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Distance in Fuzzy Cone Metric Spaces and Common Fixed Point Theorems

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Abstract

The main contribution in this paper is to introduce an idea of fuzzy c-distance in fuzzy cone metric space. A common fixed point theorem for contraction mapping is established in fuzzy cone metric space by using fuzzy c-distance. Lastly the theorem is justified by a suitable example.

Keywords: *Fuzzy Cone Metric Space, Fuzzy c-distance, Common Fixed Point.*

1 Introduction

The idea of cone metric space and cone normed linear space are recent development in functional analysis. The idea of cone metric space was introduced by H.Long-Guang et al.[13]. The definition of cone normed linear space is introduced by T.K.Samanta et al.[15] and M.Eshaghi Gordji et al. [7]. In earlier papers [2, 3], the author introduced the idea of fuzzy cone metric space as well as fuzzy cone normed linear space and studied some basic results. The study of common fixed points for mappings satisfying certain contractive conditions is now a vigorous research activity. Different authors developed several results regarding common fixed point theorem by using different types of contractive conditions for noncommuting mappings in metric spaces (for references please see [4, 5, 6, 9, 10, 11]).

On the other hand M. Abbas & G.Jungck[1] developed common fixed point results for noncommuting mappings in cone metric spaces. Recently Shenghua Wang et al.[16] have been developed a distance called c-distance on a cone metric space and prove a new common fixed point theorem by using this concept.

In this paper, following the idea of c-distance introduced by Shenghua Wang et al.[16], an idea of fuzzy c-distance in fuzzy cone metric space is introduced and by using this concept, one common fixed point theorem is established. There is an advantage to use fuzzy c-distance to establish common fixed point theorem, since it is not required that contraction mapping be weakly compatible.

The organization of the paper is as follows:

Section 2, comprises some preliminary results which are used in this paper. An idea of fuzzy c-distance in fuzzy cone metric space is introduced in Section 3. One common fixed point theorem is established in Section 4.

2 Some Preliminary Results

A fuzzy number is a mapping $x : R \to [0\ ,\ 1]$ over the set R of all reals. A fuzzy number x is convex if $x(t) \geq min\ (x(s)\ ,\ x(r))$ where $s \leq t \leq r$. If there exists $t_0 \in R$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy number (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha\ ,\ b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha\ ,\ b_\alpha]$ means the interval $(-\infty\ ,\ b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.
A fuzzy number x is called non-negative if $x(t) = 0,\ \forall t < 0$.
Kaleva (Felbin) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by *E (R(I))* and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$.

A partial ordering " $\preceq$ " in E is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0\ ,\ 1]$ where $[\eta]_\alpha = [a^1_\alpha\ ,\ b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha\ ,\ b^2_\alpha]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0\ ,\ 1]$.

Fuzzy real number $\bar{0}$ is defined as $\bar{0}(t) = 1$ if $t = 0$ and $\bar{0}(t) = 0$ otherwise.

According to Mizumoto and Tanaka [14] , the arithmetic operations $\oplus$, $\ominus$ on $E \times E$ are defined by

$$(x \oplus y)(t) = Sup_{s \in R} min\ \{x(s)\ ,\ y(t-s)\},\quad t \in R$$
$$(x \ominus y)(t) = Sup_{s \in R} min\ \{x(s)\ ,\ y(s-t)\},\quad t \in R$$

Proposition 2.1 *[14] Let* $\eta\ ,\ \delta \in E(R(I))$ *and* $[\eta]_\alpha = [a^1_\alpha\ ,\ b^1_\alpha]$, $[\delta]_\alpha = [a^2_\alpha\ ,\ b^2_\alpha]$, $\alpha \in (0\ ,\ 1]$. *Then*

$$[\eta \oplus \delta]_\alpha = [a^1_\alpha + a^2_\alpha \ , \ b^1_\alpha + b^2_\alpha]$$
$$[\eta \ominus \delta]_\alpha = [a^1_\alpha - b^2_\alpha \ , \ b^1_\alpha - a^2_\alpha]$$

