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Proof of Relative Class Number One for Almost All Real Quadratic Fields and a Counterexample for the Rest

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Abstract

Let $\varepsilon_D = v + u\sqrt{D}$ be the fundamental unit of $\mathbb{Z}[\sqrt{D}]$ with $\mathbb{Z}$ being the ordinary integers, or maximal order, in the rational field $\mathbb{Q}$. We prove that for any square-free integer $D > 1$, with D not dividing u, there exists a prime f_D such that the relative class number $H_D(f_D) = h_{f_D^2 D}/h_D = 1$, *where h_D is the ideal class number of $\mathbb{Z}[\sqrt{D}]$ and $h_{f_D^2 D}$ is the ideal class number of $\mathbb{Z}[f_D\sqrt{D}]$, the order of index f_D in the maximal order $\mathbb{Z}[\sqrt{D}]$ of $\mathbb{Q}(\sqrt{D})$. For the remaining case we provide a counterexample to class number one. This completely settles an open question left by Dirichet for any real quadratic field. This vastly generalizes recent results in the literature and does so with chiefly results by Thomas Muir from 1874 that have long gone unrecognized.*

Keywords: *Continued fractions, palindromes, Pell equations, quadratic orders, relative class numbers.*

1. Introduction

In 1856, Dirichlet showed that for certain D there exist an infinite number of f_D such that $h(f_D^2 D) = h_D$, but it remained open as to whether there exist such an f_D for *each* D.[1] It is well-known that if D is not a perfect square then the continued

[1] In Cohn's book [3, Sec. 2, p. 219], he says that "Dirichlet showed in 1856 that for certain D there exists an infinite number of f_D for which $h(f_D^2 D) = h(D)$. It is not known if such an f_D

fraction expansion is given by

$$\sqrt{D} = \left\langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0} \right\rangle, \tag{1.1}$$

where $q_0 = \lfloor \sqrt{D} \rfloor$ and $q_1 q_2 \ldots q_{\ell-1}$ is a palindrome, [2] where $\ell = \ell(\sqrt{D})$ is the period length of the simple continued fraction expansion of $\sqrt{D}$. Recently, in [8], the authors showed that for $q_j = a \in \mathbb{N}$ for all $j = 1, 2, \ldots, \ell - 1$, there is a square-free D and a prime f_D for which $H_D(f_D) = 1$. This therefore covered all period lengths $\ell = 1, 2, 3$. However,the results in [8] are a one-line consequence of results by Muir—see Theorem 2.1 below. We show herein that, indeed, Theorem 2.1 can be used to include this result, namely the relative class number is one. This means that for a prime f_D then $H_D(f_D) = h_{f_D^2 D}/h_D = 1$, called the *relative class number*, for all square-free $D > 1$ with D not dividing B where $\varepsilon_D = A + B\sqrt{D}$ is the fundamental unit of $Q(\sqrt{D})$.[3]

2. Preliminaries

Some basic facts on continued fractions which we will need are given as follows. This may be found in most introductory number theory texts such as [14]. The jth convergent for $\sqrt{D}$ for any non-negative integer j is given by $\frac{A_j}{B_j} = \langle q_0; q_1, q_2, \ldots, q_j \rangle$, where

$$A_j = q_j A_{j-1} + A_{j-2}, B_j = q_j B_{j-1} + B_{j-2},$$

with $A_{-2} = 0, A_{-1} = 1, B_{-2} = 1$, and $B_{-1} = 0$. Also, and for any $j \in \mathbb{N}$,

$$A_j B_{j-1} - A_{j-1} B_j = (-1)^{j-1}. \tag{2.2}$$

$$A_{\ell-1} = q_0 B_{\ell-1} + B_{\ell-2}, \tag{2.3}$$

and[4]

$$A_{\ell-1}^2 - B_{\ell-1}^2 D = (-1)^{\ell}. \tag{2.4}$$

exists for each D."

[2]Indeed, Lagrange proved in 1770 that if $a, b \in \mathbb{Z}$ with $0 < b < a$ and a/b not a perfect square, then there exists an $\ell \in \mathbb{N}$ such that $\sqrt{a/b} = \langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0} \rangle$—see [14, Exercise 5.16, p. 231].

