Gen. Math. Notes, Vol. 18, No. 2, October, 2013, pp. 104-119
ISSN 2219-7184; Copyright © ICSRS Publication, 2013
www.i-csrs.org
Available free online at http://www.geman.in

Compact Finite Difference Methods for the Solution of One Dimensional Anomalous Sub-Diffusion Equation

Faoziya S. Al-Shibani[1], Ahmed I. Md. Ismail[2] and Farah A. Abdullah[3]

[1, 2, 3] School of Mathematical Sciences
Universiti Sains Malaysia, 11800 Penang, Malaysia
[1] E-mail: fsmm11_mah020@student.usm.my
[2] E-mail: izani@cs.usm.my
[3] E-mail: farahaini@usm.my

(Received: 9-8-13 / Accepted: 14-9-13)

Abstract

In this paper, two high order compact finite difference schemes are formulated for solving the one dimensional anomalous subdiffusion equation. The Grünwald-Letnikov formula is used to discretize the temporal fractional derivative. The truncation error and stability of the two methods are discussed. The feasibility of the compact schemes is investigated by application to a model problem.

Keywords:*Compact finite difference method, Time fractional diffusion, Grünwald-Letnikov method.*

1 Introduction

Diffusionis a phenomenathat has been rigorously and extensively studied. This phenomenon is modelled bythe diffusion equation which is a partial differential equation that describes the spread of a substance whose particles move, due to the random motion, from a region of higher concentration to a region of lower concentration. The dependent variable in the equation is concentration of the substance and the independent variables are spatial and temporal variables. The diffusion equation is derived using Fick's law which assumes a homogenous environment. In the case of non-homogenous environments (for example, in a cell) then the fractional diffusion equation is used to model the diffusion process.Anomalous diffusion describes the spread of a particle plume at a rate incompatible with the classical Brownian motion mode.Furthermore, the plume may be asymmetric.When a cloud of particles spreads slower than classical diffusion, it leads to anomalous subdiffusion [2].

The study of fractional partial differential equations has increased in recent years. A comprehensive background on this topic can be found in books by Das[14] and Podlubny[6]. Compact finite difference schemes are sometimes preferred because their accuracy and high computational efficiency.Several papers have recently been published on compact finite difference methods for solving theanomalous diffusion equation. Gao and Sun [5] have presented a high order compact finite difference scheme for the fractional subdiffusion equations.They first transformed the problem usingthe Caputo definition and other analytical theories then thel_1 discretization was applied to approximate the temporal fractional derivative. The stability and convergence of the method were analyzed by the energy method. A compact finite difference method was also discussed by Cui [8] for the anomaloussubdiffusion equation. The Grünwald-Letnikov formulawas used to approximate the time fractional derivative. The second order spatial derivative in the problem was approximated with fourth order accuracy. The stability and the convergence of the proposed method were investigated.Richardson extrapolation was appliedto increase the accuracy in time as the generating function used in the Grünwald-Letnikovformula givesfirst order accuracy in time.It was pointed out that the use of the "Short Memory" principlemakes the solution inefficient for the given example.Du et al.[11]developed a compact finite difference scheme for the time fractional diffusion equation when $1 < \alpha < 2$ so that the fractional wave equation was also considered. The Caputo definition was applied to discretize the time fractional derivative. The stability and convergence of the scheme were analyzed in l_∞ norm by using the energy method. It was found that the scheme is unconditionally stable and has $\mathrm{O}(\tau^{3-\alpha} + \mathrm{h}^2)$ convergence.

The purpose of this article is to develop two high order compact finite difference methods for solving the one-dimensional fractional anomalous sub-diffusion equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\,,\ (x,t) \in (0,1)\times(0,T] \tag{1}$$

With the initial condition

$$u(x,0) = 0,\ 0 < x < 1 \tag{2}$$

And the boundary conditions

$$u\,(0,t) = f_1(t), \quad u\,(1,t) = f_2(t)\,, \quad 0 \le t \le T \tag{3}$$

Where$f$, f_1 and f_2 are known functions, and the function u is the unknown function to be determined.We consider the case when 0 < α < 1 as we wish to study the anomalous subdiffusion equation.We shall take h =1/Min the x direction with $x_i = ih$, $i = 0,1 \ldots M$, $t_n = n\tau$ and $n = 0, 1\ldots N$.High order discretizations require an additional number of grid points which induces more computational effort. The two compact finite difference methods which will be described in this article overcome these difficulties by involving derivatives of the function values at the nodes of the corresponding independent variables. We eventually eliminate the derivatives to get formulas with three points. This then leads to block tridiagonal systems.

