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Results on Coupled Fixed Point in Partially Ordered Metric Spaces

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Abstract

In this paper, we prove some unique coupled fixed point theorem in partially ordered metric space. Also for the effectiveness of result we have given an example.

Keywords: *Coupled fixed point, Mixed monotone property, Complete metric space.*

1 Introduction

The Banach contraction principle is one of the simplest and most applicable result of fixed point theorem. It has become a very popular tool in solving the existence problems in many branches of nonlinear analysis. Several mathematicians have extended it and have been interested in fixed point theory in some metric spaces. One of these is partially ordered metric space, that is, metric spaces endowed with a partial ordering. The first result in this direction was given by Turinici, where he extended the Banach contraction principle in partially ordered sets. Ran and Reurings presented some applications of Turinici's theorem to matrix equations. The results were then extended by many authors.

The concept of coupled fixed point was recently introduced by Bhaskar and Lakshmikantham [2]. They established some coupled fixed point theorem on ordered metric spaces and give some application in the existence and uniqueness of a solution for periodic boundary value problem. Several papers have been devoted to the study of coupled fixed points in partially ordered metric spaces [1], [3], [4], [5], [6], [7], [8].

The purpose of this paper is to present some unique coupled fixed point theorems in ordered metric space. An example is also given in order to illustrate the effectiveness of our result at the end of this paper.

2 Preliminaries

In this section, we give some definitions which are useful for main result in this paper.

Definition 2.1. *Let X be a non empty set. Then $(X, d, \leq)$ is called an ordered (partial) metric space if*

(i) $(X, \leq)$ is a partially ordered set and (ii) (X, d) is a metric space.

Definition 2.2. *Let $(X, \leq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \leq y$ or $y \leq x$ holds.*

Definition 2.3. *[2], [4] An element $(x, y) \in X \times X$ is said to be coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x, F(y, x) = y$.*

Definition 2.4. *[2] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \implies F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \implies F(x, y_1) \geq F(x, y_2).$$

3 Main Theorem

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set endowed with a metric d such that (X, d) is complete. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X and there exist $x_0, y_0 \in X$, such that $x_0 \leq F(x_0,\ y_0)$ and $y_0 \geq F(y_0,\ x_0)$. Suppose there exist ψ:[0,∞)$\to$[0,∞) is a continuous and non decreasing function such that it is positive in (0,∞), ψ(0)=0 and $\lim_{t\to\infty} \psi(t) = \infty$; such that*

$$d(F(x,y),F(u,v)) \leq d(x,u) + \psi(d(y,v)) \tag{3.1}$$

for all $x, y, u, v \in X$ with $x \geq u, y \leq v$. Suppose either,

1) F is continuous or
2) X has the following properties,
   (a) if a non-decreasing sequence$\{x_n\}$ in X converges to some point $x \in X$, then $x_n \leq x$, $\forall n$,
   (b) if a non-increasing sequence $\{y_n\}$ in X converges to some point $y \in X$, then $y_n \geq y$, $\forall n$.

Then F has a coupled fixed point $(u^*, v^*) \in X \times X$.

Proof: Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Repeating this process, set $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. Then by (3.1), we have

$$d(x_n, x_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \leq d(x_{n-1}, x_n) + \psi(d(y_{n-1}, y_n)) \tag{3.2}$$

and similarly,

$$d(y_n, y_{n+1}) = d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \leq d(y_{n-1}, y_n) + \psi(d(x_{n-1}, x_n)). \tag{3.3}$$

By adding, we have

$$p_n \leq p_{n-1} + \psi(p_{n-1}). \tag{3.4}$$

Let

$$p_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1}).$$

If $\exists\ n_1 \in N^*$ such that $d(x_{n_1}, x_{n_1-1}) = 0$, $d(y_{n_1}, y_{n_1-1})$ =0, then $x_{n_1-1} = x_{n_1} = F(x_{n_1-1}, y_{n_1-1})$,
$y_{n_1-1} = y_{n_1} = F(y_{n_1}, x_{n_1-1})$ and $x_{n_1-1}; y_{n_1-1}$ is fixed point of F and the proof is finished. In other case $d(x_{n+1}, x_n) \neq 0$; $d(y_{n+1}, y_n) \neq 0$ for all $n \in N$. Then by using assumption on ψ , we have,

$$p_n \leq p_{n-1} + \psi(p_{n-1}) \leq p_{n-1} \tag{3.5}$$

p_n is a non - negative sequence and hence posses a limit p^*. Taking limit when $n \to \infty$, we get,

$$p^* \leq p^* + \psi(p^*)$$

and consequently $\psi(p^*)$=0. By our assumption on ψ , we conclude p^*=0, ie. $\lim_{n\to\infty}(p_n)$=0

$$\lim_{n\to\infty} d(x_{n+1}, x_n) + d(y_{n+1}, y_n) = 0$$
$$\implies \lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} d(y_{n+1}, y_n) = 0. \quad (3.6)$$