[3]It is of great interest in the number theory community as to when, for a given prime $p \equiv 1 \pmod 4$, the fundamental unit $\varepsilon_p = (A + B\sqrt{p})/2$ has $p \mid B$. Mordell [18] showed that for $p \equiv 5 \pmod 8$ this holds if and only if p divides the numerator of the $(p-1)/4$-th Bernoulli number. Later Ankeny and Chowla [1] strengthened this to get the same conclusion for all $p \equiv 1 \pmod 4$. Given there results of this paper it would be interesting to see if, in the case $D \equiv 5 \pmod 8$, one could rule out p dividing B when $\varepsilon_D = A + B\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$.

[4]It is a fact that the fundamental unit of $\mathbb{Z}(\sqrt{D})$ for a non-square positive integer D is given by $\varepsilon_D^i = A_{\ell-1} + B_{\ell-1}\sqrt{D}$ and $N(\varepsilon_D) = (-1)^{\ell}$ where $\ell = \ell(\sqrt{D})$, and $i = 1, 3$ where $i = 3$ is only possible when $D \equiv 5 \pmod 8$ —see [12, Theorems 2.1.3–2.1.4, pp. 51–53] .

Now we may state the aforementioned result which Perron attributes to Muir in 1874—see [20, 3, Satz 17], which is in [19].[5] We state it here in a format suitable for our purposes.

Theorem 2.1 *For a natural number $\ell \geq 2$, let $q_1, \ldots, q_{\ell-1}$ be a palindrome. If $q_0 \in \mathsf{N}$, then the following are equivalent.*

1. *For a non-square $D \in \mathsf{N}$,*

$$\sqrt{D} = \left\langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0} \right\rangle . \tag{2.5}$$

2. *There exist integers u, v, w such that the matrix equation*

$$\prod_{j=1}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B & v \\ v & w \end{pmatrix}, \tag{2.6}$$

holds. Also, vw is even[6] and for some $x \in \mathsf{Z}$,

$$q_0 = (Bx - (-1)^\ell vw)/2 \in \mathsf{Z} \tag{2.7}$$

with $B = B_{\ell-1}, v = B_{\ell-2}$, and $w = A_{\ell-2} - q_0 B_{\ell-2}$ given A_j/B_j being the j^{th} convergent of $\sqrt{D}$, described in the previous section.[7] Moreover, (2.6) *is satisfied,*

$$D = q_0^2 + xv - (-1)^\ell w^2 = \left(\frac{Bx}{2}\right)^2 + \left(v - \frac{(-1)^\ell}{2} Bvw\right) x + \left(\frac{vw}{2}\right)^2 - (-1)^\ell w^2. \tag{2.8}$$

Proof: See [20] and also [10] for a more accessible and recent interpretation. □

It is also quite worth observing another matrix sequence of values. We present this here with proof since that proof has elements that we can isolate as an important consequence.

Theorem 2.2 (Fundamental Unit Theorem for Quadratic Orders)
Suppose that (2.6) *holds. Then*

$$\prod_{j=0}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} DB_{\ell-1} & A_{\ell-1} \\ A_{\ell-1} & B_{\ell-1} \end{pmatrix}, \tag{2.9}$$

[5] The case $(1+\sqrt{D})/2$ is also covered by Muir, but we will not need it here. We refer the reader to [13] for a complete description and extended illustrations of its modern-day usage.

[6] Observe that if vw is odd, then there is no D, square-free or not, satisfying (2.5). For instance, the palindrome $2, 3, 2$ has $B = 16$, $v = 7$ and $w = 3$ in (2.7) for which we see there is no value of $x \in \mathsf{Z}$ with $q_0 \in \mathsf{Z}$.

[7] Observe that $Bx - (-1)^\ell vw > 0$ and this holds for $x \geq \lfloor vw/B \rfloor + 1$ when ℓ is even and $x \geq -\lfloor vw/B \rfloor$ when ℓ is odd.

where

$$A_{\ell-1}^2 - B_{\ell-1}^2 D = (-1)^\ell,$$

and $\varepsilon_D = A_{\ell-1} + B_{\ell-1}\sqrt{D}$ *is the fundamental unit of the order* $\mathsf{Z}[\sqrt{D}]$.