Definition 2.2 *[12] A sequence $\{\eta_n\}$ in E is said to be convergent and converges to η denoted by $\lim_{n\to\infty} \eta_n = \eta$ if $\lim_{n\to\infty} a^n_\alpha = a_\alpha$ and $\lim_{n\to\infty} b^n_\alpha = b_\alpha$ where $[\eta_n]_\alpha = [a^n_\alpha, b^n_\alpha]$ and $[\eta]_\alpha = [a_\alpha, b_\alpha]$ $\forall \alpha \in (0, 1]$.*

Note 2.3 *[12] If $\eta, \ \delta \in G(R^*(I))$ then $\eta \oplus \delta \in G(R^*(I))$.*

Note 2.4 *[12] For any scalar t, the fuzzy real number $t\eta$ is defined as $t\eta(s) = 0$ if t=0 otherwise $t\eta(s) = \eta(\frac{s}{t})$.*

Definition of fuzzy norm on a linear space as introduced by C. Felbin is given below:

Definition 2.5 *[8] Let X be a vector space over R. Let $|| \ || \ : X \to R^*(I)$ and let the mappings $L, U \ : [0 \ , \ 1] \times [0 \ , \ 1] \to [0 \ , \ 1]$ be symmetric, nondecreasing in both arguments and satisfy*

$L(0 \ , \ 0) = 0$ and $U(1 \ , \ 1) = 1$. Write $[||x||]_\alpha = [||x||^1_\alpha \ , \ ||x||^2_\alpha]$ for $x \in X, \ 0 < \alpha \leq 1$ and suppose for all $x \in X, \ x \neq \underline{0}$, there exists $\alpha_0 \in (0 \ , \ 1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(A) $||x||^2_\alpha < \infty$

(B) $\inf ||x||^1_\alpha > 0$.

The quadruple $(X \ , \ || \ ||, \ L \ , \ U)$ is called a fuzzy normed linear space and $|| \ ||$ is a fuzzy norm if

(i) $||x|| = \bar{0}$ if and only if $x = \underline{0}$;

(ii)$||rx|| = |r| ||x||, \ x \in X, \ r \in R$;

(iii) for all $x, y \in X$,

(a) whenever $s \leq ||x||^1_1, \ t \leq ||y||^1_1$ and $s + t \leq ||x + y||^1_1$, $||x + y||(s + t) \geq L(||x||(s) \ , \ ||y||(t))$,

(b) whenever $s \geq ||x||^1_1, \ t \geq ||y||^1_1$ and $s + t \geq ||x + y||^1_1$, $||x + y||(s + t) \leq U(||x||(s) \ , \ ||y||(t))$

Remark 2.6 *[8] Felbin proved that, if $L = \wedge$(Min) and $U = \vee$(Max) then the triangle inequality (iii) in the Definition 1.3 is equivalent to $||x + y|| \preceq ||x|| \oplus ||y||$.*

Further $|| \ ||^i_\alpha; \ i = 1, 2$ are crisp norms on X for each $\alpha \in (0 \ , \ 1]$. In that case we simply denote $(X \ , \ || \ ||)$.

Definition 2.7 *[2] Let $(E, || \ ||)$ be a fuzzy real Banach space (Felbin sense) where $|| \ || : E \to R^*(I)$. Denote the range of $|| \ ||$ by $E^*(I)$. Thus $E^*(I) \subset R^*(I)$.*

Definition 2.8 *[2] A subset F of $E^*(I)$ is said to be fuzzy closed if for any sequence $\{\eta_n\}$ such that $\lim_{n\to\infty} \eta_n = \eta$ implies $\eta \in F$.*

Definition 2.9 *[2] A subset P of $E^*(I)$ is called a fuzzy cone if*
(i) P is fuzzy closed, nonempty and $P \neq \{\bar{0}\}$;
(ii) $a, b \in R,\ a, b \geq 0,\ \eta, \delta \in P \Rightarrow a\eta \oplus b\delta \in P$;
(iii) $\eta \in P$ and $-\eta \in P \ \Rightarrow \eta = \bar{0}$.