2 Fractional Derivatives and Integrals

In this section we give some relevant background on fractional derivatives and related integral. Let $f$be a function with respect to one independent variable t, the fractional derivative ${}_0D_t^{\alpha}$ of $f(t)$can be defined by Riemann-Liouville formula as (see [12]).

$${}_0D_t^{\alpha} f(t) = \frac{d}{dt}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f(\iota)d\iota}{(t-\iota)^{\alpha}}\right] 0 < \alpha < 1 \tag{4}$$

Where $\Gamma(\cdot)$ is the Gamma function and $0 \le t \le N$.

The above derivative is related to the Riemann-Liouville fractional integral, which is defined as

$${}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\iota)d\iota}{(t-\iota)^{1-\alpha}}\, 0 < \alpha < 1 \tag{5}$$

Where ${}_0D_t^{\alpha}\,{}_0I_t^{\alpha} f(t) = f(t)$ (6)

Fractional derivatives can also be represented by the Grünwald-Letnikovformula definedas (see [10]):

$${}_0D_t^{\alpha} f(t) = \frac{1}{\tau^{\alpha}}\sum_{k=0}^{\left[\frac{t}{\tau}\right]} \omega_k^{(\alpha)} f(t-k\tau) + O(\tau^p)\,, \qquad t \ge 0 \tag{7}$$

Where $\frac{t}{\tau}$ is integer and k = 0, 1, 2… $\frac{t}{\tau}$ and $\omega_0^{(\alpha)} = 1$, $\omega_k^{(\alpha)} = (1 - \frac{\alpha+1}{k})\omega_{k-1}^{(\alpha)}$.

The coefficients $\omega_k^{(\alpha)}$ are the coefficients of the power series of the generating function $\omega(z,\alpha) = (1-z)^\alpha$ and are also the coefficients of the two-point backward difference approximation of the first order derivative.

where $$(1-z)^\alpha = \sum_{k=0}^{\infty} \binom{k-\alpha-1}{k} z^k = \sum_{k=0}^{\infty} \omega_k^{(\alpha)} z^k \tag{8}$$

It is possible for the function $\omega(z,\alpha)$ to generate coefficients of the high order approximation, for instance if the generationg function is

$$\omega(z,\alpha) = (\frac{25}{12} - 4z + 3z^2 - \frac{4}{3}z^3 + \frac{1}{4}z^4)^\alpha \tag{9}$$

then we get the Grünwald-Letnikovformula of a fourth order accuracy (i.e., P= 4)(see [6]).
The coefficients $\omega_k^{(\alpha)}$ can be expressed in term of the Fourier transform (see [6]).

If $z = e^{-i\varphi}$ in Eq.(2.5) then $\omega_k^{(\alpha)} = \frac{1}{2\pi i}\int_0^{2\pi} f_\alpha(\varphi) e^{-ik\varphi} d\varphi$ and $f_\alpha(\varphi) = (1-e^{-i\varphi})^\alpha$.

3 Compact Finite Difference Method with the Grünwald-Letnikov Formula

Let us define the central, forward and backward difference operators

$$\delta_x u_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2h},\ \left(\frac{\partial_+ u}{\partial x}\right)_i^n = \frac{u_{i+1}^n - u_i^n}{h},\left(\frac{\partial_- u}{\partial x}\right)_i^n = \frac{u_i^n - u_{i-1}^n}{h} \tag{10}$$

We can represent u_{i+1}^n, u_{i-1}^n about (x_i, t_n) by Taylor seriesas (see [15])

$$\left.\begin{aligned} u_{i+1}^n &= u_i^n + h\left(\frac{\partial u}{\partial x}\right)_i^n + \frac{(h)^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{(h)^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \cdots \\ u_{i-1}^n &= u_i^n - h\left(\frac{\partial u}{\partial x}\right)_i^n + \frac{(h)^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n - \frac{(h)^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \cdots \end{aligned}\right\} \tag{11}$$

Analogously, the first and second derivatives of $u_{i+1}^n, u_{i-1}^n$are

$$\left.\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_{i+1}^n &= \left(\frac{\partial u}{\partial x}\right)_i^n + h\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{(h)^2}{2!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \frac{(h)^3}{3!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + \cdots \\ \left(\frac{\partial u}{\partial x}\right)_{i-1}^n &= \left(\frac{\partial u}{\partial x}\right)_i^n - h\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{(h)^2}{2!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n - \frac{(h)^3}{3!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + \cdots \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i+1}^n &= \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + h\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \frac{(h)^2}{2!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + \frac{(h)^3}{3!}\left(\frac{\partial^5 u}{\partial x^5}\right)_i^n + \cdots \\ \left(\frac{\partial^2 u}{\partial x^2}\right)_{i-1}^n &= \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n - h\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \frac{(h)^2}{2!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n - \frac{(h)^3}{3!}\left(\frac{\partial^5 u}{\partial x^5}\right)_i^n + \cdots\end{aligned}\right\} \tag{13}$$

