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On Subclass of Meromorphic Univalent Function Defined by Derivative Operator

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Abstract

In the present paper, we introduce a new class of meromorphic univalent functions defined by derivative operator. We obtain some geometric properties like coefficient inequality, distortion and growth theorems. Hadmard product or (convolution), radii of starlikeness and convexity, partial sums, integral operator and closure theorem for the functions in the class $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$.

Keywords: *Meromorphic univalent function, Hadmard product, Convex function, Starlike function.*

1 Introduction

Historically [1], the classical polylogarithm function was invented in 1696, by Leibniz and Bernoulli, as mentioned in [2]. For |z|<1 and c a natural number with c≥2, the polylogari-thm function (which is also known as Jonquiere's function) is defined by the absolutely convergent series: $\mathrm{Li}_c(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^c}$.

Later on, many mathematicians studied the polylogarithm function such as Euler, Spence, Able, Lobachevsky, Rogers, Ramanujun and many others [5], where they discovered many functional identities by using polylogarithm function. However, the work employing polylogarithm has been stopped many decades, later. During the past four decades, the work using polygarithm has again been in intensified vividly due to its importance in many fields of math-ematic, such as complex analytic, algebra, geometry, topology and mathematical physics (quantum field theory) ([3], [9], [10]). In 1996, Ponnusamy and Sabapathy discussed the geometric mapping properties of the generalized polylogarithm [10]. Recently, Al-Shaqsi and Darus [13] generalized Ruscheweyh and salagean operators, using polylogarithm function on class $\mathcal{A}$ of analyticfunctions in the open unit disk$U = \{z \in \mathbb{C}: |z|<1\}$. By making use of the generalized operator they introduced related properties.

A year later, same authors again employed the nth order .Polylogarithm function to define a multiplier transformation on the class $\mathcal{A}$ in U [14].

Now, we will redefine the polylogarithm function to be on meromorphic type.

LetΩ denoted the class of function of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \quad ,$$

Which are analytic and meromorphic univalent in the unit disk

$$U^* = \{z \in \mathbb{C}: 0 \leq |z|<1\} = U \setminus \{0\}.$$

Let A be a subclass of Ω of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \quad (a_k \geq 0, k \in \mathbb{N}) \ . \ (1)$$

A function $f \in A$ is meromorphicstarlike function of orderρ , $(0 \leq \rho < 1)$ if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \rho (z \in U^*).$$

The class of all meromorphically starlike functions is denoted by $A^*(\rho)$. A function $f \in A$ is meromorphic convex function of orderρ , $(0 \leq \rho < 1)$ if

$$-\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \rho (z \in U^*).$$

The Hadmard product or (convolution) of two functions, *f* given by (1) and

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \; (b_k \geq 0, k \in \mathbb{N}) \;, \qquad (2)$$

is defined

$$(f * g)(z) = \frac{1}{z} + \sum_{k=2}^{\infty} a_k b_k z^k \quad , \quad (3)$$

see [12], which are analytic and univalent in U^*.

Liu and Srivastava to defined a function $h_p(\alpha_{1}, \ldots . . \alpha_q; \beta_1, \ldots \ldots . \beta_s \,; z)$by multiplying the well-known generalized hyprogometric function qF_s with z^{-p} as follows:

$$h_p\big(\alpha_{1}, \ldots . . \alpha_q; \beta_1, \ldots \ldots . \beta_s \,; z\big) = z^{-p} qF_s\big(\alpha_{1}, \ldots . . \alpha_q; \beta_1, \ldots \ldots . \beta_s \,; z\big).$$

Where $\alpha_{1}, \ldots . . \alpha_q; \beta_1, \ldots \ldots . \beta_s are complex$ parameters and $q \leq s + 1. p \in \mathbb{N}$. analogous to Liu and Srivastava work [6] and corresponding function $\emptyset_c(z)$ given by

$$\emptyset_c(z) = z^{-2} Li_c(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{1}{(k+2)^c} z^k \quad (4)$$