Given a fuzzy cone $P \subset E^*(I)$, define a partial ordering $\leq$ with respect to P by $\eta \leq \delta$ iff $\delta \ominus \eta \in P$ and $\eta < \delta$ indicates that $\eta \leq \delta$ but $\eta \neq \delta$ while $\eta \ll \delta$ will stand for $\delta \ominus \eta \in$IntP where IntP denotes the interior of P.

The fuzzy cone P is called normal if there is a number $K > 0$ such that for all $\eta, \delta \in E^*(I)$, with $\bar{0} \leq \eta \leq \delta$ implies $\eta \preceq K\delta$. The least positive number satisfying above is called the normal constant of P. The fuzzy cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{\eta_n\}$ is a sequence such that $\eta_1 \leq \eta_2 \leq .... \leq \eta_n \leq .... \leq \eta$ for some $\eta \in E^*(I)$, then there is $\delta \in E^*(I)$ such that $\eta_n \to \delta$ as $n \to \infty$. Equivalently, the fuzzy cone P is regular if every decreasing sequence which is bounded below is convergent. It is clear that a regular fuzzy cone is a normal fuzzy cone.

In the following we always assume that E is a fuzzy real Banach (Felbin sense) space, P is a fuzzy cone in E with IntP $\neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.10 *[2] Let X be a nonempty set. Suppose the mapping $d :\ X \times X \to E^*(I)$ satisfies*
(Fd1) $\bar{0} \leq d(x, y)\ \forall x, y \in X$ and $d(x, y) = \bar{0}$ iff $x = y$;
(Fd2) $d(x, y) = d(y, x)\ \ \forall x, y \in X$;
(Fd3) $d(x, y) \leq d(x, z) \oplus d(z, y)\ \ \forall x, y, z \in X$. Then d is called a fuzzy cone metric and (X, d) is called a fuzzy cone metric space.

Definition 2.11 *[2] Let (X, d) be a fuzzy cone metric space. Let$\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\bar{0} \ll ||c||$ there is a positive integer N such that for all $n > N,\ d(x_n, x) \ll ||c||$, then $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. We denote it by $\lim_{n\to\infty} x_n = x$.*

Lemma 2.12 *[2] Let (X, d) be a fuzzy cone metric space and P be a normal fuzzy cone with normal constant K. Let$\{x_n\}$ be a sequence in X. If $\{x_n\}$ is convergent then its limit is unique.*

Definition 2.13 *[2] Let (X, d) be a fuzzy cone metric space and $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $\bar{0} \ll ||c||$, there exists a natural number N such that $\forall m, n > N,\ d(x_n, x_m) \ll ||c||$, then $\{x_n\}$ is called a Cauchy sequence in X.*

Definition 2.14 *[2] Let (X, d) be a fuzzy cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete fuzzy cone metric space.*

Definition 2.15 *[1] Let f and g be self mappings defined on a set X. If $w = f(x) = g(x)$ for some $x \in X$, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.*

Proposition 2.16 *[1] Let f and g be weakly compatible self-mappings of a set X. If f and g have a unique point of coincidence $w = f(x) = g(x)$, then w is the unique common fixed point of f and g.*

3 Fuzzy c-Distance in Fuzzy Cone Metric Spaces

In this Section, idea of fuzzy c-distance in fuzzy cone metric spaces is introduced. Here $(E, || \, ||)$ is a fuzzy normed linear space (Felbin sense) and $\leq$ is the partial ordering defined w.r.t. the fuzzy cone P of E.

Definition 3.1 *Let $(X \, , \, d)$ be a fuzzy cone metric space. Then the mapping $Q: \; X \times X \to E^*(I)$ is called a c-fuzzy distance on X if the following conditions hold:*

(Q1) $\bar{0} \leq Q(x, y) \;\; \forall x, y \in X$;
(Q2) $Q(x, z) \leq Q(x, y) \oplus Q(y, z) \;\; \forall x, y, z \in X$;
(Q3) $\forall x \in X$, if $Q(x, y_n) \leq \eta$ for some $\eta = \eta(x) \in P, \; n \geq 1$,
then $Q(x, y) \leq \eta$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(Q4) $\forall c \in E$ with $\bar{0} \ll ||c||$, $\exists e \in E$ with $\bar{0} \ll ||e||$ such that $Q(z, x) \ll ||e||$ and
$Q(z, y) \ll ||e||$ imply $d(x, y) \ll ||c||$.