We approximate the first and second spatial derivativesusing Eq.(11), (12) and (13)toget

$$\left(\frac{\partial u}{\partial x}\right)_i^n = \frac{\delta_x u_i^n}{\left(1+\frac{h^2}{6}\frac{\partial_+}{\partial x}\frac{\partial_-}{\partial x}\right)} + O(h^4) \tag{14}$$

And

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n = \frac{\frac{\partial_+}{\partial x}\left(\frac{\partial_- u}{\partial x}\right)_i^n}{\left(1+\frac{h^2}{12}\frac{\partial_+}{\partial x}\frac{\partial_-}{\partial x}\right)} + O(h^4) \tag{15}$$

To obtain the approximate solution of Eq.(1), we replace the second order derivative in space by the finite difference approximation in Eq.(15) and discretize the time fractional derivative by using Grünwald-Letnikovdefinition (7).
When $p = 1$. We obtain

$$\begin{cases}\dfrac{1}{\tau^\alpha}\displaystyle\sum_{k=0}^{n}\omega_k^{(\alpha)}u_i^{n-k} = \dfrac{\frac{\partial_+}{\partial x}\left(\frac{\partial_- u}{\partial x}\right)_i^n}{\left(1+\frac{h^2}{12}\frac{\partial_+}{\partial x}\frac{\partial_-}{\partial x}\right)} + f_i^n, \\ i=1,2,\ldots,M-1, \quad n=1,2,\ldots,N, \\ u_i^0 = 0, \quad i=1,2,\ldots,M-1, \\ u_0^n = f_1(t_n), \quad u_M^n = f_2(t_n), \quad n=1,2,\ldots,N.\end{cases} \tag{16}$$

Thecompact finite difference method with Grünwald-Letnikovformula for Eq. (1), (2) and (3) is in the form

$$\begin{cases}\displaystyle\sum_{k=0}^{n}\omega_k^{(\alpha)}(u_{i-1}^{n-k}+10u_i^{n-k}+u_{i+1}^{n-k}) = 12S(u_{i-1}^n - 2u_i^n + u_{i+1}^n) + \tau^\alpha(f_{i-1}^n + 10 f_i^n + f_{i+1}^n) \\ i=1,2,\ldots,M-1, \quad n=1,2,\ldots,N, \\ u_i^0 = 0, \quad i=1,2,\ldots,M-1, \\ u_0^n = f_1(t_n), \quad u_M^n = f_2(t_n), \quad n=1,2,\ldots,N.\end{cases} \tag{17}$$

Where $S = \frac{\tau^\alpha}{h^2}$.

We can represent the equations defined in Eq. (17) in a linear system of equations of the form

$$\underline{A}\underline{U}^{n} = \underline{B}\underline{U}^{n-1} + \sum_{k=2}^{n} \underline{C}\underline{U}^{n-k} + \tau^{\alpha}\underline{F}^{n}, 1 \le i \le M-1, n = 2,3,...N \tag{18}$$

Where $\underline{U}^{n} = [u_1^n, u_2^n, ..., u_{M-1}^n]^T$.

The matrices in Eq. (18) are tridiagonal and are defined as follows

$$\underline{A} = \begin{pmatrix} -(24S+10) & 12S-1 & & & \\ 12S-1 & -(24S+10) & 12S-1 & & \\ & \ddots & \ddots & \ddots & \\ & & 12S-1 & -(24S+10) & 12S-1 \\ & & & 12S-1 & -(24S+10) \end{pmatrix}_{(M-1)\times(M-1)},$$

$$,\underline{B} = -\alpha \begin{pmatrix} 10 & 1 & & & \\ 1 & 10 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 10 & 1 \\ & & & 1 & 10 \end{pmatrix}_{(M-1)\times(M-1)} ,\underline{C} = \omega_k^{(\alpha)} \begin{pmatrix} 10 & 1 & & & \\ 1 & 10 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 10 & 1 \\ & & & 1 & 10 \end{pmatrix}_{(M-1)\times(M-1)}$$

$, k = 2,3,...,n, \quad 0 < \alpha < 1$

and the vectors $\underline{F}^n$ for $2 \le n \le N$ are in the form

$$\underline{F}^{n} = \begin{pmatrix} \tau^{-\alpha}(\alpha u_0^{n-1} + (12S-1)u_0^n - \sum_{k=2}^{n} \omega_k^{(\alpha)} u_0^{n-k}) + f_0^n + 10 f_1^n + f_2^n \\ f_1^n + 10 f_2^n + f_3^n \\ \vdots \\ f_{M-3}^n + 10 f_{M-2}^n + f_{M-1}^n \\ \tau^{-\alpha}(\alpha u_M^{n-1} + (12S-1)u_M^n - \sum_{k=2}^{n} \omega_k^{(\alpha)} u_M^{n-k}) + f_{M-2}^n + 10 f_{M-1}^n + f_M^n \end{pmatrix}$$