Next, we prove that $\{x_n\}$, $\{y_n\}$ are cauchy sequences. Suppose that at least one $\{x_n\}$ or $\{y_n\}$ be not a cauchy sequence. Then $\exists\ \varepsilon > 0$ and two subsequence of integers n_k, m_k with $n_k > m_k \geq k$, such that

$$r_k = d(x_{m_k}, x_{n_k}) + d(y_{m_k}, y_{n_k}) \geq \varepsilon,\ \forall k = 1, 2, 3.... \quad (3.7)$$

Further, corresponding to m_k, we can choose n_k in such a way that it is smallest integer with $n_k > m_k \geq k$ satisfying equation (3.7), we have

$$d(x_{m_k}, x_{n_k-1}) + d(y_{m_k}, y_{n_k-1}) < \varepsilon. \quad (3.8)$$

Using (3.7) and (3.8) and triangle inequality, we get

$$\begin{aligned} \varepsilon \leq r_k &= d(x_{m_k}, x_{n_k}) + d(y_{m_k}, y_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) + d(y_{m_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}) \\ &= d(x_{m_k}, x_{n_k-1}) + d(y_{m_k}, y_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) + d(y_{n_k-1}, y_{n_k}) \\ &< \varepsilon + p_{n_k-1}. \end{aligned} \quad (3.9)$$

Letting $k \to \infty$ and using (3.6), we have

$$\lim_{n,m\to\infty} r_k = \varepsilon > 0. \quad (3.10)$$

Now, we get

$$\begin{aligned} d(x_{m_k+1}, x_{n_k+1}) &= d(F(x_{m_k}, y_{m_k}), F(x_{n_k}, y_{n_k})) \\ &= d(F(x_{n_k}, y_{n_k}), F(x_{m_k}, y_{m_k})) \\ &\leq d(x_{n_k}, x_{m_k}) + \psi(p(y_{n_k}, y_{m_k})). \end{aligned} \quad (3.11)$$

Similarly,

$$\begin{aligned} d(y_{m_k+1}, y_{n_k+1}) &= d(F(y_{m_k}, x_{m_k}), F(y_{n_k}, x_{n_k})) \\ &= d(F(y_{n_k}, x_{n_k}), F(y_{m_k}, x_{m_k})) \\ &\leq d(y_{n_k}, y_{m_k}) + \psi(d(x_{n_k}, x_{m_k})). \end{aligned} \quad (3.12)$$

Using (3.11) and (3.12), we get

$$r_{k+1} \leq r_k + \psi(r_k) \quad (3.13)$$

$\forall k \in 1, 2, 3, ...$ taking $k \to \infty$ of both sides of equation (3.13) and from equation(3.10), it follows that

$$\varepsilon = \lim_{k\to\infty} r_{k+1} \leq \lim_{k\to\infty} r_k + \psi(r_k) < \varepsilon$$

which is a contraction. Therefore $\{x_n\}$ and $\{y_n\}$ are cauchy sequences. We now prove that $F(u^*, v^*) = u^*, F(v^*, u^*) = v^*$. We shall distinguish the cases (1), 2(a) and 2(b) of the Theorem 3.1.

Since X is a complete metric space, $\exists\ u^*, v^* \in X$ such that $\lim_{n\to\infty} x_n = u^*, \lim_{n\to\infty} y_n = v^*$. We now show that if the assumption (1) holds, then (u^*, v^*) is coupled fixed point of F.
As, we have

$$u^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(u^*, v^*)$$

and

$$v^* = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(v^*, u^*).$$

Therefore, (u^*, v^*) is coupled fixed point of F.