Proof: Using (2.6), we get:

$$\prod_{j=0}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_0^2 B + 2q_0 v + w & q_0 B + v \\ q_0 B + v & B \end{pmatrix}, \tag{2.10}$$

where $B = B_{\ell-1}$, and by (2.3), $A_{\ell-1} = q_0 B + v = q_0 B_{\ell-1} + B_{\ell-2}$. We now show that upper left entries in the matrices (2.9)–(2.10) agree. By looking at D as given in (2.8), we see that we must show

$$xvB - (-1)^\ell w^2 B = 2q_0 v + w.$$

However, from (2.7), we deduce that we only need to verify that

$$v^2 - Bw = (-1)^\ell. \tag{2.11}$$

We have

$$v^2 - Bw = B_{\ell-2}^2 - B_{\ell-1}(A_{\ell-2} - q_0 B_{\ell-2}) = B_{\ell-2}^2 - B_{\ell-1}A_{\ell-2} + q_0 B_{\ell-1} B_{\ell-2} =$$
$$B_{\ell-2}A_{\ell-1} - B_{\ell-2}(B_{\ell-2} + q_0 B_{\ell-1}) = B_{\ell-2}A_{\ell-1} - B_{\ell-2}A_{\ell-1} = (-1)^\ell$$

where the penultimate equality follows from (2.3) and the last equality follows from (2.2). That $A_{\ell-1} + B_{\ell-1}\sqrt{D}$ is indeed the fundamental unit is discussed in Footnote 4. □

Corollary 2.1 *Given*

$$\sqrt{D} = \left\langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0} \right\rangle,$$

$$Bw = B_{\ell-1}(A_{\ell-2} - q_0 B_{\ell-2}) = B_{\ell-2}^2 - (-1)^\ell = v^2 - (-1)^\ell.$$

Proof: This is (2.11) in the proof above. □

In [7] Friesen proved the following which is related to the above.

Theorem 2.3 (Friesen [7]) *Let* $D \in \mathsf{N}$, $q_0 = \lfloor \sqrt{D} \rfloor$, *and* $q_1, \ldots, q_{\ell-1}$ *any palindrome. Then the equation*

$$\sqrt{D} = \left\langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0} \right\rangle,$$

has infinitely many square-free integers D as solutions whenever[8]

$$\textit{either } (B_{\ell-2}^2 - (-1)^\ell)/B_{\ell-1} \textit{ or } B_{\ell-2} \textit{ is even.} \tag{2.12}$$

[8]Note that from Theorem 2.1, $v = B_{\ell-2}$, $w = (B_{\ell-2}^2 - (-1)^\ell)/B_{\ell-1}$, and since $v^2 - (-1)^\ell = vw$ from (2.11), then wv even, so we have the condition given in Theorem 2.1.

Lastly, for this section, we require the following elucidation on relative class numbers.

When $D > 1$ is squarefree, then $\mathcal{O}_D = \mathsf{Z}[\sqrt{D}]$ when $D \not\equiv 1 \pmod 4$, respectively $\mathcal{O}_D = Z[(1+\sqrt{D})/2]$ when $D \equiv 1 \pmod 4$, is called the *ring of integers* or *maximal order* of the real quadratic field $\mathsf{Q}(\sqrt{D})$, with radicand D and *discriminant* $4D$, respectively D. For $f_D \in \mathsf{N}$, $\mathcal{O}_{f_D^2 D} = \mathsf{Z}[f_D\sqrt{D}]$, respectively $\mathcal{O}_{f_D^2 D} = \mathsf{Z}[f_D(1+\sqrt{D})/2]$ is called an *order* in $\mathcal{O}_D$ of index f_D since $|\mathcal{O}_D : \mathcal{O}_{f_D^2 D}| = f_D$. In this case, the index f_D is called the *conductor* of $\mathcal{O}_{f_D^2 D}$ in the maximal order.[9] Now we look at the ideal class number relation between the two orders, namely h_D of $\mathcal{O}_D$ and $h_{f_D^2 D}$ of $\mathcal{O}_{f_D^2 D}$. If v is a unit in an order $\mathcal{O}_{f_D^2 D}$ and $\varepsilon_{f_D^2 D}$ is the *fundamental unit* of $\mathcal{O}_{f_D^2 D}$, this means that $v = \pm\varepsilon_D^m$ for some $m \in \mathsf{Z}$. Therefore, $\varepsilon_{f_D^2 D}$ is a unit in the maximal order $\mathcal{O}_D$ whose fundamental unit is ε_D. Hence, the *unit index* of $\mathcal{O}_{f_D^2 D}$ in $\mathcal{O}_D$ is that integer $u_{f_D} \in \mathsf{N}$ such that