We consider liner operator $\Sigma_c f(z) \colon \Omega \to \Omega$

Which is defined the following Hadmard product (or convolution):

$$\Sigma_c f(z) = \emptyset_c(z) * f(z)$$

$$= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{1}{(k+2)^c} a_k z^k \;. \quad (5)$$

In [7], N.E, Cho and H. M. Srivastava a liner operator of the form:

$$\mathcal{T}(m,n) f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{l+k}{l+1}\right)^m a_k \quad z^k m \in \mathbb{Z}\,, l \geq 0. \quad (6)$$

Further , denoted by D^{λ}: $S \to S$ *the Rucheweyh dervitive of order* λ defined by [11]:

$$D^{\lambda,1} f(z) = \frac{z^{-1}}{(1+z)^{\lambda+1}} * f(z) \;, \lambda > -1$$

$$=\frac{1}{z}+\sum_{n=2}^{\infty} a_n \delta(n,k) z^n\,,(\lambda > -1, f(z) \in S),$$

$$where \delta(n,k) = \binom{n+k-1}{n},$$

This function yields the following family of liner operators

$$I^{\lambda} f(z) = (D^{\lambda,1} * \mathcal{T}(m,n) f(z)$$

$$=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^{m} \delta(n,k) a_k \quad z^k m \in \mathbb{Z}\,, l \geq 0. \qquad (7)$$

Now, we define the linear operator by

$$H_c^{\lambda} f(\mathrm{z}) = \Omega_c f(z) * I^{\lambda} f(z)$$

$$=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^{m} \frac{1}{(k+2)^c} \delta(n,k) a_k \quad z^k (m \in \mathbb{Z}\,, l \geq 0). \qquad (8)$$

Now we define the following:

Definition 1.1: *A function* $f \in A$ *of the form (1) is in the class* $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$ *if satisfies the following condition:*

$$\left| \frac{\dfrac{z\left(H_c^{\lambda,1} f(\mathrm{z})\right)''}{\left(H_c^{\lambda,1} f(\mathrm{z})\right)'} + 2}{\dfrac{z\left(H_c^{\lambda,1} f(\mathrm{z})\right)''}{\left(H_c^{\lambda,1} f(\mathrm{z})\right)'} + 2\alpha} \right| < \beta, \qquad (9)$$

where$\lambda \geq -1 \quad 0 \leq \alpha < 1, \; 0 < \beta \leq 1 \;, c \geq 2$.

Special cases of this class was studied by many researchers like A.R .S. Juma and H. Zirar [4]. Also Kh. R. Alhindi and Maslina Darus [1] investigate in the same filed.

In this paper we obtain coefficient estimates for the class $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$, Hadmard product, growth and distortion theorems, radii of starlikeness and convexity.

2 Coefficient Inequality

Theorem 2.1: *A function f defined by (1) is in the class $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$, if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^m \frac{\delta(n,k)}{(k+2)^c} k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)]a_k \le 2\beta(1-\alpha)\ . \quad (10)$$

The result is sharp for the function

$$f_k(z)=\frac{1}{z}+\frac{(k+2)^c 2\beta(1-\alpha)}{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)]}z^k, k\ge 1 \quad (11)$$

Proof: To proof the sufficient part, let the inequality (10) holds true and let $|z|=1$, by (9), we have

$$\left|\frac{z\left(H_c^{\lambda,1}f(\mathrm{z})\right)''}{\left(H_c^{\lambda,1}f(\mathrm{z})\right)'}+2\right|-\beta\left|\frac{z\left(H_c^{\lambda,1}f(\mathrm{z})\right)''}{\left(H_c^{\lambda,1}f(\mathrm{z})\right)'}+2\alpha\right|<0,$$

$$\left|z\left(H_c^{\lambda,1}f(\mathrm{z})\right)''+2\left(H_c^{\lambda,1}f(\mathrm{z})\right)'\right|-\beta\left|z\left(H_c^{\lambda,1}f(\mathrm{z})\right)''+2\alpha\left(\left(H_c^{\lambda,1}f(\mathrm{z})\right)'\right)\right|,$$