Suppose now that the condition 2(a) and 2(b) of the theorem holds.
The sequence $\{x_n\}\to u^*$, $\{y_n\}\to v^*$

$$\begin{aligned} d(F(u^*, v^*), u^*) &\leq d(F(u^*, v^*), x_{n+1}) + d(x_{n+1}, u^*) \\ &= d(F(u^*, v^*), F(x_n, y_n)) + d(x_{n+1}, u^*) \\ &\leq d(u^*, x_n) + \psi(d(v^*, y_n)) + d(x_{n+1}, u^*). \end{aligned}$$

Letting $n\to\infty$, we have

$$d(F(u^*, v^*), u^*) \leq 0 + \psi(0) = 0.$$

This implies that $F(u^*, v^*) = u^*$, similarly, we can show that $F(v^*, u^*) = v^*$. This completes the theorem.

Theorem 3.2. *Let the hypotheses of Theorem 3.1 hold. In addition, suppose that there exists $z \in X$ which is comparable to u and v for all $u, v \in X$. Then F has a unique coupled fixed point.*

Proof: Suppose that there exists $(u', v'), (u^*, v^*) \in X \times X$ are coupled fixed points of F.
Consider the following two cases:

Case 1: (u', v') and (u^*, v^*) are compareable. We have

$$d(u', u^*) = d(F(u', v'), F(u^*, v^*)) \leq d(u', u^*) + \psi(d(v', v^*))$$

similarly,

$$d(v', v^*) = d(F(v', u'), F(v^*, u^*)) \leq d(v', v^*) + \psi(d(u', u^*)).$$

It follows that

$$d(u', u^*) + d(v', v^*) \leq d(u', u^*) + d(v', v^*) + \psi[d(v', v^*) + d(u', u^*)]$$
$$\implies d(u', u^*) + d(v', v^*) = 0.$$

So, $u^* = u', v^* = v'$. The proof is complete.

Case 2: Suppose now that (u', v') and $(u^*, \ v^*)$ are not compareable. Choose an element $(w, z) \in$ X compareable with both of them.

Monotonicity $\implies$ $(F^n(w, z), F^n(z, w))$

$$\begin{aligned} d\begin{pmatrix}(u^*, v^*)\\(u', v')\end{pmatrix} &= d\Big(\begin{pmatrix}F^n(u^*, v^*)\\F^n(v^*, u^*)\end{pmatrix}, \begin{pmatrix}F^n(u', v')\\F^n(v', u')\end{pmatrix}\Big)\\ &\leq d\Big(\begin{pmatrix}F^n(u^*, v^*)\\F^n(v^*, u^*)\end{pmatrix}, \begin{pmatrix}F^n(w, z)\\F^n(z, w)\end{pmatrix}\Big) + d\Big(\begin{pmatrix}F^n(w, z)\\F^n(z, w)\end{pmatrix}, \begin{pmatrix}F^n(u', v')\\F^n(v', u')\end{pmatrix}\Big)\\ &\leq d(u^*, w) + \psi(d(v^*, z))) + (d(v^*, z) + \psi(d(u^*, w)))\\ &\quad + (d(w, u') + \psi(d(z, v'))) + (d(z, v') + \psi(d(w, u')))\\ &= 0 \end{aligned}$$

so $u^* = u', v^* = v'$. The proof is complete.

Example 3.3. *Let $X = [0, \infty)$ be endowed with the standard metric $d(x, y) = |x - y|, \forall x, y \in X$. Then (X, d) is complete metric space.*

Consider the mapping $F : X \times X \to X$ defined by $F(x, y) = \frac{x-2y}{3}$; $x \geq 2y$. Let us take $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) = \frac{2t}{3}$.

Clearly F is continous and has the mixed monotone property. Also there are $x_0 = 0; y_0 = 0$ in X such that
$x_0 = 0 \leq F(0, 0) = F(x_0, y_0)$ and $y_0 = 0 \geq F(0, 0) = F(y_0, x_0)$.
Then it is obvious that (0, 0) is the coupled fixed point of F.
Now, we have following possibility for values of (x, y) and (u, v) such that $x \geq u$, $y \leq v$,

$$\begin{aligned} d(F(x, y), F(u, v)) &= d(\tfrac{x-2y}{3}, \tfrac{u-2v}{3})\\ &= \tfrac{1}{3}|(x - 2y) - (u - 2v)|\\ &= \tfrac{1}{3}|(x - u) - 2(y - v)|\\ &\leq \tfrac{1}{3}|(x - u)| + \tfrac{2}{3}|(y - v)|\\ &\leq |(x - u)| + \tfrac{2}{3}|(y - v)|\\ &= d(x, u) + \psi(d(y, v)). \end{aligned}$$

Thus all the conditions of theorem 3.1 are satisfied.
Therefore F has a coupled fixed point in X.

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