4 Compact Finite Difference Method with Right-Shifted Grünwald-Letnikov Formula

To estimate the left-handed time fractional derivative in Eq. (1), we use the right-shifted Grünwald-Letnikovformula which is defined as(see [9]).

$$\frac{\partial^{\alpha} u(x_i, t_n)}{\partial t^{\alpha}} = \frac{1}{\tau^{\alpha}} \sum_{k=0}^{n+1} \omega_k^{(\alpha)} u_i^{n-k+1} + O(\tau) \tag{19}$$

where p = 1.

This formula is shifted by one grid point to the right to get the fractional forward difference formula.

We use finite difference approximation in Eq. (15) to discretize the second order spatial derivative in Eq. (1) to obtain

$$\begin{cases} \dfrac{1}{\tau^{\alpha}} \displaystyle\sum_{k=0}^{n+1} \omega_k^{(\alpha)} u_i^{n-k+1} = \dfrac{\dfrac{\partial_+}{\partial x}\left(\dfrac{\partial_- u}{\partial x}\right)_i^n}{\left(1 + \dfrac{h^2}{12}\dfrac{\partial_+}{\partial x}\dfrac{\partial_-}{\partial x}\right)} + f_i^n, \\ i = 1, 2, ..., M-1, \quad n = 1, 2, ..., N, \\ u_i^0 = 0, \quad i = 1, 2, ..., M-1, \\ u_0^n = f_1(t_n), \quad u_M^n = f_2(t_n), \quad n = 1, 2, ..., N. \end{cases} \tag{20}$$

Hence the compact finite difference method with right-shifted Grünwald-Letnikovformulafor Eq. (1), (2) and (3) is given as:

$$\begin{cases} \displaystyle\sum_{k=0}^{n+1} \omega_k^{(\alpha)} (u_{i-1}^{n-k+1} + 10u_i^{n-k+1} + u_{i+1}^{n-k+1}) = 12S(u_{i-1}^n - 2u_i^n + u_{i+1}^n) + \tau^{\alpha}(f_{i-1}^n + 10 f_i^n + f_{i+1}^n) \\ i = 1, 2, ..., M-1, \quad n = 1, 2, ..., N, \\ u_i^0 = 0, \quad i = 1, 2, ..., M-1, \\ u_0^n = f_1(t_n), \quad u_M^n = f_2(t_n), \quad n = 1, 2, ..., N. \end{cases} \tag{21}$$

The linear system of Eq. (4.3) is

$$\underline{A}\underline{U}^{n+1} = \underline{B}\underline{U}^n - \sum_{k=2}^{n+1} \underline{C}\underline{U}^{n-k+1} + \tau^{\alpha} \underline{F}^n, \quad 1 \le i \le M-1, \ 1 \le n \le N \tag{22}$$

where $\underline{U}^n = [u_1^n, u_2^n, ...,u_{M-1}^n]^T$ and the vector $\underline{F}^n$ isdefined as

$$\underline{F}^n = [(a + f_0^n + 10 f_1^n + f_2^n), (f_1^n + 10 f_2^n + f_3^n), ..., (f_{M-3}^n + 10 f_{M-2}^n + f_{M-1}^n), (b + f_{M-2}^n + 10 f_{M-1}^n + f_M^n)]^T$$

where $a = \tau^{-\alpha}((\alpha + 12S)u_0^n - u_0^{n+1} - \sum_{k=2}^{n+1} \omega_k^{(\alpha)} u_0^{n-k+1})$ and

$b = \tau^{-\alpha}((\alpha + 12S)u_M^n - u_M^{n+1} - \sum_{k=2}^{n+1} \omega_k^{(\alpha)} u_M^{n-k+1})$ for $1 \le n \le N$.

The entries $a_{i,j}, b_{i,j}, c_{i,j}$ and $d_{i,j}$ for $i = 1, 2, ...M - 1$ and $j = 1, 2, ...M - 1$ can be defined as

$$a_{i,j} = \begin{cases} 1 \quad \text{when } \ i = j-1 \text{ and } i = j+1 \\ 10 \quad \text{when } \ i = j \\ 0 \quad \text{otherwise} \end{cases} , \quad b_{i,j} = \begin{cases} \alpha + 12S \quad \text{when } \ i = j-1 \text{ and } i = j+1 \\ 10\alpha - 24S \quad \text{when } \ i = j \\ 0 \quad \text{otherwise} \end{cases}$$

$$c_{i,j} = \begin{cases} \omega_k^{(\alpha)} \quad \text{when } \ i = j-1 \text{ and } i = j+1 \\ 10\omega_k^{(\alpha)} \quad \text{when } \ i = j \\ 0 \quad \text{otherwise} \end{cases} , k = 2, 3, ...n+1, \quad 0 < \alpha < 1$$