$$=\left|\sum_{k=1}^{\infty}k(k+1)\left(\frac{l+k}{l+1}\right)^m\frac{\delta(n,k)}{(k+2)^c}a_k z^{k-1}\right|$$

$$-\beta\left|\frac{2(1-\alpha)}{z^2}+\sum_{k=1}^{\infty}k(k-1+2\alpha)\left(\frac{l+k}{l+1}\right)^m\frac{\delta(n,k)}{(k+2)^c}a_k z^{k-1}\right|,$$

$$\le\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^m\frac{\delta(n,k)}{(k+2)^c}k\left[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)\right]a_k-2\beta(1-\alpha)\le 0\ .$$

Thus by the maximum modulus theorem, we have $f\in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$.

Conversely, suppose the f of the form (1) is in the class $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$, then by (9) we have

$$\left|\frac{\frac{z\left(H_c^{\lambda,1}f(z)\right)''}{\left(H_c^{\lambda,1}f(z)\right)'}+2}{\frac{z\left(H_c^{\lambda,1}f(z)\right)''}{\left(H_c^{\lambda,1}f(z)\right)'}+2\alpha}\right|<\beta,$$

$$\left|\frac{\sum_{k=1}^{\infty}k(k+1)\left(\frac{l+k}{l+1}\right)^m\frac{\delta(n,k)}{(k+2)^c}a_kz^{k-1}}{\frac{2(1-\alpha)}{z^2}+\sum_{k=1}^{\infty}k(k-1+2\alpha)\left(\frac{l+k}{l+1}\right)^m\frac{\delta(n,k)}{(k+2)^c}a_kz^{k-1}}\right|<\beta.$$

Since $|\Re(z)|\le|z|$ for all z , we get

$$\Re\left\{\frac{\sum_{k=1}^{\infty}k(k+1)\left(\frac{l+k}{l+1}\right)^m\frac{\delta(n,k)}{(k+2)^c}a_kz^{k-1}}{\frac{2(1-\alpha)}{z^2}+\sum_{k=1}^{\infty}k(k-1+2\alpha)\left(\frac{l+k}{l+1}\right)^m\frac{\delta(n,k)}{(k+2)^c}a_kz^{k-1}}\right\}<\beta. \qquad (12)$$

Now, by choosing the value of z on the real axis so that the value of $\frac{z\left(H_c^{\lambda,1}f(z)\right)''}{\left(H_c^{\lambda,1}f(z)\right)'}$ is real, then clearing the denominator of (12) and letting $z\to 1-$ through real value, we get the inequality (10).

The result is sharp for the function

$$f_k(z)=\frac{1}{z}+\frac{(k+2)^c2\beta(1-\alpha)}{\left(\frac{l+k}{l+1}\right)^m\delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)]}z^k, k\ge 1$$

Corollary 2.1: *If* $f\in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$, *then*

$$a_k\le\frac{(k+2)^c2\beta(1-\alpha)}{\left(\frac{l+k}{l+1}\right)^m\delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)]},$$

where $\lambda\ge -1 \quad 0\le\alpha<1, \qquad 0<\beta\le 1 \ , c\ge 2$.

Corollary 2.2: *If* $f\in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$,

$$a_k\le\frac{(3)^c2\beta(1-\alpha)}{\delta(n,1)(1+\ \beta\alpha)},$$

and equality holds for $f_k(z)=\frac{1}{z}+\frac{(3)^c2\beta(1-\alpha)}{\delta(n,1)(1+\ \beta\alpha)}z$

Corollary 2.3: *If* $f \in R_c^{\lambda,1}(\alpha, 1, c, \lambda)$,

$$a_k \le \frac{(k+2)^c 2\,(1-\alpha)}{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[2(k+\alpha)]},$$

and equality holds for

$$f_k(z) = \frac{1}{z} + \frac{(k+2)^c 2\,(1-\alpha)}{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[2(k+\alpha)]} z^k, k \ge 1$$

