Gen. Math. Notes, Vol. 26, No. 1, January 2015, pp.74-83
ISSN 2219-7184; Copyright ©ICSRS Publication, 2015
www.i-csrs.org
Available free online at http://www.geman.in

Ruled Surfaces With the Bishop Frame in Euclidean 3-Space

Yılmaz Tunçer

Usak University, Faculty of Sciences and Arts
Department of Mathematics, Usak-Turkey
E-mail: yilmaz.tuncer@usak.edu.tr

(Received: 9-7-14 / Accepted: 7-9-14)

Abstract

In this study, we examined the ruled surfaces in Euclidean 3-space with the Bishop frame of the base curve in two cases and obtained some characterizations on the ruled surfaces by using its directrix, Bishop curvatures, shape operator and Gauss curvature. Furthermore, we calculated Bishop Steiner rotation, Bishop tranlation vectors, the pitch and angle of pitch of the ruled surfaces and gave some sufficient and necessary conditions for developable ruled surface with respect to the Bishop frame of the base curve by using Bishop curvatures and curvatures of the ruled surface along the generating lines.

Keywords: *Ruled surface, Bishop frame, Gauss curvature.*

1 Preliminaries

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. we can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis ($N_1(s)$ and $N_2(s)$) for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(s)$ at each point. If the derivatives of ($N_1(s)$ and $N_2(s)$) depend only on $T(s)$ and not each other we can make and $N_1(s)$ vary $N_2(s)$ smoothly throughout the path regardless of the curvature. Therefore, we have the alternative frame equations

$$\begin{bmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{bmatrix} \tag{1}$$

where

$$\kappa(s) = \sqrt{k_1^2 + k_2^2}, \quad \theta(s) = \arctan\left(\tfrac{k_2}{k_1}\right), \quad \tau(s) = -\tfrac{d\theta(s)}{ds}$$

[1,2,3], so that $k_1(s)$ and $k_2(s)$ effectively correspond to a Cartesian coordinate system for the polar coordinates $(\kappa(s),\theta(s))$, with $\theta(s) = -\int \tau(s)\, ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation.

2 Introduction

A straight line X in IR^3, such that it is strictly connected to Frenet frame the curve $\alpha(s)$, is represented uniquely with respect to this frame, in the form

$$X(s) = f_1(s)T(s) + f_2(s)N_1(s) + f_3(s)N_2(s) \tag{2}$$

As X move along $\alpha(s)$, it generates a ruled surface given by the regular parametrization

$$\varphi(s,v) = \alpha(s) + vX(s) \tag{3}$$

where the components $f_i(s)$, $i = 1,2,3$ are scalar functions of the arc-length parameter of the curve $\alpha(s)$. This surface will be denoted by M. the curve $\alpha(s)$ is called a base curve and the various positions of the generating line X are called the rulings of the surface M. If consecutive rulings of ruled surface in IR^3 intersect, then the surface is to be developable. All the other ruled surfaces are called skew surfaces. If there exists a common perpendicular to two constructive rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a striction point. The set of striction points on the ruled surface defines the striction curve[6]. The striction curve of M can be written in terms of the base curve $\alpha(s)$ as

$$\overline{\alpha}(s) = \alpha(s) - \frac{\langle T, D_T X\rangle}{\|D_T X\|^2} X(s) \tag{4}$$

If $\|D_T X\| = 0$ then the ruled surface doesn't have any striction curve. This case characterizes the ruled surface is cyclindirical. Thus the base curve can take as a striction curve. Let P_x be distribution parameter of M, then

$$P_X = \frac{det(T, X, D_T X)}{\|D_T X\|^2} \tag{5}$$

where D is Levi-Civita connection on E^3 [4]. If the base curve is periodic then M is closed ruled surface. Let M be a closed ruled surface and W be Darboux vector then the Steiner rotation vector and Steiner translation vector are

$$D = \oint_{(\alpha)} W d\alpha, \quad V = \oint_{(\alpha)} d\alpha \tag{6}$$

respectively. Furthermore, the pitch and angle of pitch of M are

$$L_X = \langle V, X \rangle, \quad \lambda_X = \langle D, X \rangle \tag{7}$$

respectively.