Example 3.2 *Let $(X \, , \, d)$ be a fuzzy cone metric space and P be a fuzzy normal cone given by $\{\eta \in E^*(I) : \; \eta \geq \bar{0}\}$.*
If we put $Q(x, y) = d(x, y) \;\; \forall x, y \in X$, then Q is a fuzzy c-distance.

Solution: (Q1) and (Q2) are obvious.
For (Q3), let $\{y_n\}$ be a sequence in X converging to a point $y \in X$ such that $Q(x, y_n) \leq \eta$ for some $\eta = \eta(x) \in P$.
Thus $\eta \ominus Q(x, y_n) \in P \;\; \forall n$
$\Rightarrow \eta \ominus d(x, y_n) \in P \;\; \forall n$
$\Rightarrow \lim_{n \to \infty} (\eta \ominus d(x, y_n)) \in P$ (since P is closed)
$\Rightarrow \eta \ominus d(x, y) \in P$
$\Rightarrow d(x, y) \leq \eta$

i.e. $Q(x,y) \leq \eta$. Thus (Q3) holds.
Let $c \in E$ with $\bar{0} \ll ||c||$ and put $||e|| = \frac{||c||}{2}$, $e \in E$.
Suppose that $Q(z,x) \ll ||e||$ and $Q(z,y) \ll ||e||$.
Then $d(x,y) = Q(x,y) \leq Q(x,z) \oplus Q(z,y) \ll ||e|| \oplus ||e|| = ||c||$.
So Q satisfies (Q4). Hence Q is a fuzzy c-distance.

Example 3.3 *Let $E = C[0,1]$ and $||x||' = \bigvee_{0 \leq t \leq 1} |x(t)|$. Then $(E, ||\ ||')$ is a Banach space.*

Define $||\ || :\ E \to R^*(I)$ by

$$||x||(t) = \begin{cases} \frac{||x||'}{t} & \text{if } t \geq ||x||' \\ 1 & \text{if } t = ||x||' = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $[||x||]_\alpha = [||x||', \frac{||x||'}{\alpha}]$ $\forall \alpha \in (0,1]$.
It can be verified that $(E, ||\ ||)$ is a fuzzy normed linear space (Felbin's sense).
Now we show that $(E, ||\ ||)$ is complete.
Let $\{x_n\}$ be a Cauchy sequence in $(E, ||\ ||)$.
Thus $||x_n - x_m|| \to \bar{0}$ as $m, n \to \infty$
$\Rightarrow ||x_n - x_m||^i_\alpha \to 0$ as $m, n \to \infty$, $\forall \alpha \in (0,1]$ and for $i = 1, 2$
$\Rightarrow ||x_n - x_m||' \to 0$ as $m, n \to \infty$
$\Rightarrow \{x_n\}$ is a Cauchy sequence in $(E, ||\ ||')$.
Since $(E, ||\ ||')$ is complete, $\exists x \in E$ such that $||x_n - x||' \to 0$ as $n \to \infty$.
$\Rightarrow ||x_n - x||^i_\alpha \to 0$ as $n \to \infty$, $\forall \alpha \in (0,1]$ and for $i = 1, 2$
$\Rightarrow ||x_n - x|| \to \bar{0}$ as $n \to \infty$.
So $(E, ||\ ||)$ is a complete fuzzy normed linear space.
Now define $d : E \times E \to E^*(I)$ by $d(x,y) = ||x - y||$.
If we take the ordering $\leq$ as $\preceq$ then it can be shown that (E, d) is a fuzzy cone metric space. Also $P = \{\eta \in E^*(I) : \eta \succeq \bar{0}\}$ is a cone of E. Again since $||x|| \leq ||y||$ implies $||x|| \preceq ||y||$ $\forall x, y \in P$, thus P is a fuzzy normal cone with normal constant 1.
Now define $Q(x,y) = d(x,y)$ $\forall x, y \in E$. We show that Q is a c-distance.
In fact (Q1) and (Q2) are obvious. Now we verify (Q3).
Let $\{y_n\}$ be a sequence in E converging to $y \in E$.
Now $Q(x, y_n) \leq \eta$ for some $\eta(x) \in P$, $n \geq 1$
$\Rightarrow d(x, y_n) \preceq \eta$ $n \geq 1$
$\Rightarrow d^i_\alpha(x, y_n) \leq \eta^i_\alpha$ for $i = 1, 2$ and $\forall \alpha \in (0,1]$
$\Rightarrow d^i_\alpha(x, y) \leq \eta^i_\alpha$ for $i = 1, 2$ and $\forall \alpha \in (0,1]$
$\Rightarrow d(x,y) \leq \eta$.
Let $c \in E$ with $\bar{0} \ll ||c||$ be given. Put $||e|| = \frac{||c||}{2}$.
Suppose $Q(z,x) \ll ||e||$ and $Q(z,y) \ll ||e||$.
Then $d(x,y) = Q(x,y) \preceq Q(x,z) \oplus Q(z,y) << ||e|| \oplus ||e|| = ||c||$.
This shows that Q satisfies (Q4) and hence Q is a fuzzy c-distance of (X, d).

