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$\mathcal{I}_2$-Cauchy Double Sequences of Fuzzy Numbers

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Abstract

In this paper, we introduce the concepts of $\mathcal{I}_2$-Cauchy, $\mathcal{I}_2^*$-Cauchy double sequence of fuzzy numbers and study their some properties and relations, where $\mathcal{I}_2$ denotes the ideal of subsets of $\mathbb{N} \times \mathbb{N}$.

Keywords: *Ideal, double sequences, $\mathcal{I}$-convergence, fuzzy valued sequences.*

1 Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [10] and Schoenberg [32]. A lot of developments have been made in this area after the works of Šalát [28] and Fridy [11, 13]. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [10, 11, 13, 26]. This concept was extended to the double sequences by Mursaleen and Edely [19]. Çakan and Altay [3] presented multidimensional analogues of the results given by Fridy and Orhan [12].

The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [18] and proved some basic theorems for sequences of fuzzy numbers. Nanda [20] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Recently, Nuray and Savaş [24] defined the concepts of statistical convergence and statistically Cauchy for sequence of fuzzy numbers.

They proved that a sequence of fuzzy numbers is statistically convergent if and only if it is statistically Cauchy. Nuray [23] introduced Lacunary statistical convergence of sequences of fuzzy numbers whereas Savaş [30] studied some equivalent alternative conditions for a sequence of fuzzy numbers to be statistically Cauchy. A lot of developments have been made in this area after the works of [2, 27, 31, 33].

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [14] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Nuray and Ruckle [22] indepedently introduced the same with another name generalized statistical convergence. Das et al. [4] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied its some properties. Nabiev et al. [21] proved a decomposition theorem for $\mathcal{I}$-convergent sequences and introduced the notions of $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence, and then studied their certain properties. Also some results on ideal convergence may be found in [5, 6, 15, 16, 29, 34].

Kumar and Kumar [17] studied the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence and $\mathcal{I}$-Cauchy sequence for sequences of fuzzy numbers where $\mathcal{I}$ denotes the ideal of subsets of $\mathbb{N}$. Dündar and Talo [8] introduced the concepts of $\mathcal{I}_2$-convergence, $\mathcal{I}_2^*$-convergence for double sequences of fuzzy numbers and studied their some properties and relations.

In this paper, we introduce the concepts of $\mathcal{I}_2$-Cauchy, $\mathcal{I}_2^*$-Cauchy double sequences of fuzzy numbers, where $\mathcal{I}_2$ denotes the ideal of subsets of $\mathbb{N} \times \mathbb{N}$. Also, we study their some properties and relations.

2 Definitions and Notations

Now, we recall the concept of ideal, convergence, statistical convergence, ideal convergence of sequence, double sequence and fuzzy numbers and some basic definitions (See [1, 4, 7, 8, 9, 10, 14, 19, 25, 31, 34]).

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ whenever $m, n > N_\varepsilon$. In this case we write $\lim_{m,n\to\infty} x_{mn} = L$.

Let $K \subset \mathbb{N} \times \mathbb{N}$ and K_{mn} be the number of $(j, k) \in K$ such that $j \le m$, $k \le n$. If the sequence $\{K_{mn}/(mn)\}$ converges in Pringsheim's sense then we say that K has double natural density and is denoted by

$$d_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{mn}.$$

A double sequence $x = (x_{mn})$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) =$

$\{(m,n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$,

(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,

(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,

(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,

(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1 *[14] If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X and is called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is also admissible.

Let $\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N}), (i, j \geq m(A) \Rightarrow (i,j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Let (X, ρ) be a metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2$ and we write $\mathcal{I}_2 - \lim_{m,n \to \infty} x_{mn} = L$.

If $\mathcal{I}_2$ is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then Pringsheim's convergence implies $\mathcal{I}_2$-convergence.

A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2$-Cauchy if for every $\varepsilon > 0$ there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, x_{st}) \geq \varepsilon\} \in \mathcal{I}_2.$$

A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2^*$-Cauchy sequence if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that for every

$\varepsilon > 0$ and for $(m,n),(s,t) \in M$, $m,n,s,t > k_0 = k_0(\varepsilon)$, $\rho(x_{mn}, x_{st}) < \varepsilon$. In this case we write

$$\lim_{\substack{m,n,s,t\to\infty \\ (m,n),(s,t)\in M}} \rho(x_{mn}, x_{st}) = 0.$$

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2), if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \to [0,1]$ which satisfies the following four conditions:

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex, i.e., $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 := \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, (cf. Zadeh [35]), where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and called it as the space of fuzzy numbers. α-level set $[u]_\alpha$ of $u \in E^1$ is defined by

$$[u]_\alpha := \begin{cases} \{t \in \mathbb{R} : x(t) \geq \alpha\} & , \quad (0 < \alpha \leq 1), \\ \\ \overline{\{t \in \mathbb{R} : x(t) > \alpha\}} & , \quad (\alpha = 0). \end{cases}$$

The set $[u]_\alpha$ is closed, bounded and non-empty interval for each $\alpha \in [0,1]$ which is defined by $[u]_\alpha := [u^-(\alpha), u^+(\alpha)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) := \begin{cases} 1 & , \quad (x = r), \\ 0 & , \quad (x \neq r). \end{cases}$$

Theorem 2.2 *[9] Let $[u]_\alpha = [u^-(\alpha), u^+(\alpha)]$ for $u \in E^1$ and for each $\alpha \in [0,1]$. Then the following statements hold:*

(i) u^- is a bounded and non-decreasing left continuous function on $(0,1]$.

(ii) u^+ is a bounded and non-increasing left continuous function on $(0,1]$.

(iii) The functions u^- and u^+ are right continuous at the point $\alpha = 0$.

(iv) $u^-(1) \leq u^+(1)$.

Conversely, if the pair of functions u^- and u^+ satisfy the conditions (i)-(iv), then there exists a unique $u \in E^1$ such that $[u]_\alpha := [u^-(\alpha), u^+(\alpha)]$ for each $\alpha \in [0,1]$. The fuzzy number u corresponding to the pair of functions u^- and u^+ is defined by $u : \mathbb{R} \to [0,1]$, $u(x) := \sup\{\alpha : u^-(\alpha) \leq x \leq u^+(\alpha)\}$.

Let $u, v, w \in E^1$ and $k \in \mathbb{R}$. Then the operations addition, scalar multiplication and product defined on E^1 by

$$\begin{aligned} u + v = w \quad &\Longleftrightarrow \quad [w]_\alpha = [u]_\alpha + [v]_\alpha, \quad \text{for all } \alpha \in [0,1] \\ &\Longleftrightarrow \quad w^-(\alpha) = u^-(\alpha) + v^-(\alpha) \text{ and } w^+(\alpha) = u^+(\alpha) + v^+(\alpha), \\ &\qquad\;\; [ku]_\alpha = k[u]_\alpha, \quad \text{for all } \alpha \in [0,1] \end{aligned}$$

and

$$uv = w \Longleftrightarrow [w]_\alpha = [u]_\alpha [v]_\alpha, \quad \text{for all } \alpha \in [0,1],$$

where it is immediate that

$$\begin{aligned} w^-(\alpha) &= \min\{u^-(\alpha)v^-(\alpha), u^-(\alpha)v^+(\alpha), u^+(\alpha)v^-(\alpha), u^+(\alpha)v^+(\alpha)\}, \\ w^+(\alpha) &= \max\{u^-(\alpha)v^-(\alpha), u^-(\alpha)v^+(\alpha), u^+(\alpha)v^-(\alpha), u^+(\alpha)v^+(\alpha)\}, \end{aligned}$$

for all $\alpha \in [0,1]$.

Let W be the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e. $A := [\underline{A}, \overline{A}]$. Define the relation d on W by

$$d(A, B) := \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can easily be observed that d is a metric on W and (W, d) is a complete metric space, (cf. Nanda [20]). Now, we can define the metric D on E^1 by means of the Hausdorff metric d as

$$D(u, v) := \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha) := \sup_{\alpha \in [0,1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\}.$$

One can see that

$$D(u, \overline{0}) = \sup_{\alpha \in [0,1]} \max\{|u^-(\alpha)|, |u^+(\alpha)|\} = \max\{|u^-(0)|, |u^+(0)|\}. \tag{1}$$

The partial ordering relation $\preceq$ on E^1 is defined as follows:

$$u \preceq v \Leftrightarrow u^-(\alpha) \leq v^-(\alpha) \text{ and } u^+(\alpha) \leq v^+(\alpha) \quad \text{for all} \quad \alpha \in [0,1].$$

Now, we may give:

Proposition 2.3 *[2] Let $u, v, w, z \in E^1$ and $k \in \mathbb{R}$. Then,*

(i) (E^1, D) *is a complete metric space.*

(ii) $D(ku, kv) = |k| D(u, v)$.