Corollary 2.4: *If* $f \in R_2^{\lambda,1}(\alpha, 1, 2, \lambda)$,

$$a_k \le \frac{(k+2)^2 2\,(1-\alpha)}{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[2(k+\alpha)]},$$

and equality holds for

$$f_k(z) = \frac{1}{z} + \frac{(k+2)^2 2\,(1-\alpha)}{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[2(k+\alpha)]} z^k, k \ge 1$$

Corollary 2.5: *If* $f \in R_2^{\lambda,1}(\alpha, 1, 2, \lambda)$,

$$a_k \le \frac{18(1-\alpha)}{\delta(n,1)[2(1+\alpha)]},$$

and equality holds for

$$f(z) = \frac{1}{z} + \frac{18(1-\alpha)}{\delta(n,1)[2(1+\alpha)]} \mathrm{z}$$

3 Growth and Distortion Theorems

A growth and distortion property for the function$f \in R_c^{\lambda,1}(\alpha, \beta, c, \lambda)$ is given as follows

Theorem 3.1: *A function* f *defined by (1) is in the class* $R_c^{\lambda,1}(\alpha, \beta, c, \lambda)$ *, then for*

$0 < |z| = r < 1$, we have

$$\frac{1}{r} - \frac{(3)^c \beta(1-\alpha)}{\delta(n,1)(1+\beta\alpha)} r \le |f(z)| \le \frac{1}{r} + \frac{(3)^c \beta(1-\alpha)}{\delta(n,1)(1+\beta\alpha)} r$$

with equality for

$$f(z)=\frac{1}{z}+\frac{(3)^c\beta(1-\alpha)}{\delta(n,1)(1+\beta\alpha)}\mathrm{z}\quad .$$

Proof: Since $\in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$, we have form Theorem 2.1 the inequality

$$\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^m\frac{\delta(n,k)}{(k+2)^c}k\left[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)\right]a_k\leq 2\beta(1-\alpha)\ ,$$

then

$$|f(z)|\leq\frac{1}{|z|}+\sum_{k=1}^{\infty}a_k|z|^k,$$

for $0<|z|=r<1$, we have

$$|f(z)|\leq\frac{1}{r}+r\sum_{k=1}^{\infty}a_k,\leq\frac{1}{r}+\frac{(3)^c\beta(1-\alpha)}{\delta(n,1)(1+\beta\alpha)}r.$$

Also

$$|f(z)|\geq\frac{1}{|z|}-\sum_{k=1}^{\infty}a_k|z|^k,\geq\frac{1}{r}-\frac{(3)^c\beta(1-\alpha)}{\delta(n,1)(1+\beta\alpha)}r,|z|=r.$$

Hence the proof is complete.

Theorem 3.1: A function f defined by (1) is in the class $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$, then for

$0<|z|=r<1$, we have

$$\frac{1}{r^2}-\frac{(3)^c\beta(1-\alpha)}{\delta(n,1)(1+\beta\alpha)}\leq|f(z)'|\leq\frac{1}{r^2}+\frac{(3)^c\beta(1-\alpha)}{\delta(n,1)(1+\beta\alpha)}r,$$

with equality for

$$f(z)=\frac{1}{z}+\frac{(3)^c\beta(1-\alpha)}{\delta(n,1)(1+\beta\alpha)}z.$$

Proof: Form Theorem 2.1 we have

$$\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^m\frac{\delta(n,k)}{(k+2)^c}k\left[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)\right]a_k\leq 2\beta(1-\alpha)\,.$$

Thus

$$|f(z)'| \leq \left|\frac{-1}{z^2}\right| + \sum_{k=1}^{\infty} ka_k|z|^{k-1},$$

for $0 < |z| = r < 1$, we get:

$$|f(z)'| \leq \left|\frac{1}{r^2}\right| + \sum_{k=1}^{\infty} ka_k, \leq \left|\frac{1}{r^2}\right| + \frac{(3)^c\beta(1-\alpha)}{\delta(n,1)(1+\beta\alpha)},$$

and

$$|f(z)'| \geq \left|\frac{-1}{z^2}\right| - \sum_{k=1}^{\infty} ka_k|z|^{k-1}, |f(z)'| \geq \left|\frac{1}{r^2}\right| - \sum_{k=1}^{\infty} ka_k \ , \geq \frac{1}{r^2} - \frac{(3)^c\beta(1-\alpha)}{\delta(n,1)(1+\beta\alpha)}.$$