Theorem 2.1 *A ruled surface is a developable surface if and only if the distribution parameter of the ruled surface is zero [4].*

3 One-Parameter Spatial Bishop Motion in $\mathbf{E}^3$

Let $\alpha : I \to E^3$ be a regular curve and $\{T, N_1, N_2\}$ be Bishop Frame. The two coordinate systems $\{O,\ T, N_1, N_2\}$ and $\{0'; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are orthogonal coordinate systems in E^3 which represent the moving space H and the fixed space H', respectively. Let us express the displacements of H with respect to H' as (H/H'). In generally, during the one parameter spatial motion (H/H'), each line X of the moving space H, generates a ruled surface in the fixed space H'. We assume that X be fixed unit vector such that the coefficients f_1, f_2 and f_3 in (2.1) are constant. We can obtain the distribution parameter of the ruled surface generated by line X of the moving space H. From (2) we obtain,

$$D_T X = f_1 D_T T + f_2 D_T N_1 + f_3 D_T N_2 \tag{8}$$

and substituting (1) into (8), we obtain

$$D_T X = (-k_1 f_2 - k_2 f_3)\, T + k_1 f_1 N_1 + k_2 f_1 N_2 \tag{9}$$

Thus distribution parameter of the ruled surface generated by line X is

$$P_X = \frac{f_1 f_2 k_2 - f_1 f_3 k_1}{\left\| D_T X \right\|^2} \tag{10}$$

We can state $P_X = 0$ if and only if following equation satisfies.

$$\frac{k_1}{k_2} = \frac{f_2}{f_3}$$

On the other side, (6) and (7) brings us to definition of the Bishop Steiner rotation and Bishop tranlation vectors for constant f_1, f_2 and f_3 are

$$D = -N_1 \oint_{(\alpha)} k_2 ds + N_2 \oint_{(\alpha)} k_1 ds, \quad V = \oint_{(\alpha)} T ds \tag{11}$$

respectively. After rutin calculations we obtain the pitch and angle of pitch of M are

$$L_X = f_1 L_T, \quad \lambda_X = f_3 \oint_{(\alpha)} k_1 ds - f_2 \oint_{(\alpha)} k_2 ds \tag{12}$$

respectively. In the special cases that $X = T$, $X = N_1$ and $X = N_2$ we can obtained following equations by using (7), (11) and (12).

$$L_T = \oint_{(\alpha)} ds, \quad L_{N_1} = L_{N_2} = 0, \quad L_X = f_1 L_T$$

and

$$\lambda_T = 0, \quad \lambda_{N_1} = -\oint_{(\alpha)} k_2 ds, \quad \lambda_{N_2} = \oint_{(\alpha)} k_1 ds, \quad \lambda_X = f_3 \oint_{(\alpha)} k_1 ds - f_2 \oint_{(\alpha)} k_2 ds$$

For the rest of our study, we assume that X be unit vector such that the coefficients $f_1 = 0$, f_2 and f_3 in (2) are nonconstant differentiable functions and ξ be unit normal vector field of M. Thus we can set up the orthonormal system $\{T, X, \xi\}$. Since ξ normal to T then ξ lies in the plane $Sp\{N_1, N_2\}$ and ξ can be choose as $\xi = T\Lambda X$ along the line X depending orientation of M. Therefore, in this case we shall use the special functions f_2 and f_3.Thus ξ and X can be written as

$$\xi = \cos\psi N_1 + \sin\psi N_2 \quad , \quad X = \sin\psi N_1 - \cos\psi N_2 \tag{13}$$

where $\psi = \psi(s)$ is the angle between ξ and N_1 along α and $f_2(s) = \sin\psi(s)$, $f_3(s) = -\cos\psi(s)$. We obtained the distribution parameter of the ruled surface M by using (1) and (5). From (13)

$$D_T X = -\lambda T + \psi' \xi \tag{14}$$

and putting this equation in (5), we obtained by direct computation that the distribution parameter is

$$P_X = \frac{\psi'}{\lambda^2 + \psi'^2} \tag{15}$$

which we need to determine the conditions of the ruled surface is developable. We know that the ruled surface developable if and only if P_X is zero so we can state the following theorem.

Theorem 3.1 *A ruled surface M is a developable if and only if its generating lines has to be constant with respect to the Bishop frame along the base curve.*

Corollary 3.2 *A ruled surface is a developable if and only if the angle between the vector fields ξ and N_1 is constant along the base curve.*

We can give the following remark for the special cases that $X = T$, $X = N_1$ and $X = N_2$.

Corollary 3.3 *Ruled surfaces which has the generating line is T, N_1 or N_2, are developable. This implies that $P_T = P_{N_1} = P_{N_2} = 0$.*

Proof: It is easly see that from (2) and (10), we obtain $P_T = P_{N_1} = P_{N_2} = 0$ then the ruled surface which has the generating line is one of T, N_1 or N_2, are developable.