$$\varepsilon_{f_D^2 D} = \varepsilon_D^{u_{f_D}}. \tag{2.13}$$

Then the following relation holds between the two class numbers:[10]

$$h_{f_D^2 D} = h_D \psi_D(f_D)/u_{f_D}, \tag{2.14}$$

where

$$\psi_D(f_D) = f_D \prod (1 - (D/p)/p), \tag{2.15}$$

with the product ranging over all the distinct primes p dividing f_D and $(*/*)$ is the Kronecker symbol. This shows that $h_D \mid h_{f^2 D}$. We will be looking at when they are equal, so it suffices to look at

$$H_D(f_D) = \frac{h_{f_D^2 D}}{h_D} = \frac{\psi_D(f_D)}{u_{f_D}}, \tag{2.16}$$

and it is this latter relation we shall examine throughout.

The following reduces the problem in this paper to one case and shows that a positive density of square-free D for which the relative class number is 1 exists.

[9] For background details and an overview of arbitrary quadratic orders, see [12, §1.5, pp. 23–30].

[10] There are various formulations of the class number of an arbitrary order $\mathcal{O}_{f_D^2 D}$ such as Borevich-Shafarevich [2, Exercise 11, p. 152-153], Cox [5, Corollary 7.28, and Exercise 7.30, pp. 146-158] (for complex quadratic orders), and Cohn [3, Theorem 2, p. 217]. It is the latter that we prefer for our purposes here. Note that h_D for square-free D will always refer to the ideal class number of the maximal order. Furthermore, when Gauss and Dirichlet spoke in the language of quadratic forms, later reformulated into the language of quadratic fields via Dirichlet's introduction of ideals, the distinction may be viewed as follows. If h'_D is the cardinality of the form class group, then $h'_D = h_D$ unless $D > 0$ and $N(\varepsilon_D) = 1$, in which case $h'_D = 2h_D$. Also, h'_D may be shown to be the same as the so-called *narrow ideal class number* of $\mathcal{O}_D$, while h_D is called the *wide ideal class number* of $\mathcal{O}_D$. All of this and intimate connections may be found in [12, Appendix E, pp. 347-354] or in [15, §3.2, pp. 105–117].

Theorem 2.4 *If D is square-free and either*

1. $D \equiv 1 \pmod 8$ *or,*
2. $D \equiv 5 \pmod 8$, *and the fundamental unit has form* $\varepsilon_D = (a + b\sqrt{D})/2$ *with ab odd,*

then $H_D(2) = 1$.

Proof: If $D \equiv 1 \pmod 8$, then $\varepsilon_D \in \mathsf{Z}[\sqrt{D}]$ so $u_{f_D} = 1$ if $f_D = 2$ in (2.16). Also, since $(D/2) = 1$, then $\psi_D(2) = 1$ so $H_D(2) = 1$. Thus, the problem left open by Dirichlet is solved for square-free $D \equiv 1 \pmod 8$.[11] An interesting point about $D \equiv 5 \pmod 8$ and the interplay between the maximal order, or ring of integers, $\mathcal{O}_D = \mathsf{Z}[(1 + \sqrt{D})/2]$ and the order $\mathsf{Z}[\sqrt{D}]$ of index $f_D = 2$ in $\mathcal{O}_D$ is that $H_D(2) = 1$ if and only if $[4, 1 + \sqrt{D}]$ is principal in $\mathsf{Z}[\sqrt{D}]$, and when this does not occur then $H_D(2) = 3$, and the latter occurs exactly when $u_{f_D} = 1$ and $f_D = 2$.[12] $\square$

Theorem 2.5 *If $D > 1$ is a non-square integer and $p > 2$ is a prime not dividing D, then* $u_D \mid (p - (D/p))/2$.