This completes the proof

4 Hadmard Product

Theorem 4.1: *Let functions*

$f, g \in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$. Then $(f * g) \in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$ for

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \ ,$$

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \ ,$$

and

$$(f * g)(z) = \frac{1}{z} + \sum_{k=2}^{\infty} a_k b_k z^k \ ,$$

where

$$\gamma = \frac{2\beta^2(1-\alpha)(\ k+1)}{2\beta^2(1-\alpha)(k+2\alpha-1) - \left(\frac{l+k}{l+1}\right)^m \frac{\delta(n,k)}{(k+2)^c} k\left[k(1+\beta) + \left(1+\beta(2\alpha-1)\right)\right]^2}$$

Proof: Since $f, g \in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$, then by Theorem 2.1 we have

$$\sum_{k=1}^{\infty} \frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta) + \left(1+\beta(2\alpha-1)\right)]}{(k+2)^c 2\beta(1-\alpha)} a_k \leq 1,$$

and

$$\sum_{k=1}^{\infty} \frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)]}{(k+2)^c 2\beta(1-\alpha)} b_k \leq 1.$$

We must find the largest δsuch that

$$\sum_{k=1}^{\infty} \frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)]}{(k+2)^c 2\beta(1-\alpha)} a_k b_k \leq 1.$$

By Cauchy-Schwarz inequality, we get

$$\sum_{k=1}^{\infty} \frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)]}{(k+2)^c 2\beta(1-\alpha)} \sqrt{a_k b_k} \leq 1, \qquad (13)$$

To prove the theorem it is enough to show that

$$\sum_{k=1}^{\infty} \frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\gamma)+\big(1+\gamma(2\alpha-1)\big)]}{(k+2)^c 2\gamma(1-\alpha)} a_k b_k$$

$$\leq \sum_{k=1}^{\infty} \frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)]}{(k+2)^c 2\beta(1-\alpha)} \sqrt{a_k b_k} \ ,$$

Which is equivalent to

$$\sqrt{a_k b_k} \leq \frac{\gamma[k(1+\gamma)+\big(1+\gamma(2\alpha-1)\big)]}{\beta[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)]}.$$

From (13), we have

$$\sqrt{a_k b_k} \leq \frac{(k+2)^c 2\beta(1-\alpha)}{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)]}.$$

We must show that

$$\frac{(k+2)^c 2\beta(1-\alpha)}{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k\big[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)\big]} \leq \frac{\gamma\big[k(1+\gamma)+\big(1+\gamma(2\alpha-1)\big)\big]}{\beta\big[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)\big]},$$

Which gives

$$\gamma \leq \frac{2\beta^2(1-\alpha)(\,k+1)}{2\beta^2(1-\alpha)(k+2\alpha-1)-\left(\frac{l+k}{l+1}\right)^m \frac{\delta(n,k)}{(k+2)^c} k\big[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)\big]^2}.$$

Hence the proof is complete.

Theorem 4.2: *Let the functions* $f_i(i=1,2,\dots.)$ *defined by*

$$f_i(z)=\frac{1}{z}+\sum_{k=1}^{\infty} a_{k,i} z^k \quad ,(a_{k,i}\geq 0, \mathrm{i}=1{,}2)$$

be in the class $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$. Then the function$g$ defined by

$$g(z)=\frac{1}{z}+\sum_{k=1}^{\infty}({a^2}_{k,1}+{a^2}_{k,2})z^k \,,$$

Is in the class $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$, where

$$\emptyset=\frac{4\beta^2(1-\alpha)(\,k+1)}{4\beta^2(1-\alpha)(k+2\alpha-1)-\left(\frac{l+k}{l+1}\right)^m \frac{\delta(n,k)}{(k+2)^c} k\big[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)\big]^2}$$