(iii) $D(u + v, w + v) = D(u, w)$.

(iv) $D(u + v, w + z) \leq D(u, w) + D(v, z)$.

(v) $|D(u, \overline{0}) - D(v, \overline{0})| \leq D(u, v) \leq D(u, \overline{0}) + D(v, \overline{0})$.

Following Matloka [18], we give some definitions concerning with the sequences of fuzzy numbers below, which are needed in the text.

A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$ into the set E^1. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the k^{th} term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy numbers.

A sequence $(u_n) \in w(F)$ is called convergent with limit $u \in E^1$, if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $D(u_n, u) < \varepsilon$ for all $n \geq n_0$.

A double sequence $u = (u_{nk})$ of fuzzy real numbers is defined by a function u from the set $\mathbb{N} \times \mathbb{N}$ into the set E^1. The fuzzy number u_{nk} denotes the value of the function at $(n, k) \in \mathbb{N} \times \mathbb{N}$.

A double sequence $u = (u_{mn})$ of fuzzy real numbers is said to be convergent in the Pringsheim's sense or P-convergent to a fuzzy number u_0, if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $D(u_{mn}, u_0) < \varepsilon$, for all $m, n \geq k$ and denoted by $P - \lim_{m,n\to\infty} u_{mn} = u_0$. The fuzzy real number u_0 is called the Pringsheim limit of u.

Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $u = (u_{mn})$ of fuzzy numbers is said to be $\mathcal{I}_2$-convergent to a fuzzy number u_0, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : D(u_{mn}, u_0) \geq \varepsilon\} \in \mathcal{I}_2$. In this case we say that u is $\mathcal{I}_2$-convergent and we write $\mathcal{I}_2 - \lim_{m,n\to\infty} u_{mn} = u_0$.

Lemma 2.4 ([6], Theorem 3.3) *Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N} \times \mathbb{N}$ such that $P_i \in F(\mathcal{I}_2)$ for each i, where $\mathcal{F}(\mathcal{I}_2)$ is a filter associated with a strongly admissible ideal $\mathcal{I}_2$ with the property $(AP2)$. Then there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and the set $P \backslash P_i$ is finite for all i.*

3 Main Results

Definition 3.1 *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $u = (u_{mn})$ of fuzzy numbers is said to be $\mathcal{I}_2$-Cauchy double sequence, if for each $\varepsilon > 0$, there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that*

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : D(u_{mn}, u_{st}) \geq \varepsilon\} \in \mathcal{I}_2.$$

Theorem 3.2 *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $u = (u_{mn})$ of fuzzy numbers is $\mathcal{I}_2$-convergent if and only if it is $\mathcal{I}_2$-Cauchy double sequence.*

Proof. Suppose that a double sequence $u = (u_{mn})$ of fuzzy numbers is $\mathcal{I}_2$-convergent to u_0. Fix $\varepsilon > 0$. Then we have

$$A\left(\frac{\varepsilon}{2}\right) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}_2.$$

This implies that

$$A^c\left(\frac{\varepsilon}{2}\right) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_0) < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}_2)$$

and therefore is nonempty. So, we can choose positive integers k, l such that $(k,l) \notin A\left(\frac{\varepsilon}{2}\right)$, but then we have $D(u_{kl}, u_0) < \varepsilon/2$. Now, we define the set

$$B(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_{kl}) \geq \varepsilon\}.$$

Now we show that $B(\varepsilon) \subset A(\varepsilon/2)$. Let $(m,n) \in B(\varepsilon)$ then we have

$$\varepsilon \leq D(u_{mn}, u_{kl}) \quad \leq \quad D(u_{mn}, u_0) + D(u_{kl}, u_0) < D(u_{mn}, u_0) + \frac{\varepsilon}{2}.$$

This implies that

$$\frac{\varepsilon}{2} < D(u_{mn}, u_0)$$

and therefore $(m,n) \in A(\varepsilon/2)$. Hence we have $B(\varepsilon) \subset A(\varepsilon/2)$. This shows that $u = (u_{mn})$ is $\mathcal{I}_2$-Cauchy double sequence.