On the other hand, from (4) the striction curve of M is

$$\overline{\alpha}(s) = \alpha(s) + \frac{\lambda}{\lambda^2 + (\psi')^2} X(s)$$

where $\lambda = k_1 \sin\psi - k_2 \cos\psi$. In the case of M is a cylindirical ruled surface, we can give the following theorem.

Corollary 3.4 *If M is a cylindirical ruled surface then $\lambda = 0$. In this case, the base and striction curves are coincide.*

In the case $\psi \neq (2k+1)\frac{\pi}{2}$, $k \in Z$ then we can write following equation by using corollary 3

$$\frac{k_2}{k_1} = \tan\psi.$$

Thus we proved the following corollary.

Corollary 3.5 *A cylindirical ruled surface is developable if and only if $\frac{k_2}{k_1}$ is constant. In this case the base curve is the planar.*

Corollary 3.6 *Let $\overline{\alpha}(s)$ be striction curve of M then following equation satisfies*

$$\frac{\lambda}{\lambda^2 + (\psi')^2} = constant.$$

Proof: Since the tangent vector field of the striction curve is normal to X then $\left\langle X, \frac{d\overline{\alpha}}{ds} \right\rangle = 0$.

$$\begin{aligned} \left\langle X, \frac{d\overline{\alpha}}{ds} \right\rangle &= \left\langle X, T + \frac{\lambda}{\lambda^2 + (\psi')^2} D_T X + \frac{d}{ds}\left(\frac{\lambda}{\lambda^2 + (\psi')^2}\right) X \right\rangle \\ &= \langle X, T\rangle + \frac{\lambda}{\lambda^2 + (\psi')^2} \langle X, D_T X\rangle + \frac{d}{ds}\left(\frac{\lambda}{\lambda^2 + (\psi')^2}\right) \langle X, X\rangle \end{aligned}$$

then

$$\left\langle X, \frac{d\overline{\alpha}}{ds} \right\rangle = \frac{d}{ds}\left(\frac{\lambda}{\lambda^2 + (\psi')^2}\right)$$

thus we get

$$\frac{d}{ds}\left(\frac{\lambda}{\lambda^2+(\psi')^2}\right)=0$$

and we obtain

$$\frac{\lambda}{\lambda^2+(\psi')^2}=constant.$$

Corollary 3.7 *Let M be a ruled surface given in the form (3) then $\varphi(t,v_o)$ is a striction point if and only if $D_T X$ is normal to tangent plane at that point on M, where v_o is*

$$v_o=\frac{\lambda}{\lambda^2+(\psi')^2}$$

Proof: Let suppose that $\varphi(t,v_o)$ be a striction point on the base curve $\alpha(t)$ of M then we must show that $\langle D_T X, X\rangle=0$ and $\langle D_T X, A\rangle=0$. Since the vector field X is unit vector then $T[\langle X, X\rangle]=0$ and we get $\langle D_T X, X\rangle=0$. On the other side, from (13) and (14) we obtain,

$$\langle D_T X, A\rangle=\left\{\lambda^2+(\psi')^2\right\}v_o-\lambda.$$

Now, by making the substitution

$$v_o=\frac{\lambda}{\lambda^2+\psi'^2}$$

we have that the last equation reduces to $\langle D_T X, A\rangle=0$. This means that $D_T X$ is normal to tangent plane at the striction point $\varphi(t,v_o)$ on M.

Conversely, since $D_T X$ is normal to tangent plane at the point $\varphi(t,v_o)$ on M then $\langle D_T X, A\rangle=0$ and so we obtain

$$\langle D_T X, A\rangle=\left\{\lambda^2+(\psi')^2\right\}v_o-\lambda=0$$

thus we get

$$v_o=\frac{\lambda}{\lambda^2+(\psi')^2}$$

so we proved that $\varphi(t,v_o)$ is a striction point on M.

Theorem 3.8 *Absolute value of the Gauss curvature of M is maximum at the striction points on the generating line X and*

$$|K|_{\max}=\frac{\{\lambda^2+(\psi')^2\}^2}{(\psi')^2}$$

Proof: Let M be a ruled surface that given in the form (3) and let Φ be base of the tangent space which is spanning by the unit vectors A_o and X where A_o is the tanget vector of the curve $\varphi(s, v = const.)$ with the arc-lengh parameter s^*. Hence we write

$$A_o = \frac{d\varphi}{ds^*} = \frac{d\varphi}{ds}\frac{ds}{ds^*}$$

where $\frac{d\varphi}{ds} = A$, $A_o = \frac{1}{\|A\|}A$ and $\frac{ds}{ds^*} = \frac{1}{\|A\|}$. Thus we obtain the following equations after the routine calculations.