Proof: Since $f_i \in R_c^{\lambda,1}(\alpha,\beta,c,\lambda),(i=1{,}2)$, then by Theorem 2.1 we have:

$$\sum_{k=1}^{\infty} \frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)]}{(k+2)^c 2\beta(1-\alpha)} a_{k,i} \leq 1, i=1{,}2, \dots.$$

Hence

$$\sum_{k=1}^{\infty}\left[\frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)]}{(k+2)^c 2\beta(1-\alpha)}\right]^2 {a^2}_{k,i} \leq 1,$$

$$\leq\left[\sum_{k=1}^{\infty} \frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)]}{(k+2)^c 2\beta(1-\alpha)} a_{k,i}\right]^2 \leq 1, i=1{,}2.$$

Thus

$$\sum_{k=1}^{\infty}\frac{1}{2}\left[\frac{\left(\frac{l+k}{l+1}\right)^{m}\delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)]}{(k+2)^{c}2\beta(1-\alpha)}\right]^{2}({a^{2}}_{k,1}+{a^{2}}_{k,2})\leq 1,$$

To prove the theorem, we must find the largest ∅ such that

$$\frac{[k(1+\emptyset)+\left(1+\emptyset(2\alpha-1)\right)]}{\emptyset}\leq\frac{\left(\frac{l+k}{l+1}\right)^{m}\delta(n,k)k\left[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)\right]^{2}}{(k+2)^{c}4\beta(1-\alpha)},$$

So that

$$\emptyset\leq\frac{4\beta^{2}(1-\alpha)(k+1)}{4\beta^{2}(1-\alpha)(k+2\alpha-1)-\left(\frac{l+k}{l+1}\right)^{m}\frac{\delta(n,k)}{(k+2)^{c}}k\left[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)\right]^{2}}.$$

Hence the proof is complete.

Theorem 4.3: $If\ f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_{k}z^{k}\in R_{c}^{\lambda,1}(\alpha,\beta,c,\lambda),(and g(z)=\frac{1}{z}+\sum_{k=1}^{\infty}b_{k}z^{k}$ with $|b_{k}|\leq 1$ is in the class $R_{c}^{\lambda,1}(\alpha,\beta,c,\lambda), then f(z)*g(z)\in R_{c}^{\lambda,1}(\alpha,\beta,c,\lambda)$.

Proof: From Theorem 2.1 we have

$$\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^{m}\frac{\delta(n,k)}{(k+2)^{c}}k\left[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)\right]a_{k}\leq 2\beta(1-\alpha)\ .$$

Since

$$\sum_{k=1}^{\infty}\frac{\left(\frac{l+k}{l+1}\right)^{m}\delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)]}{(k+2)^{c}2\beta(1-\alpha)}|a_{k}b_{k}|\ ,$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{l+k}{l+1}\right)^{m}\delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)]}{(k+2)^{c}2\beta(1-\alpha)}a_{k}|b_{k}|\ ,$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{l+k}{l+1}\right)^{m}\delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)]}{(k+2)^{c}2\beta(1-\alpha)}a_{k}\leq 1.$$

$$\text{Thus} f(z) * g(z) \in R_c^{\lambda,1}(\alpha,\beta,c,\lambda).$$

Corollary 4.1: *If* $f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k \in R_c^{\lambda,1}(\alpha,\beta,c,\lambda).$ *, and* $g(z)=\frac{1}{z}+\sum_{k=1}^{\infty}b_k z^k$ *with* $0\le b_k<1$ *is in the class* $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$*, then* $f(z)*g(z)\in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$.

