi\left(\max\left\{d(\theta,\theta^*),d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*)\right\}\right)$$
$$\therefore \xi\left(\max\left\{d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*)\right\}\right)$$
$$\leq \xi\left(\max\left\{d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*)\right\}\right)$$
$$-\phi\left(\max\left\{d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*)\right\}\right)$$
$$\therefore \phi\left(\max\left\{d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*)\right\}\right) = 0$$
$$\therefore \max\left\{d(\alpha,\alpha^*),d(\beta,\beta^*),d(\gamma,\gamma^*),d(\theta,\theta^*)\right\} = 0$$
$$\Rightarrow d(\alpha,\alpha^*) = d(\beta,\beta^*) = d(\gamma,\gamma^*) = d(\theta,\theta^*) = 0$$
$$\therefore \alpha = \alpha^*, \quad \beta = \beta^*, \quad \gamma = \gamma^*, \quad \theta = \theta^*$$

Thus $(\alpha,\beta,\gamma,\theta)$ is a unique common quadruple fixed point of F,G,f,g.

Corollary 3.2: *Let (X,≤) be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space .Suppose* $F,G : X^4 \to X$ and $g,f : X \to X$ *are such that F,G are continuous.* $k \in [0,1)$

(i) $d(F(x,y,z,w),G(p,q,r,s)$
$$\leq k\left(\max\left\{d(fx,gp),d(fy,gq),d(fz,gr),d(fw,gs))\right\}\right)$$
for all $x,y,z,w,p,q,r,s \in X$.

(ii) $F(X^4) \subset g(X), G(X^4) \subset f(X)$,

(iii) *the pairs (F, f) and (G, g) are W - compatible* .

Then F, G, f and g have a unique common coupled fixed point in X^4 and also they have a unique common fixed point in X.

Proof: Substituting $\xi(t) = t$ and $\phi(t) = (1-k)\,t$ and L=0, in the main theorem we get the proof.

Corollary 3.3: *Let (X,≤) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space .Suppose* $F,G : X^4 \to X$ and $g,f : X \to X$ *are such that F, G are continuous.*

(i) $d(F(x,y,z,w),G(p,q,r,s)) \leq \frac{k}{4}\left(d(fx,gp)+d(fy,gq)+d(fz,gr)+d(fw,gs)\right)$

for all $x,y,z,w,p,q,r,s \in X$.

(ii) $F(X^4) \subset g(X), G(X^4) \subset f(X)$,
(iii) the pairs (F, f) and (G, g) are W-compatible .

Then F, G, f and g have a unique common coupled fixed point in X^4 and also they have a unique common fixed point in X.

Proof: We know that

$$\frac{1}{4}\{d(fx,gp)+d(fy,gq)+d(fz,gr)+d(fw,gs)\} \le \max\{d(fx,gp),d(fy,gq),d(fz,gr),d(fw,gs)\}$$

So we get the proof from Corollary 3.1.

Corollary 3.4: *Let $(X,\le)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space .Suppose* $F,G : X^4 \to X$ and $g,f : X \to X$ *are such that F, G are continuous.* $\phi\in\Phi$

(i) $d(F(x,y,z,w),G(p,q,r,s)$
$$\le \max\{d(fx,gp),d(fy,gq),d(fz,gr),d(fw,gs))\} - \phi\left(\max\{d(fx,gp),d(fy,gq),d(fz,gr),d(fw,gs))\}\right)$$
for all $x,y,z,w,p,q,r,s \in X$.

(ii) $F(X^4) \subset g(X), G(X^4) \subset f(X)$,

(iii) the pairs (F, f) and (G, g) are W-compatible .

Then F, G, f and g have a unique common coupled fixed point in X^4 and also they have a unique common fixed point in X.

Proof: Substituting $\xi(t) = t$ and L=0, in the main theorem we get the proof.

Corollary 3.5: *Let $(X,\le)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space .Suppose* $F: X^4 \to X$ and $f : X \to X$ *are such that F,G are continuous. k is a constant such that* $k\in[0,1)$

(i) $d(F(x,y,z,w),F(p,q,r,s) \le k \max\{d(fx,fp),d(fy,fq),d(fz,fr),d(fw,fs))\}$
for all $x,y,z,w,p,q,r,s \in X$.

(ii) $F(X^4) \subset f(X)$,
(iii) the pairs (F, f) are W-compatible .

Then F, f has a unique common coupled fixed point in X^4 and also they have a unique common fixed point in X.

Proof: Substituting $F = G$ and $f = g$ in Corollary 3.1, we get the proof.

Example 3.6: Let X= [0, 1] and $d(x, y) = |x - y|$, (X,d) is a complete metric space

$F(x, y, z, w) = \max\{x, y, z, w\}$ $\quad G(x, y, z, w) = \frac{1}{2}\max\{x, y, z, w\}$

$\xi(t) = \frac{1}{5}t$ $\quad \phi(t) = \frac{1}{6}t$

$fx = 4x$ $\quad gx = 3x$

Thus $F, G : X^4 \to X$ and $g, f : X \to X$ and F, G are continuous. $\xi, \phi \in \Phi$ and $L \geq 0$ such that $F(X^4) \subset g(X), G(X^4) \subset f(X)$, and the pairs (F, f) and (G, g) are W-compatible . Thus all the conditions of the theorem are satisfied .The unique fixed point is (0, 0, 0, 0).

Conclusion

A unique common quadruple fixed point theorem by introducing a new contractive condition for two altering distance functions and four maps satisfying w-compatible condition in pairs has been proved without using monotone property.

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