Assume that $u = (u_{mn})$ is $\mathcal{I}_2$-Cauchy double sequence. We prove that $u = (u_{mn})$ is $\mathcal{I}_2$-convergent. To this effect, let (ε_{pq}) be a strictly decreasing sequence of numbers converging to zero. Since $u = (u_{mn})$ is $\mathcal{I}_2$-Cauchy double sequence, there exist two strictly increasing sequences (k_p) and (l_q) of positive integers such that

$$A(\varepsilon_{pq}) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_{k_p l_q}) \geq \varepsilon_{pq}\} \in \mathcal{I}_2, (p,q \in \mathbb{N}).$$

This implies that

$$\emptyset \neq \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_{k_p l_q}) < \varepsilon_{pq}\} \in \mathcal{F}(\mathcal{I}_2), (p,q \in \mathbb{N}). \quad (2)$$

Let p, q, s, t be four positive integers such that $p \neq q$ and $s \neq t$. By (2), both the sets

$$C(\varepsilon_{pq}) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_{k_p l_q}) < \varepsilon_{pq}\}$$

and

$$D(\varepsilon_{st}) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : D(u_{mn}, u_{k_s l_t}) < \varepsilon_{st}\}$$

are nonempty sets in $\mathcal{F}(\mathcal{I}_2)$. Since $\mathcal{F}(\mathcal{I}_2)$ is a filter on $\mathbb{N} \times \mathbb{N}$, therefore

$$\emptyset \neq C(\varepsilon_{pq}) \cap D(\varepsilon_{st}) \in \mathcal{F}(\mathcal{I}_2).$$

Thus for each pair (p,q) and (s,t) of positive integers with $p \neq q$ and $s \neq t$, we can select a pair $(m_{(p,q),(s,t)}, n_{(p,q),(s,t)}) \in \mathbb{N} \times \mathbb{N}$ such that

$$D(u_{m_{pqst} n_{pqst}}, u_{k_p l_q}) < \varepsilon_{pq} \text{ and } D(u_{m_{pqst} n_{pqst}}, u_{k_s l_t}) < \varepsilon_{st}.$$

It follows that

$$\begin{aligned} D(u_{k_p l_q}, u_{k_s l_t}) &\leq D(u_{m_{pqst} n_{pqst}}, u_{k_p l_q}) + D(u_{m_{pqst} n_{pqst}}, u_{k_s l_t}) \\ &\leq \varepsilon_{pq} + \varepsilon_{st} \to 0 \end{aligned}$$

as $p,q,s,t \to \infty$. This implies for $p,q \in \mathbb{N}$ that $(u_{k_p l_q})$ is a Cauchy double sequence of fuzzy numbers. Thus the sequence $(u_{k_p l_q})$ converges to a finite limit u_0 (say).i.e.,

$$\lim_{p,q \to \infty} u_{k_p l_q} = u_0.$$

Also, we have $\varepsilon_{pq} \to 0$, as $p,q \to \infty$, so for each $\varepsilon > 0$ we can choose the positive integers p_0 and q_0 such that

$$\varepsilon_{p_0 q_0} < \frac{\varepsilon}{2} \text{ and } D(u_{k_p l_p}, u_0) < \frac{\varepsilon}{2}, \quad (\text{for } p > p_0 \text{ and } q > q_0). \tag{3}$$

Now, we define the set

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : D(u_{mn}, u_0) \geq \varepsilon\}.$$

We prove that $A(\varepsilon) \subset A(\varepsilon_{pq})$. Let $(m,n) \in A(\varepsilon)$, then by second half of (3) we have

$$\begin{aligned} \varepsilon \leq D(u_{mn}, u_0) &\leq D(u_{mn}, u_{k_{p_0} l_{q_0}}) + D(u_{k_{p_0} l_{q_0}}, u_0) \\ &< D(u_{mn}, u_{k_{p_0} l_{q_0}}) + \frac{\varepsilon}{2}. \end{aligned}$$

This implies that

$$\frac{\varepsilon}{2} < D(u_{mn}, u_{k_{p_0} l_{q_0}})$$

and therefore by first half of (3)

$$\varepsilon_{p_0 q_0} < D(u_{mn}, u_{k_{p_0} l_{q_0}}).$$

Thus, we have $(m,n) \in A(\varepsilon_{pq})$ and therefore $A(\varepsilon) \subset A(\varepsilon_{pq})$. Since $A(\varepsilon_{pq}) \in \mathcal{I}_2$ so, $A(\varepsilon) \in \mathcal{I}_2$ by property of ideal. Hence, $(u_{k_p l_q})$ is $\mathcal{I}_2$-convergent to u_0.