5 Radii of Starlikeness and Convexity

Theorem 5.1: *Let the function* $f(z)$ *defined by (1) be in the clas* $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$ *.Then* f *is meromorphically starlike of order* $\rho(0\le\rho<1)$ *in the disk* $|z|<r_1 R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$*, where*

$$r_1(\alpha,\beta,c,\lambda,\rho)=\inf_k\left\{\frac{\left(\frac{l+k}{l+1}\right)^m\delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)](1-\rho)}{(k+2)^c 2\beta(k+2-\rho)(1-\alpha)}\right\}^{\frac{1}{n-1}}.$$

The result is sharp for the function is given by (11).

Proof: It is enough to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right|\le 1-\rho\,,$$

$$\left|\frac{zf'(z)}{f(z)}+1\right|=\left|\frac{\sum_{k=1}^{\infty}(k+1)a_k z^k}{z^{-1}+\sum_{k=1}^{\infty}a_k z^k}\right|\le\frac{\sum_{k=1}^{\infty}(k+1)a_k|z|^{k+1}}{1-\sum_{k=1}^{\infty}a_k|z|^{k+1}}.$$

This will be bounded by $1-\rho$,

$$\frac{\sum_{k=1}^{\infty}(k+1)a_k|z|^{k+1}}{1-\sum_{k=1}^{\infty}a_k|z|^{k+1}}\le 1-\rho\,,$$

$$\sum_{k=1}^{\infty}(k+2-\rho)a_k|z|^{k+1}\le 1-\rho\,,$$

By theorem 2.1, we have

$$\sum_{k=1}^{\infty}\frac{\left(\frac{l+k}{l+1}\right)^m\delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)]}{(k+2)^c 2\beta(1-\alpha)}a_k\le 1.$$

Hence

$$|z|^{k+1} \leq \frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)](1-\rho)}{(k+2)^c 2\beta(k+2-\rho)(1-\alpha)},$$

$$|z| \leq \left\{\frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)](1-\rho)}{(k+2)^c 2\beta(k+2-\rho)(1-\alpha)}\right\}^{\frac{1}{n+1}}.$$

This completes the proof of the theorem.

Theorem 5.1: *Let the function $f(z)$ defined by (1) be in the class$R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$ .Then f is meromorphically convex of order $\delta(0 \leq \delta < 1)$ in the disk $|z| < r_2(\alpha,\beta,c,\lambda,\delta), where$*

$$r_2(\alpha,\beta,\gamma,\lambda,\delta) = \inf_k \left\{\frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)\,(1-\delta)[k(1+\beta)+\left(1+\beta(2\alpha-1)\right)]}{(k+2)^c 2\beta(k+2-\delta)(1-\alpha)}\right\}^{\frac{1}{n-1}}.$$

The result is sharp for the function is given by (11).

Proof: By using the same technique in the proof of Theorem 5.1 we can show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \leq 1-\delta \ ,(0 \leq \delta < 1).$$

For $|z| < r_2$ with the aid of Theorem 2.1, we have the assertion of Theorem 5.2.

6 Closure Theorem

Theorem 6.1: *Let the function*

$$f_n(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,n} z^k \ ,\left(a_{k,n} \geq 0, \mathrm{k} \in \mathbb{N} = \{1,2,\ldots\}\right),$$

be in the class$\mathcal{AR}(\alpha,\beta,\gamma,\lambda)$ for every n=1,2,….i. Then the function h defined by

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} e_k z^k \quad (e_n \geq 0\,, k \in \mathbb{N})$$

also belong to the class $R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$, where

$$e_k = \frac{1}{m}\sum_{k=1}^{\ell} a_{k,n} z^k \;\; (k = 1,2,\dots).$$

Proof: Since$f_n \in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$ it follows from Theorem 2.1 that

$$\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^m \frac{\delta(n,k)}{(k+2)^c} k\big[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)\big]a_{k,n} \le 2\beta(1-\alpha) \;,$$

for every n=1,2,….. m

Hence

$$\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^m \frac{\delta(n,k)}{(k+2)^c} k\big[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)\big]e_k$$

$$\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^m \frac{\delta(n,k)}{(k+2)^c} k\big[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)\big]\left(\frac{1}{m}\sum_{k=1}^{\ell} a_{k,n} z^k\right)$$

$$= \frac{1}{m}\sum_{k=1}^{\ell}\left(\sum_{k=1}^{\infty}\left(\frac{l+k}{l+1}\right)^m \frac{\delta(n,k)}{(k+2)^c} k\big[k(1+\beta)+\big(1+\beta(2\alpha-1)\big)\big]a_{k,n}\right) \le 2\beta(1-\alpha).$$

By Theorem 2.1, it follows that $h \in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$

7 Integral Operator and Partial Sums

Next, we consider some properties have been found the other class in [8].