4 Fixed Point Theorem in Fuzzy Cone Metric Spaces using Fuzzy c-Distance

In [4], some common fixed point theorems have been established in a normal fuzzy cone metric space. The statement of the Theorems are as follows.

Theorem 4.1 *[4] Let $(X\ ,\ d)$ be a fuzzy cone metric space and P be a fuzzy normal cone with normal constant K. Suppose mappings $f,\ g:\ X \to X$ satisfy $d(fx\ ,\ fy) \leq kd(gx\ ,\ gy)\ \ \forall x,y \in X$ where $k \in [0\ ,\ 1)$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have unique point of coincidence in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.*

Theorem 4.2 *[4] Let $(X\ ,\ d)$ be a fuzzy cone metric space and P be a fuzzy normal cone with normal constant K. Suppose mappings $f,\ g:\ X \to X$ satisfy the contractive condition $d(fx\ ,\ fy) \leq k(d(fx\ ,\ gx) \oplus d(fy\ ,\ gy))\ \ \forall x,y \in X$ where $k \in\ [0\ ,\ \frac{1}{2})$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.*

Here a common fixed point theorem is established by using fuzzy c-distance and it is not required that f and g are weakly compatible.

Theorem 4.3 *Let $(X\ ,\ d)$ be a fuzzy cone metric space and P be a fuzzy normal cone with normal constant K. Let $Q:\ X \times X \to E^*(I)$ be a fuzzy c-distance on X. Let $a_i \in (0,1)$ $(i = 1,2,3,4)$ be constants with $a_1 + 2a_2 + a_3 + a_4 < 1$ and $f,\ g:\ X \to X$ be two mappings satisfying the condition*

$$Q(fx,fy) \leq a_1Q(gx,gy) \oplus a_2Q(gx,fy) \oplus a_3Q(gx,fx) \oplus a_4Q(gy,fy)\ \ \forall x,y \in X \qquad (1)$$

Suppose that the range of g contains the range of f and $g(X)$ is a complete subspace of X. If f and g satisfy $inf\{Q^1_{\alpha_0}(fx,y) + Q^1_{\alpha_0}(gx,y) + Q^1_{\alpha_0}(gx,fx)\ :\ \ x \in X\} > 0$ for some $\alpha_0 \in (0,1]$, $\forall y \in X$ with $y \neq fy$ or $y \neq gy$, then f and g have a common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point. Since $f(X) \subset g(X)$, $\exists x_1 \in X$ such that $fx_0 = gx_1$.
By induction, a sequence $\{x_n\}$ can be chosen such that $fx_n = gx_{n+1}, n = 0,1,2,.....$
By (1) and (Q2) for any natural number n we have,

$$Q(gx_n, gx_{n+1}) = Q(fx_{n-1}, fx_n)$$
$$\leq a_1Q(gx_{n-1}, gx_n) \oplus a_2Q(gx_{n-1}, fx_n) \oplus a_3Q(gx_{n-1}, fx_{n-1}) \oplus a_4Q(gx_n, fx_n)$$