5 The Truncation Error

Let us consider the truncation error of Eq. (16). We have

$$T(x_i, t_n) = \frac{1}{\tau^\alpha} \sum_{k=0}^{n} \omega_k^{(\alpha)} u_i^{n-k} - \frac{\frac{\partial_+}{\partial x}\left(\frac{\partial_- u}{\partial x}\right)_i^n}{\left(1 + \frac{h^2}{12}\frac{\partial_+}{\partial x}\frac{\partial_-}{\partial x}\right)} - f_i^n \tag{23}$$

$$= \frac{1}{\tau^\alpha} \sum_{k=0}^{n} \omega_k^{(\alpha)} u_i^{n-k} - \left(\frac{\partial^\alpha u}{\partial t^\alpha}\right)_i^n + \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n - \frac{\frac{\partial_+}{\partial x}\left(\frac{\partial_- u}{\partial x}\right)_i^n}{\left(1 + \frac{h^2}{12}\frac{\partial_+}{\partial x}\frac{\partial_-}{\partial x}\right)}$$

$$= \frac{1}{\tau^\alpha} \sum_{k=0}^{n} \omega_k^{(\alpha)} u_i^{n-k} - \left(\frac{\partial^\alpha u}{\partial t^\alpha}\right)_i^n + \left(-\frac{h^4}{240}\left(\frac{\partial^6 u}{\partial x^6}\right)_i^n + ...\right)$$

$= O(\tau^p) + O(h^4)$

Since we use the generating function $(1-z)^\alpha$ then the Grünwald-Letnikovformula has an accuracy of order $p = 1$.

Hence

$$T(x_i, t_n) = O(\tau) + O(h^4) \tag{24}$$

The same result will be obtained if the truncation error of Eq. (20) is investigated. Therefore both of the propsed fractional compact finite difference methodsfor the time fractional diffusion equation have accuracy of order $O(\tau + h^4)$

6 The Stability Analysis of the Two Compact Finite Difference Methods

In this section the von Neumann (or Fourier) method will be used to analyze the stability of CFD methodwith the Grünwald-Letnikovformula and CFDmethod with the right-shifted Grünwald-Letnikovformula.

We start with Eq. (17) and for simplicity let us write this formula with no source. Cui [8] introduced the following lemma about properties of $\omega_k^{(\alpha)}$

Lemma: *The coefficients* $\omega_k^{(\alpha)}$ $(k = 0,1,..)$ *satisfy (see [8])*

(1) $\omega_0^{(\alpha)} = 1;\ \omega_1^{(\alpha)} = -\alpha;\ \omega_k^{(\alpha)} < 0,\ k = 1,2,...$

(2) $\sum_{k=0}^{\infty} \omega_k^{(\alpha)} = 0;\ \forall m \in N^+, -\sum_{k=1}^{m} \omega_k^{(\alpha)} < 1$

Proposition 6.1: *The compact finite difference scheme with the Grünwald-Letnikov formula defined in Eq. (17) is unconditionally stable.*

Proof: Suppose U_i^n is the approximate solution of Eq. (17) and so the error we obtain from the difference between the theoretical and numerical solutions can be addressed as

$$\varepsilon_i^n = u_i^n - U_i^n, \quad i = 0,1,...,M, \quad n = 0,1,...,N.$$

This error can be presented by the same compact finite differencemethod with the Grünwald-Letnikovformula.

We obtain

$$\sum_{k=0}^{n}\omega_k^{(\alpha)}(\varepsilon_{i-1}^{n-k}+10\varepsilon_i^{n-k}+\varepsilon_{i+1}^{n-k})=12S(\varepsilon_{i-1}^{n}-2\varepsilon_i^{n}+\varepsilon_{i+1}^{n}),\ i=1,2,\ldots,M-1,$$

$$n=1,2,\ldots,N \tag{25}$$

$$\varepsilon_0^n=\varepsilon_m^n=0,\ \ n=1,2,\ldots,N \tag{26}$$

According to the Fourier method, the errors $\varepsilon^n(x_i)$ at the grid points in the range $x\in[0,1]$ and a given time level can be expressed in terms of a finite Fourier series with the complex exponential form(see [4])

$$\varepsilon^n\left(x_i\right)=\sum_{m=0}^{M}A_m e^{\sqrt{-1}q_m x_i}\ ,\ i=0,1,\ldots,M \tag{27}$$

where $q_m=m\pi/l\ ,\ l=Mh$

To study the propagation of errors as *t* increases, let us omit the summation and the constant A_m, taking only a single term $e^{\sqrt{-1}q_m x_i}$.