$$D_{A_o}T = \frac{1}{\|A\|}\{\lambda X + \mu\xi\} \tag{16}$$

$$D_{A_o}\xi = \frac{1}{\|A\|}\{-\mu T - \psi' X\} \tag{17}$$

$$D_{A_o}A = \frac{1}{\|A\|}\left\{\begin{array}{c} \left\{(1-v\lambda)' - v\psi'\mu\right\}T + \left\{(1-v\lambda)\lambda - v(\psi')^2\right\}X \\ +\left\{(1-v\lambda)\mu + (v\psi')'\right\}\xi \end{array}\right\} \tag{18}$$

where $\lambda = k_1 \sin\psi - k_2\cos\psi$ and $\mu = k_1\cos\psi + k_2\sin\psi$. On the other hand, we denote $\xi_{\varphi(s,v)}$ as the unit normal vector at the points $\varphi(s,v)$ then from (2) and (13) we get

$$\xi_{\varphi(s,v)} = \frac{1}{\|A\|}\{(-v\psi')T + (1-v\lambda)\xi\} \tag{19}$$

By differentiating both side of (19) with respect to the parameter s we get

$$\begin{aligned} \frac{d\xi_{\varphi(s,v)}}{ds} = & -\left\{\left((v\psi')' + \mu(1-v\lambda)\right)\frac{1}{\|A\|} + v\psi'\left(\frac{1}{\|A\|}\right)'\right\}T \\ & +\left\{(1-v\lambda)\left(\frac{1}{\|A\|}\right)' + \left((1-v\lambda)' - v\mu\psi'\right)\frac{1}{\|A\|}\right\}\xi \\ & -\psi'\frac{1}{\|A\|}X \end{aligned}$$

Let S be shape operator of M at the points $\varphi(s,v)$ then we can easly obtained that the matrix S_Φ is following with respect to base Φ.

$$S_\Phi = \begin{bmatrix} \langle S(A_o), A_o\rangle & \langle S(A_o), X\rangle \\ \langle S(X), A_o\rangle & \langle S(X), X\rangle \end{bmatrix}$$

Since $\langle S(X), X\rangle = 0$ and $\langle S(A_o), X\rangle = \langle S(X), A_o\rangle$ then the Gauss curvature is

$$K(s,v) = \det S_\Phi = -\left(\langle S(A_o), X\rangle\right)^2 \tag{20}$$

Suppose that s^* is arc-length parameter of A_o then we can get

$$S\left(A_o\right) = D_{A_o}\xi_{\varphi(s,v)} = \frac{d\xi_{\varphi(s,v)}}{ds^*} = \frac{d\xi_{\varphi(s,v)}}{ds}\frac{ds}{ds^*} = \frac{1}{\|A\|}\frac{d\xi_{\varphi(s,v)}}{ds}$$

From (17) and (19) we obtained

$$\begin{aligned} S\left(A_o\right) &= -\frac{1}{\|A\|}\left\{\left((v\psi')' + \mu\left(1 - v\lambda\right)\right)\frac{1}{\|A\|} + v\psi'\left(\frac{1}{\|A\|}\right)'\right\}T \\ &\quad + \left\{(1 - v\lambda)\left(\frac{1}{\|A\|}\right)' + \left((1 - v\lambda)' - v\mu\psi'\right)\frac{1}{\|A\|}\right\}\xi \\ &\quad - \psi'\frac{1}{\|A\|}X \end{aligned}$$

Hence the Gauss curvature is

$$K(s,v) = -\frac{\left(\psi'\right)^2}{\|A\|^4}. \tag{21}$$

We differentiated both side of (21) with respect to v for find the maximum value of the Gauss curvature along the X on M. Thus we obtain

$$\frac{\partial K(s,v)}{\partial v} = \frac{2\left(\psi'\right)^2\left\{v\left(\lambda^2 + (\psi')^2\right) - \lambda\right\}}{\left\{v^2\left(\lambda^2 + (\psi')^2\right) - 2v\lambda + 1\right\}^3} = 0$$

and we get the value of v is

$$v = \frac{\lambda}{\lambda^2 + \left(\psi'\right)^2}. \tag{22}$$

It is easly to see that, $\varphi\left(s,v\right)$ is the striction point and we can say that the Gauss curvature of M is minimum at the striction points on the directrix X. Finally, by substuting (21) in (22) then we get

$$|K|_{\max} = \frac{\left\{\lambda^2 + (\psi')^2\right\}^2}{(\psi')^2}. \tag{23}$$

We can write the relation between the Gauss curvature and the distribution parameter by using (15) and (23) is following

$$|K|_{\max} = \frac{1}{(P_X)^2}.$$

Thus we give the following remark.