Proof: See Dick Lehmer's result from 1926, given for convenient reference in his collected works [11, Theorem 5, p. 226]. $\square$

Remark 2.1 *The crucial importance of Theorem 2.5 is that it says,* $H_D(p) > 1$ *for all odd primes p not dividing D. This allows us to provide counterexamples to the open question as to whether, for* every *square-free $D > 1$, there an $f_D > 1$ with $H_D(f_D) = 1$. First, in what follows, we show that almost all real quadratic fields* do *satisfy the open question in the affirmative.*

3. Relative Class Number Equality

For a given square-free $D > 1$, with

$$\sqrt{D} = \left\langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0} \right\rangle, \tag{3.17}$$

let B, v, w be given as above in what follows.

[11] An instance of the above is $H_{17}(2) = 1$ since $|\mathsf{Z}[(1 + \sqrt{17})/2] : \mathsf{Z}[\sqrt{17}]| = 2$ and $\psi_{17}(2) = 1$ since $2(1 - (17/2)/2) = 2(1/2) = 1$ and $u_{17} = 1$ since $\varepsilon_{17} = 4 + \sqrt{17} \in \mathsf{Z}[\sqrt{17}]$.

[12] Hence, the open problem left by Dirichlet is solved when $D \equiv 5 \pmod 8$ and $u_{f_D} = 3$ for $f_D = 2$, since $H_D(2) = 1$ in this case. Hence, for $D \equiv 1 \pmod 4$ we need only look at those values $D \equiv 5 \pmod 8$ with $\varepsilon_D \in \mathsf{Z}[\sqrt{D}] = \mathcal{O}_D$—see [12, Theorem 2.1.4, p. 53]. This is all intimately linked to the solvability of the Diophantine equation $x^2 - Dy^2 = \pm 4$ for relatively prime x, y—see [12, Exercise 2.1.16, p. 61] for instance. In any case, the above are all special instances of the more general question of paramount importance in our quest, namely, when is $u_p = 1$, which implies that $p \mid B_{\ell-1}$ when $A_{\ell-1} + B_{\ell-1}\sqrt{D} \in \mathsf{Z}[\sqrt{D}]$. This is a special case of the following result from Lucas-Lehmer theory which will be important in what follows.

Theorem 3.6 *Given any square-free integer $D > 1$ and that D does not divide $B_{\ell-1}$ where $\varepsilon_D = A_{\ell-1} + B_{\ell-1}\sqrt{D} \in \mathsf{Z}[\sqrt{D}] = \mathcal{O}_D$, there exists a prime f_D such that $H_D(f_D) = 1$.*[13]

Proof: As noted in the previous section, given any square-free integer D, the continued fraction expansion of $\sqrt{D}$ must be of the form (3.17). Let f_D be any prime dividing D but not $B = B_{\ell-1}$. We now show that such an f_D must exist. If $\ell = 1$, and then by [12, Theorem 3.2.1, p. 78], $D = q_0^2+1$ and $\varepsilon_D = q_0+\sqrt{D}$, so $B_{\ell-1} = 1$, so we may assume that $\ell \geq 2$. By Theorem 2.1, and Footnote 7, there is an integer x so that (2.7) is satisfied. If $x \geq \lfloor vw/B \rfloor + 2$ when ℓ is even and $x \geq -\lfloor vw/B \rfloor$ when ℓ is odd then the following argument holds—see Footnote 7. If $B > D$, then $B > q_0^2$ since by definition $q_0 = \lfloor \sqrt{D} \rfloor$, so $B > (Bx - (-1)^\ell vw)/2)^2$. If ℓ is odd then $B > B^2x^2/4$ forcing $x = 1$ and $D < B \leq 3$, a contradiction. If ℓ is even, and $x \geq \lfloor vw/B \rfloor + 2$, then

$$B > [(B(vw/B + 1) - vw)/2]^2 = B^2/4,$$

so $B \leq 3$, again a contradiction. Now since D is square-free and we have shown that $D > B$, then there is a prime f_D dividing D and not dividing B. Thus, by (2.14),

$$H_D(f_D) = h_{f_D^2 D}/h_D = \psi_D(f_D)/u_D.$$

Since f_D does not divide $B = B_{\ell-1}$, then $u_D = f_D$ is forced. Also, since $\psi_D(f_D) = f_D$, given that $f_D \mid D$, imples $(D/f_D) = 0$, then $H_D(f_D) = 1$.