Suppose the solution of equations (25) and (26) in the form

$$\varepsilon_i^n=e^{\beta n\tau}e^{\sqrt{-1}qih} \tag{28}$$

where $e^{\beta n\tau}$ is termed the temporal or amplification factor and β is complex temporal number which depends upon *q*

We note that at *n* = 0, the solution of error reduces to $e^{\sqrt{-1}qih}$. For the von Neumann method, a scheme is said to be stable if and only if $\left|e^{\beta\tau}\right|\le 1$ for all γ . Substituting Eq.(28) into Eq.(25) we obtain

$$\sum_{k=0}^{n}\omega_k^{\alpha}e^{-k\beta\tau}=\frac{-12S\sin^2\left(\frac{qh}{2}\right)}{3-\sin^2\left(\frac{qh}{2}\right)} \tag{29}$$

To study the stability of Eq.(29) let us consider the iteration when *n* → ∞; so we obtain

$$\lim_{n\to\infty}\sum_{k=0}^{n}\omega_k^{\alpha}e^{-k\beta\tau}=\frac{-12S\sin^2\left(\frac{qh}{2}\right)}{3-\sin^2\left(\frac{qh}{2}\right)} \tag{30}$$

Hence $$\sum_{k=0}^{\infty}\omega_k^{\alpha}e^{-k\beta\tau}=\frac{-12S\sin^2\left(\frac{qh}{2}\right)}{3-\sin^2\left(\frac{qh}{2}\right)} \tag{31}$$

Then from Eq.(8) we can obtain

$$\left(1-e^{-\beta\tau}\right)^{\alpha}=\frac{-12S\sin^2\left(\frac{qh}{2}\right)}{3-\sin^2\left(\frac{qh}{2}\right)} \tag{32}$$

Since the maximum value of $\sin(qh)=1$, then

$$e^{\beta\tau}=\frac{1}{1+(6S)^{\frac{1}{\alpha}}} \tag{33}$$

Therefore according to the von Neuman method the condition for stability $\left|e^{\beta\tau}\right|\le 1$ is satisfied by $\left|\frac{1}{1+(6S)^{\frac{1}{\alpha}}}\right|\le 1$ for all values of $S>0$ and $0<\alpha<1$. Hence the compact finite difference method with the Grünwald-Letnikovformula (17) is unconditionally stable for $S=\frac{\tau^{\alpha}}{h^2}>0$ and $0<\alpha<1$. Now we study the stability of the compact finite difference method with the right-shifted Grünwald-Letnikovformula.

Proposition 6.2: *The compact finite difference scheme with the right-shiftedGrünwald-Letnikovformula defined in Eq.(21) is stable if* $S\le\frac{2^{\alpha-1}}{3}$ *for* $0<\alpha<1$

Proof: The error ε_i^n satisfies the equations (21). This gives

$$\sum_{k=0}^{n+1}\omega_k^{(\alpha)}\left(\varepsilon_{i-1}^{n-k+1}+10\varepsilon_i^{n-k+1}+\varepsilon_{i+1}^{n-k+1}\right)=12S\left(\varepsilon_{i-1}^{n}-2\varepsilon_i^{n}+\varepsilon_{i+1}^{n}\right),\ n=0,1,\ldots,N \tag{34}$$

$$\varepsilon_0^n=\varepsilon_M^n=0,\ n=0,1,\ldots,N \tag{35}$$

We assume that the solution of the equation (34) takes the form $\varepsilon_i^n=e^{\beta n\tau}e^{\sqrt{-1}qih}$.

Inserting the above solution into Eq.(34), we obtain

$$e^{\beta\tau} = \frac{-4S\sin^2\left(\frac{qh}{2}\right)}{\left(1-\frac{1}{3}\sin^2\left(\frac{qh}{2}\right)\right)\sum_{k=0}^{n+1}\omega_k^{(\alpha)}e^{-k\beta\tau}} \tag{36}$$

If $e^{\beta\tau} > 1$ for all β then the errors will propagate exponentially and the equation of the error will be unstable

Suppose that $\left|e^{\beta\tau}\right| \leq 1$ then $-1 \leq e^{\beta\tau} \leq 1$

when $e^{\beta\tau} \leq 1$ then from Eq.(36) we have

$$\frac{-4S\sin^2\left(\frac{qh}{2}\right)}{\left(1-\frac{1}{3}\sin^2\left(\frac{qh}{2}\right)\right)\sum_{k=0}^{n+1}\omega_k^{(\alpha)}e^{\beta\tau}} \leq 1 \tag{37}$$

taking the maximume value of $e^{\beta\tau} = 1$ we get

$$\frac{-4S\sin^2\left(\frac{qh}{2}\right)}{\left(1-\frac{1}{3}\sin^2\left(\frac{qh}{2}\right)\right)\sum_{k=0}^{n+1}\omega_k^{(\alpha)}} \leq 1$$