Corollary 3.9 *Distribution parameter of the ruled surface is depent only generating lines.*

Morover, the Darboux frame of the surface along base curve is

$$\begin{bmatrix} D_T T \\ D_T X \\ D_T \xi \end{bmatrix} = \begin{bmatrix} 0 & \lambda & \mu \\ -\lambda & 0 & \psi' \\ -\mu & -\psi' & 0 \end{bmatrix} \begin{bmatrix} T \\ X \\ \xi \end{bmatrix}$$

and the Bishop Darboux vector is

$$W = (\psi')\,T - \mu X + \lambda \xi$$

where $\lambda = k_1 \sin\psi - k_2 \cos\psi$ and $\mu = k_1 \cos\psi + k_2 \sin\psi$. If $\psi = 0$ then we obtain the same Bishop Darboux vector as in [3]. Thus we obtain geodesic curvature, geodesic torsion and normal curvature of the ruled surface along its generating lines as follows

$$\kappa_g = \lambda, \quad \tau_g = \psi', \quad \kappa_\xi = \mu \tag{24}$$

respectively. Note also that if the ruled surface is constant curvature surface with nonzero geodesic curvature then P_X is constant and from (15) and (24) we obtain $\frac{\tau_g}{\kappa_g}$ is constant. Hence we can give the following theorem.

Theorem 3.10 *A ruled surface is constant curvature surface with nonzero geodesic curvature if and only if* $\frac{\tau_g}{\kappa_g}$ *is constant.*

In the case that the base curve is the geodesic of the ruled surface then $\tau_g = 0$ and ψ =constant. In this case, the ruled surface is developable. On the other side, the Bishop Steiner rotation and Bishop tranlation vectors are

$$D = \psi T - \left(\oint_{(\alpha)} \lambda ds \right) X + \left(\oint_{(\alpha)} \mu ds \right) \xi, \quad V = \oint_{(\alpha)} T ds \tag{25}$$

respectively, furthermore, from (7) the pitch and angle of pitch of M are

$$L_X = 0, \quad \lambda_X = -\oint_{(\alpha)} \lambda ds \tag{26}$$

respectively. From (13) and (26), we can obtained following equations in the special cases that $X = T$, $X = N_1$ and $X = N_2$

$$L_T = L_{N_1} = L_{N_2} = 0$$

and

$$\lambda_T = \psi$$

$$\lambda_{N_1} = \left(\oint\limits_{(\alpha)} \mu ds \right) \cos\psi - \left(\oint\limits_{(\alpha)} \lambda ds \right) \sin\psi$$

$$\lambda_{N_2} = \left(\oint\limits_{(\alpha)} \mu ds \right) \sin\psi + \left(\oint\limits_{(\alpha)} \lambda ds \right) \cos\psi.$$

Thus we conclude that if the ruled surface is developable then following equation

$$(\lambda_{N_1})^2 + (\lambda_{N_2})^2 = \left(\oint\limits_{(\alpha)} k_1 ds \right)^2 + \left(\oint\limits_{(\alpha)} k_2 ds \right)^2$$

is satisfies.

References

[1] A.J. Hanson and H. Ma, Parallel transport approach to curve framing, Indiana University, *Techreports-TR425*, January 11 (1995).

[2] L.R. Bishop, There is more than one way to frame a curve, *Amer. Math. Monthly*, 82(3) (1975), 246-251.

[3] B. Bükçü and M.K. Karacan, Special Bishop motion and Bishop darboux rotation axis of the space curve, *Journal of Dynamical Systems and Geometric Theories*, 6(1) (2008), 27-34.

[4] H.H. Hacısalihoğlu, *Diferensiyel Geometri*, Ankara Üniversitesi Fen Fakültesi, Cilt. 2, Ankara, (1993).

[5] S. Izumiya and N. Takeuchi, Special curves and ruled surfaces, *Beitrage zur Algebra und Geometrie*, 44(1) (2003), 203-212.

[6] E. Kasap, A method of the determination of a Geodesic curve on ruled surface with time-like rulings, *Novi Sad J. Math.*, 35(2) (2005), 103-110.

[7] K. Nomizu, Kinematics and differential geometry of submanifolds, *Tohoku Math. Journ.*, 30(1978), 623-637.

[8] B. O'Neill, *Semi-Riemannian Geometry With Applications to Relativity*, Academic Press, New York, (1983).

[9] Y. Tunçer, Y. Yayli and M.K. Sağel, On differential geometry and kinematics of Euclidean submanifolds, *Balkan Journal of Geometry and Its Applications*, 13(2) (2008), 102-111.