Definition 3.3 *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $u = (u_{mn})$ of fuzzy numbers is said to be $\mathcal{I}_2^*$-Cauchy double sequence, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that for every $\varepsilon > 0$ and for $(m,n),(s,t) \in M$, $m,n,s,t > k_0 = k_0(\varepsilon)$*

$$D(u_{mn}, u_{st}) < \varepsilon.$$

In this case we write

$$\lim_{\substack{m,n,s,t\to\infty \\ (m,n),(s,t)\in M}} D(u_{mn}, u_{st}) = 0.$$

Theorem 3.4 *$\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If a double sequence $u = (u_{mn})$ of fuzzy numbers is an $\mathcal{I}_2^*$-Cauchy sequence, then it is $\mathcal{I}_2$-Cauchy.*

Proof. Suppose that $u = (u_{mn})$ is an $\mathcal{I}_2^*$-Cauchy sequence. Then, there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e.,$H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that $D(u_{mn}, u_{st}) < \varepsilon$ for every $\varepsilon > 0$ and for all $(m,n),(s,t) \in M$, $m,n,s,t \geq k_0$ and $k_0 = k_0(\varepsilon) \in \mathbb{N}$. Then

$$\begin{aligned} A(\varepsilon) &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_{st}) \geq \varepsilon\} \\ &\subset H \cup \big[M \cap \big((\{1,2,3,\ldots,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,3,\ldots,(k_0-1)\})\big)\big]. \end{aligned}$$

Since $\mathcal{I}_2$ be a strongly admissible ideal, then

$$H \cup \big[M \cap \big((\{1,2,3,\ldots,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,3,\ldots,(k_0-1)\})\big)\big] \in \mathcal{I}_2.$$

Therefore, we have $A(\varepsilon) \in \mathcal{I}_2$. This shows that the double sequence $u = (u_{mn})$ of fuzzy numbers is $\mathcal{I}_2$-Cauchy.

Theorem 3.5 *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property (AP2) and $u = (u_{mn})$ be a double sequence of fuzzy numbers. Then the concepts $\mathcal{I}_2$-Cauchy double sequence of fuzzy numbers and $\mathcal{I}_2^*$-Cauchy double sequence of fuzzy numbers coincide.*

Proof. If a double sequence is $\mathcal{I}_2^*$-Cauchy, then it is $\mathcal{I}_2$-Cauchy by Theorem 3.4, where $\mathcal{I}_2$ need not have the property $(AP2)$.

Now, it is sufficient to prove that a double sequence $u = (u_{mn})$ of fuzzy numbers is a $\mathcal{I}_2^*$-Cauchy double sequence under assumption that it is an $\mathcal{I}_2$-Cauchy double sequence. Let $u = (u_{mn})$ be an $\mathcal{I}_2$-Cauchy double sequence of fuzzy numbers. Then, there exists $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : D(u_{mn}, u_{st}) \geq \varepsilon\} \in \mathcal{I}_2$$

for every $\varepsilon > 0$. Let

$$P_i = \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : D(u_{mn}, u_{s_i t_i}) < \frac{1}{i} \right\}; \quad (i \in \mathbb{N}),$$

where $s_i = s(1\backslash i), t_i = t(1\backslash i)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I}_2)$ for all $i \in \mathbb{N}$. Since $\mathcal{I}_2$ has the property $(AP2)$, then by Lemma 2.4 there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and $P\backslash P_i$ is finite for all $i \in \mathbb{N}$. Now we show that

$$\lim_{m,n,s,t \to \infty} D(u_{mn}, u_{st}) = 0$$

for $(m,n),(s,t) \in P$. To prove this, let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > 2/\varepsilon$. If $(m,n),(s,t) \in P$ then $P\backslash P_i$ is a finite set, so there exists $N = N(j)$ such that $(m,n),(s,t) \in P_j$ for all $m,n,s,t > N(j)$. Therefore,

$$D(u_{mn}, u_{s_i t_i}) < \frac{1}{j} \quad \text{and} \quad D(u_{st}, u_{s_i t_i}) < \frac{1}{j}$$

for all $m,n,s,t > N(j)$. Hence it follows that

$$\begin{aligned} D(u_{mn}, u_{st}) &\leq D(u_{mn}, u_{s_i t_i}) + D(u_{st}, u_{s_i t_i}) \\ &< \frac{1}{j} + \frac{1}{j} = \frac{2}{j} < \varepsilon \end{aligned}$$

for all $n,m,s,t > N(j)$. Thus, for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for $m,n,s,t > N(\varepsilon)$ and $(m,n),(s,t) \in P$ we have

$$D(u_{mn}, u_{st}) < \varepsilon.$$

This shows that the sequence (u_{mn}) of fuzzy numbers is an $\mathcal{I}_2^*$-Cauchy.

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