From the Lemma we have $\sum_{k=0}^{m}\omega_k^{(\alpha)} > -1 \ \forall\, m \in N^{+}$ then

$S\sin^2\left(\frac{qh}{2}\right) > 0$ is always true if S > 0

when $e^{\beta\tau} \geq -1$ we have

$$\frac{-4S\sin^2\left(\frac{qh}{2}\right)}{\left(1-\frac{1}{3}\sin^2\left(\frac{qh}{2}\right)\right)\sum_{k-0}^{n+1}\omega_k^{(\alpha)}e^{-k\beta\tau}} \geq -1$$

Taking the limits of the summation when $n\to\infty$ we obtain

$$\frac{4S\sin^2\left(\frac{qh}{2}\right)}{\left(1-\frac{1}{3}\sin^2\left(\frac{qh}{2}\right)\right)\sum_{k-0}^{\infty}\omega_k^{(\alpha)}e^{-k\beta\tau}} \geq 1$$

then from equation (8) we have

$$4S\sin^2\left(\tfrac{qh}{2}\right) \le \left(1-\frac{1}{3}\sin^2\left(\tfrac{qh}{2}\right)\right)\left(1-e^{-\beta\tau}\right)^{\alpha}$$

Consider the extreme value $e^{\beta\tau} = -1$ then

$$4S\sin^2\left(\tfrac{qh}{2}\right) \le 2^{\alpha}\left(1-\frac{1}{3}\sin^2\left(\tfrac{qh}{2}\right)\right)$$

Since the maximum value of $\sin\left(\tfrac{qh}{2}\right)=1$, then we arrive at the condition that

$$S \le \frac{2^{\alpha-1}}{3} .$$

7 Numerical Experiments

Let us consider the equation from Takaci et al. [3]

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{2e^{x}t^{2-\alpha}}{\Gamma(3-\alpha)} - t^2e^{x} \text{ for } \alpha = 0.5 \tag{38}$$

With the initial condition

$$u(x,0) = 0, 0 < x < 1 \tag{39}$$

And the boundary conditions

$$u(0,t) = t^2 , \qquad u(1,t) = et^2, \quad 0 \le t \le T \tag{40}$$

The exact solution of Eq. (38) is

$$u(x,t) = t^2e^{x} \tag{41}$$

We implement the two numerical methods discussed in this paper for solving the above example and compare their solutions with the exact solution.Figure 1 shows the comparison of $u(x,t)$ values between the solutions of the two methodsand the exact solutionfor a certain set of parameter values which satisfy the stability condition of the CFD scheme with the right-shifted Grünwald-Letnikovformula.

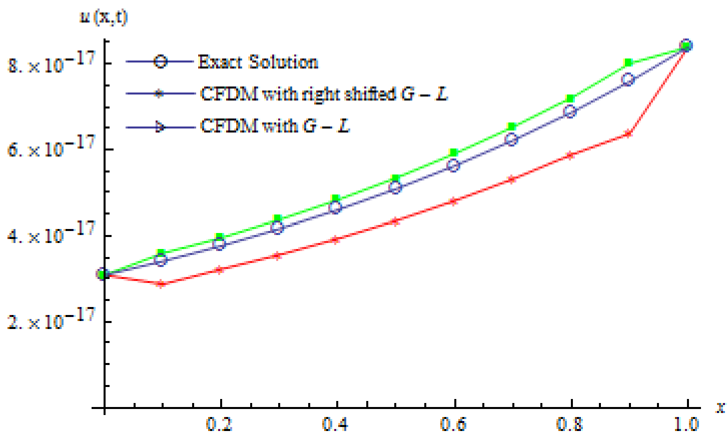

Fig.1: Comparison between the results of the CFD method with G-L formula and CFD method with right-shifted G-L formulaand the exact solution at $\tau = 5.55556 \times 10^{-10}$ and $h = 0.1$

To test the order of convergence of our schemes firstwe estimate the error in l^∞ norm where $\|e\|_{l^\infty} = \max_{1\le i\le M-1} |u_i^N - U_i^N|$ at $T = 1, h = \tau = 0.2$ and then decrease the mesh size of h to half and τ to 0.125τ. The results of the maximum errors and the order of convergence of the compactDu Fort Frankel method are listed in table 1.

The order of convergence $r(\tau, h)$ is evaluated by the formula

$$r(\tau,h) = \log_2\left(\|e(8\tau,2h)\|_{l^\infty} / \|e(\tau,h)\|_{l^\infty}\right)$$

Table 2 represents the maximum errors and the order of convergence obtained by the CFD method with the right-shifted Grünwald-Letnikovformula, here the step sizesτ, h are shosen to satisfy the stability condition of this method.