$= a_1Q(gx_{n-1}, gx_n) \oplus a_2Q(gx_{n-1}, gx_{n+1}) \oplus a_3Q(gx_{n-1}, gx_n) \oplus a_4Q(gx_n, gx_{n+1})$
$\leq a_1Q(gx_{n-1}, gx_n) \oplus a_2[Q(gx_{n-1}, gx_n) \oplus Q(gx_n, gx_{n+1})] \oplus a_3Q(gx_{n-1}, gx_n) \oplus a_4Q(gx_n, gx_{n+1})$
$= (a_1 + a_2 + a_3)Q(gx_{n-1}, gx_n) \oplus (a_2 + a_4)Q(gx_n, gx_{n+1})$.
So, $Q(gx_n, gx_{n+1}) \leq bQ(gx_{n-1}, gx_n),\ n = 1, 2, ...$ where $b = \frac{a_1+a_2+a_3}{1-a_2-a_4} \in (0, 1)$.

By induction we get,
$Q(gx_n, gx_{n+1}) \leq b^nQ(gx_1, gx_0),\ n = 1, 2, ...$ (2)

Let m, n with $m > n$ be arbitrary. From (2) and (Q2) we have,
$Q(gx_n, gx_m) \leq Q(gx_n, gx_{n+1}) \oplus Q(gx_{n+1}, fx_{n+2}) \oplus ..... \oplus Q(gx_{m-1}, gx_m)$
$\leq b^nQ(gx_0, gx_1) \oplus b^{n-1}Q(gx_0, gx_1) \oplus ..... \oplus b^{m-1}Q(gx_0, gx_1)$
$= (b^n + b^{n+1} + ... + b^{m-1})Q(gx_0, gx_1) = \frac{b^n}{1-b}Q(gx_0, gx_1)$
Thus $Q(gx_n, gx_m) \leq \frac{b^n}{1-b}Q(gx_0, gx_1)$ (3)
Since P is a normal cone with normal constant K we have
$Q(gx_n, gx_m) \preceq K\frac{b^n}{1-b}Q(gx_0, gx_1)$
$\Rightarrow Q^i_\alpha(gx_n, gx_m) \leq K\frac{b^n}{1-b}Q^i_\alpha(gx_0, gx_1)\ \ \forall\alpha \in (0, 1]$ and $i = 1, 2$.
$\Rightarrow \lim_{m,n\to\infty} Q^i_\alpha(gx_n, gx_m) = 0\ \ \forall\alpha \in (0, 1]$ and $i = 1, 2$ (since $b < 1$).
$\Rightarrow \lim_{m,n\to\infty} Q(gx_n, gx_m) = \bar{0}$.
Thus $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists some $y \in g(X)$ such that $gx_n \to y$ as $n \to \infty$.
By (3) and (Q3) we have,
$Q(gx_n, y) \leq \frac{b^n}{1-b}Q(gx_0, gx_1),\ n = 0, 1, 2, ...$ (4)
Since P is a normal cone with normal constant K, from (4) it follows that,
$Q(gx_n, y) \preceq \frac{Kb^n}{1-b}Q(gx_0, gx_1),\ n = 0, 1, 2, ...$
$\Rightarrow Q^i_\alpha(gx_n, y) \leq \frac{Kb^n}{1-b}Q^i_\alpha(gx_0, gx_1),\ \ \forall\alpha \in (0, 1]$ and $i = 1, 2$ (5)
From (1) we have,
$Q(gx_n, gx_m) \preceq \frac{Kb^n}{1-b}Q(gx_0, gx_1)$ for $m > n$.
In particular we have,
$Q(gx_n, gx_{n+1}) \preceq \frac{Kb^n}{1-b}Q(gx_0, gx_1)$ for $n = 0, 1, 2, ...$
$\Rightarrow\ Q^i_\alpha(gx_n, gx_{n+1}) \leq \frac{Kb^n}{1-b}Q^i_\alpha(gx_0, gx_1)\ \forall\alpha \in (0, 1],\ i = 1, 2$ and for $n = 0, 1, 2, ...$ (6)
If possible suppose that $y \neq gy$ or $y \neq fy$. Then by hypothesis, (5) and (6) we have
$0 <\ inf\{Q^1_{\alpha_0}(fx, y) + Q^1_{\alpha_0}(gx, y) + Q^1_{\alpha_0}(gx, fx) :\ x \in X\}$
$\leq\ inf\{Q^1_{\alpha_0}(fx_n, y) + Q^1_{\alpha_0}(gx_n, y) + Q^1_{\alpha_0}(gx_n, fx_n) :\ n \geq 1\}$
$=\ inf\{Q^1_{\alpha_0}(gx_{n+1}, y) + Q^1_{\alpha_0}(gx_n, y) + Q^1_{\alpha_0}(gx_n, gx_{n+1}) :\ n \geq 1\}$
$\leq\ inf\{\frac{Kb^{n+1}}{1-b}Q^1_{\alpha_0}(gx_1, gx_0) + \frac{Kb^n}{1-b}Q^1_{\alpha_0}(gx_1, gx_0) + \frac{Kb^n}{1-b}Q^1_{\alpha_0}(gx_1, gx_0) :\ n \geq 1\}\ = 0$.
This is a contradiction. Hence $y = gy = fy$. Thus y is a common fixed point of f and g.