The only case remaining is for $x = \lfloor vw/B \rfloor + 1$ and ℓ even. If every prime that divides D also divides B, then since D is square-free, $D \mid B$, which contradicts the hypothesis. Hence, we have shown there is always a prime dividing D that does not divide B. $\square$

Corollary 3.2 (Furness and Parker [8, Theorem 2.12, p. 1403])
If D is square-free and $\sqrt{D} = \left\langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0} \right\rangle$, where $q_j = a \in \mathsf{N}$ for all $j = 1, 2, \ldots, \ell - 1$, then there exists a prime f_D with $H_D(f_D) = 1$.

Proof: In this case, $D = n^2 + r$ where r is a proper divisor of $2n$. If $a = 2n/r$, then $\varepsilon_D = an + 1 + a\sqrt{D} \in \mathsf{Z}[\sqrt{D}]$—see [12, Exercise 3.2.7, p. 85].[14] $\square$

[13] Observe, by the discussion in the preceding section that this theorem covers all square-free $D > 1$ except those $D \equiv 5 \pmod 8$ with $u_{f_D} = 3$ where $f_D = 2$ in which case we demonstrated that $H_D(2) = 1$. Also, the values for which $D \mid B_{\ell-1}$ will be handled (in the negative) in what follows.

[14] For the benefit of the reader, in the notation of [8], P_r/Q_r is, in our notation, $(2q_0v + w)/B$, and is part of the proof of Theorem 2.2. The work they did to establish

$$D = q_0^2 + (2q_0v + w)/B$$

Remark 3.2 *The main objective of this paper may now be stated, namely all of the above points to the fact that Muir's result, and related work, is not not well-known or acknowledged. In [13], we attempted to remedy this situation by exhibiting the results and others with their force and power. The goal of this work is to further that aim and show how these classical results can be used to get much deeper results than those given in the modern setting.*

4. Counterexample to the Rest

Now we provide a counterexample to the existence of relative class number one for those D not covered by Theorem 3.6.[15] Note that if $D \mid B_{\ell-1}$ then it is a rare phenomenon. Indeed, in [21], Stephens et all found only 8 values less than 10^7 with $D \mid B_{\ell-1}$. They are $D \in \{46, 430, 1817, 58254, 209991, 1752299, 3124318, 4099215\}$. Now we show that $H_{46}(f_{46}) \neq 1$ for any $f_{46} > 1$.[16]

Theorem 4.7 [17] *There is no integer $f_{46} > 1$ such that $H_{46}(f_{46}) = 1$.*

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for the special case given in Corollary 3.2 is a simple consequence of Theorems 2.1–2.2. Furthermore, all of the values the authors consider are of the form of D given above which are ERD-types studied extensively by this author and others, in greater generality, as delineated for instance in [12]. Also, there is no reference to the work of Muir, Perron or extensive work of other authors and trivial proofs are given instead. Indeed, the exercise cited in the proof of Corollary 3.2 is given in greater generality than what is proved in [8]. Without further comment, the reader should see [22] for an apt review of [8].

[15] Observe that in the proof of that theorem, the only case left was where ℓ is even, $x = \lfloor vw/B \rfloor + 1$ and $D \mid B$. In the following examples, therefore, where $D \mid B$ in each case it is verifiable that x is indeed of this form. Of course if x is of this form that does not imply that $D \mid B$ since the theorem took care of those cases in the affirmative. For instance, for $D = 19$, $x = \lfloor vw/B \rfloor + 1 = 2$ with $B = 39$, $v = 14$, $w = 5$, and $q_0 = (Bx - vw)/2 = 4$.