Assuming that $\|e\|_{l^\infty} = C_1\tau + C_2h^4$ where C_1 and C_2 are constant, the maximum error tends to C_2h^4 as time step size becomes small enough and the order of the convergence tends to 4. Analogously, if τ is large then $\|e\|_{l^\infty} \approx C_1\tau$ and the order of the convergence tends to 1 (Hu and Zhang, 2012).

Table 1: Error inl^∞ norm and order of convergence of CF with G-L method at $\alpha = 0.5$

	$\|e\|_{l^\infty}$	order
$h = \tau = 0.2$	$2.1081E-2$	-
$h = 0.1, \tau = 0.025$	$1.8383E-3$	3.5195
$h = 0.05, \tau = 3.125E-3$	$1.0299E-4$	4.1578
$h = \tau = 0.1$	$1.0713E-2$	-

$h = 0.05, \tau = 0.0125$	$8.1616E - 4$	3.7143
$h = 0.025, \tau = 1.5625E - 3$	$3.7556E - 5$	4.4418

Table 2: Error inl^{∞} norm and order of convergence of CF with shifted G-Lmethod at $\alpha = 0.5$

$\|e\|_{l^{\infty}}$order		
$h = 0.2, \tau = 1.3889E - 3$	$3.3687E - 1$	-
$h = 0.1, \tau = 1.7361E - 11$	$1.1923E - 2$	4.8204
$h = 0.05, \tau = 2.17014E - 12$	$7.7921E - 4$	3.9356
$h = 0.1, \tau = 5.4254E - 13$	$1.1656E - 5$	-
$h = 0.05, \tau = 6.7817E - 14$	$7.6203E - 7$	3.9351
$h = 0.025, \tau = 8.4771E - 15$	$2.4355E - 8$	4.9675

These results indicate the convergence of the proposedmethods.

8 Conclusion

In this paper, two high order compact finite difference schemes have been developed. The stability of the proposod methods was investigated and it was shown that the compact finite difference scheme with the Grünwald-Letnikovformulais unconditionally stable while the compact finite difference scheme with the right-shifted Grünwald-Letnikovformula is conditionally stable. Both of these methods have an accuracy of order $O(h^4)$ in space and oforder$O(\tau)$in the temporal step. It should be noted that the proposed algorithms produced reasonable results.

References

[1] C. Chen, F. Liu and K.Burrage, Finite difference methods and a fourier analysis for the fractional reaction-subdiffusion equation, *Appl. Math. Comput.*, 198(2008), 754-769.

[2] D. Baleanu, J.A. Machado and A.C.J. Luo, *Fraction Dynamics and Control*, Springer, New York Dordrecht Heidelberg, London, (2012).

[3] D. Takaci, A. Takaci and M. Strboja, On the character of operational solutions of the time-fractional diffusion equation, *Nonlinear Anal.*, 72(2009), 2367-2374.

[4] G.D. Smith, *Numerical Solution of Partial Differential Equations (2nd Edn.)*, Clarendon Press, Oxford, (1978).

[5] G. Gao and Z. Sun, A compact finite difference scheme for the fractional sub-diffusion equations, *J. Comput. Phys.*, 230(2011), 586-595.

[6] I. Podlubny, *Fractional Differential Equations*, Academic Press, California, USA, (1999).

[7] K. Khaled and S. Momani, An approximate solution for a fractional diffusion-wave equation using the decomposition method, *Appl. Math. Comput.*, 165(2005), 473-483.

[8] M. Cui, Compact finite difference method for the fractional diffusion equation, *J. Comput. Phys.*, 228(2009), 7792-7804.

[9] M.M. Meerschaert and C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations, *Appl. Numer. Math.*, 56(2006), 80-90.

[10] N.H. Sweilam, M.M. Khader and A.M. Nagy, Numerical solution of two-sided space-fractional wave equation using finite difference method, *J. Comput. Appl. Math.*, 235(2011), 2832-2841.

[11] R. Du, W.R. Cao and Z.Z. Sun, A compact difference scheme for the fractional diffusion-wave equation, *Appl. Math. Modelling.*, 34(2010), 2998-3007.

[12] R. Klages, G. Radons and I.M. Sokolov, *Anomalous Transport*, WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim, Germany, (2008).

[13] S.B. Yuste and L. Acedo, An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations, *SIAM J. Numer. Anal.*, 42(2005), 1862-1874.

[14] S. Das, *Functional Fractional Calculus for System Identification and Controls (2th Edn.),* Verlag Berlin Heidelberg-Springer, New York, (2011).

[15] T. Petrila and D. Trif, *Basics of Fluid Mechanics and Introduction to Computational Fluid Dynamics*, Springer, USA, (2005).