Theorem 4.3 is justified by the following Example.

Example 4.4 *Let E=R (set of real numbers). Define* $||\ ||:\ E \to R^*(I)$ *by*

$$||x||(t) = \begin{cases} \frac{|x|}{t} & if\ t \geq |x|, x \neq \theta \\ 1 & if\ t = |x| = 0 \\ 0 & otherwise \end{cases}$$

Then $[||x||]_\alpha = [|x|, \frac{|x|}{\alpha}]\ \ \forall \alpha \in (0,1]$. *It can be verified that* $(E, ||\ ||)$ *is a complete fuzzy normed linear space (Felbin's sense).*

Let $X = [0, \infty)$ and $P = \{x \in E:\ ||x|| \succeq \bar{0}\}$.
Define a mapping $d:\ X \times X \to E^*(I)$ by $D(x, y) = ||x - y||\ \ \forall x, y \in X$.
If we chose the ordering $\leq$ as $\preceq$ then (X, d) is a fuzzy cone metric space with normal cone P and normal constant 1.

Again define a mapping $Q:\ X \times X \to E^*(I)$ by $Q(x, y) = ||y||\ \forall x, y \in X$.
Then Q is a fuzzy c-distance.
In fact, (Q1)-(Q2) are obvious. Let $\{y_n\}$ be a sequence in X converging to a point $y \in X$.
For all $x \in X,\ Q(x, y_n) \preceq \eta(x),\ \eta \in P$ implies $||y_n|| \preceq \eta$.
Now $||y|| \preceq ||y - y_n|| \oplus ||y_n||$
$\Rightarrow ||y||^i_\alpha \leq ||y - y_n||^i_\alpha + ||y_n||^i_\alpha$ for $i = 1, 2$ and $\alpha \in (0, 1]$
$\Rightarrow ||y||^i_\alpha \leq \lim\limits_{n\to\infty} ||y_n||^i_\alpha \leq \eta^i_\alpha$ for $i = 1, 2$ and $\alpha \in (0, 1]$
$\Rightarrow ||y|| \preceq \eta$
$\Rightarrow Q(x, y) \preceq \eta\ \ \forall x \in X$.
So (Q3) holds.
Let $||e|| \succ \bar{0}$ be given where $e \in E$. Set $||c|| = \frac{||e||}{2}$.
If $Q(z, x) = ||x|| \prec ||c||$ and $Q(z, y) = ||y|| \prec ||c||$ then
$d(x, y) = ||x - y|| \preceq ||x|| \oplus ||y|| \prec 2||c|| = ||e||$.
Thus (Q4) holds and hence Q is a fuzzy c-distance.
Define $f:\ X \to X$ by $f(x) = \frac{x}{2}\ \ \forall x \in X$ and $g(x) = x\ \ \forall x \in X$
Take $a_1 = \frac{1}{2},\ \ a_2 = \frac{1}{32},\ \ a_3 = \frac{3}{32}$ and $a_4 = \frac{5}{32}$.
Now we show that f and g satisfy the relation (1). We have
$Q^1_\alpha(fx, fy) = Q^1_\alpha(\frac{x}{2}, \frac{y}{2}) = \frac{1}{2}||y||^1_\alpha = \frac{1}{2}|y|$.
$Q^1_\alpha(gx, gy) = Q^1_\alpha(x, y) = ||y||^1_\alpha = |y|$.
$Q^1_\alpha(gx, fy) = Q^1_\alpha(x, \frac{y}{2}) = \frac{1}{2}||y||^1_\alpha = \frac{1}{2}|y|$.
$Q^1_\alpha(gx, fx) = Q^1_\alpha(x, \frac{x}{2}) = \frac{1}{2}||x||^1_\alpha = \frac{1}{2}|x|$.
$Q^1_\alpha(gy, fy) = Q^1_\alpha(y, \frac{y}{2}) = \frac{1}{2}||y||^1_\alpha = \frac{1}{2}|y|$.
Now,
$a_1 Q^1_\alpha(gx, gy) + a_2 Q^1_\alpha(gx, fy) + a_3 Q^1_\alpha(gx, fx) + a_4 Q^1_\alpha(gy, fy)$
$= \frac{1}{2}|y| + \frac{1}{64}|y| + \frac{3}{64}|x| + \frac{5}{64}|y| = \frac{38}{64}|y| + \frac{3}{64}|x|$.
So,
$a_1 Q^1_\alpha(gx, gy) + a_2 Q^1_\alpha(gx, fy) + a_3 Q^1_\alpha(gx, fx) + a_4 Q^1_\alpha(gy, fy) - Q^1_\alpha(fx, fy)$