[16] It is most interesting to note that the case where $D \mid B_{\ell-1}$ is closely linked to the study of powerful numbers, those natural numbers whose canonical prime factorizations have no primes to the first power. In other words, n is powerful if it is of the form $n = a^2b^3$. An open question in such a study is whether there exist three consecutive powerful numbers. Indeed, in [17], we proved that the existence of such a triple is tantamount to the existence of a non-square $D \in \mathsf{N}$ with $D \equiv 7 \pmod 8$ and for which $\varepsilon_D^k = T_k + U_k\sqrt{D}$ has $D \mid U_k$ with T_k powerful for some $k \in \mathsf{N}$. The first such possibility is $(8 + 3\sqrt{7})^{114254287}$ which we demonstrated does *not* produce a powerful triple. Erdos conjectured there are only finitely many such triples. Their existence remains an open question—see [12, §1.6, pp. 30–39] for an overview and background. Furthermore, $Q_{\ell/2} = 2 = Q_6$, where $Q_{\ell/2}$ is called the *central norm*, in the simple continued fraction expansion of $\sqrt{46}$. We have exhaustively studied when the central norm is 2 (where ℓ must be even) and classified when this occurs including a highly palatable generalization of a result of Lagrange, namely $Q_{\ell/2} = 2$ if and only if $A_{\ell-1} \equiv (-1)^{\ell/2} \pmod D$—see [16, Theorem 4.3, Corollary 4.4, Remark 4.6, p. 781].

[17] Although the authors of a *preprint* [9], which appeared online in late 2012, had this result, this author had it much earlier, but did not put a preprint of it online, believing that priority goes to a truly *published paper*. Furthermore, the proof herein is far simpler and more revealing than that in [9], which appeared *after* an initial version of *this* paper, with that result, had been circulated for some time.

Proof: First we show there is no prime f_{46} with $H_{46}(f_{46}) = 1$, then show that suffices to prove the result. First we note that $\ell = \ell(\sqrt{46}) = 12$ and $Q_{\ell/2} = Q_6 = 2$. Also,

$$\varepsilon_{46} = 24335 + 3588\sqrt{46}.$$

Assume there is a prime $f_{46} = p$ with $H_{46}(p) = 1$. Note that, as a result of the above, $\overline{\ell} = \ell(\sqrt{46p^2})$ must be even. Also, $Q_{\overline{\ell}/2} \mid 92p^2$.

If $p \mid D$, then by (2.16), $H_{46}(p) = p$ since the unit index $u_p = 1$ and the Kronecker symbol $(D/p) = 0 = (46/p)$. Therefore, we may assume that p does not divide D. By Theorem 2.2, $\varepsilon_{46p^2} = A_{\overline{\ell}-1} + B_{\overline{\ell}-1}\sqrt{46p^2}$, where $\overline{\ell} = \ell(\sqrt{46p^2})$. By Theorem 2.5, $u_p \mid (p - (46/p))/2$, so

$$H_{46}(p) = (p - (46/p))/u_p > 1,$$

a contradiction. Now we need only show that if there is no prime value of f_{46}, then there is no composite such value with $H_{46}(f_{46}) = 1$.

Let $f_{46} = f_D = \prod_{i=1}^{n} p_i^{a_i}$. Then $\psi_D(f_D) = \prod_{i=1}^{n} p_i^{a_i-1}(p_i - (D/p_i))$. If we set $b_i = (p_i - (D/p_i))$, then by what we have shown above, for any $i = 1, 2, \ldots, n$, $\varepsilon_D^{c_i} \in \mathbb{Z}[\sqrt{p_i^2 D}]$ where $c_i < b_i$, then $u_D < \psi_D(f_D)$, so $H_D(f_D) > 1$. $\square$

Interestingly, for $D = 1817 \equiv 1 \pmod 8$, we know from previous discussions that $H_D(2) = 1$. The cases where $D \mid B_{\ell-1}$ and D is even can probably also be shown to be counterexamples as with $D = 46$. An exhaustive search for them up to higher bounds than those given above is yet to be accomplished. This completes the resolution of the problem left open by Dirichlet on relative class number one.

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