$= \frac{38}{64}|y| + \frac{3}{64}|x| - \frac{1}{2}|y| \geq 0$
$\Rightarrow Q^1_\alpha(fx, fy) \leq a_1 Q^1_\alpha(gx, gy) + a_2 Q^1_\alpha(gx, fy)$
$$+ a_3 Q^1_\alpha(gx, fx) + a_4 Q^1_\alpha(gy, fy) \quad \forall \alpha \in (0, 1] \qquad (7)$$
Similarly we have,
$$Q^2_\alpha(fx, fy) \leq a_1 Q^2_\alpha(gx, gy) + a_2 Q^2_\alpha(gx, fy) + a_3 Q^2_\alpha(gx, fx) + a_4 Q^2_\alpha(gy, fy) \quad \forall \alpha \in (0, 1] \qquad (8)$$
From (7) and (8) we have,
$Q(fx, fy) \preceq a_1 Q(gx, gy) \oplus a_2 Q(gx, fy) \oplus a_3 Q(gx, fx) \oplus a_4 Q(gy, fy)$.
Suppose $y \neq fy$ or $y \neq gy$. That is $y \neq \underline{0}$.
We have for some α_0,
$inf\{Q^1_{\alpha_0}(fx, y) + Q^1_{\alpha_0}(gx, y) + Q^1_{\alpha_0}(gx, fx) : \ x \in X\}$
$= \ inf\{Q^1_{\alpha_0}(\frac{x}{2}, y) + Q^1_{\alpha_0}(x, y) + Q^1_{\alpha_0}(x, \frac{x}{2}) : \ x \in X\}$
$= \ inf\{|y| + |y| + \frac{|x|}{2} : \ x \in X\} \ = 2|y| > 0.$

Thus all the hypothesis of the Theorem 4.3 are satisfied. Consequently f and g have a common fixed point and this is $\underline{0}$.

5 Conclusion

In this paper, idea of fuzzy c-distance in fuzzy cone metric space is introduced. By using this concept, common fixed point theorems for contraction mapping are established in fuzzy cone metric spaces. Generally to establish common fixed point theorems, contraction mapping should be weakly compatible. Here fuzzy c-distance is used to establish common fixed point theorems and it is not required that mappings are weakly compatible. I think that there is a wide scope of research to develop fixed point results in fuzzy cone metric spaces by using fuzzy c-distance.

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