Theorem 7.1: *The$f \in R_c^{\lambda,1}(\alpha,\beta,c,\lambda)$ if and only if the function F given by*

$$F(Z) = \frac{\lambda}{z^{\lambda+1}}\int_0^z t^{\lambda} f(t)dt \quad ,(\lambda > -1, \qquad (14)$$

is in the class$R_c^{\lambda+1,1}(\alpha,\beta,c,\lambda)$.

Proof: By using of (14), we have

$$\lambda f(Z) = (\lambda+1)F(z) + zF'(z), \qquad (15)$$

Which, in the right hand of (8), implies

$$\lambda(H_{c*}^{\lambda,1}f(\mathrm{z})) = (\lambda+1)(H_{c}^{\lambda,1}F(\mathrm{z})) + z(H_{c}^{\lambda,1}F(\mathrm{z}))' = \lambda(H_{c*}^{\lambda+1,1}F(\mathrm{z})).$$

Therefore, we have

$$, \qquad H_{c*}^{\lambda,1}f(\mathrm{z}) = H_{c*}^{\lambda+1,1}F(\mathrm{z})$$

and the desired result follows at once.

Theorem 7.2: *Let* $f \in A$ *given by (2) and define the partial sums* $s_1(z)$ *and* $s_n(z)$ *by*

$$s_1(z) = \frac{1}{z} and\ s_n(z) = \frac{1}{z} + \sum_{k=1}^{n-1} a_k z^k,$$

Suppose also that

$$\sum_{k=1}^{\infty} d_k a_k \leq 1, \quad \left(d_k = \frac{\left(\frac{l+k}{l+1}\right)^m \delta(n,k)k\left[k(1+\beta) + \left(1+\beta(2\alpha-1)\right)\right]}{(k+2)^c 2\beta(1-\alpha)} \right). \quad (16)$$

Then we have

$$\mathcal{R}e\left\{\frac{f(z)}{s_n(z)}\right\} > 1 - \frac{1}{d_k} and \mathcal{R}e\left\{\frac{s_n(z)}{f(z)}\right\} > 1 - \frac{d_k}{1-d_k}. \qquad (17)$$

Each of the bounds in (17) is the best possible for$k \in \mathbb{N}$.

Proof: For the coefficient d_k given by (16), it is not difficult to verify that

$$d_{k+1} > d_k > 1, k = 1,2 \ldots\ldots .$$

Therefore, by using the hypothesis (16), we have

$$\sum_{k=1}^{n-1} a_k + d_n \sum_{k=n}^{\infty} a_k \leq \sum_{k=1}^{\infty} d_k a_k \leq 1. \qquad (18)$$

Be setting

$$g_1(z) = d_n\left(\frac{f(z)}{s_n(z)} - \left(1 - \frac{1}{d_n}\right)\right) = 1 + \frac{d_n \sum_{k=n}^{\infty} a_k z^{k+1}}{1 + \sum_{k=1}^{n-1} a_k z^{k+1}} \qquad (19)$$

And applying (18) we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_n \sum_{k=n}^{\infty} a_k}{2-2\sum_{k=1}^{n-1} a_k - d_n \sum_{k=n}^{\infty} a_k} \quad (20)$$

Which readily yields the left assertion (17). If we take

$$f(z) = \frac{1}{z} - \frac{z^n}{d_n}\ , \qquad (21)$$

then $\frac{f(z)}{s_n(z)} = 1 - \frac{z^n}{d_n} \rightarrow 1 - \frac{1}{d_n}(z \rightarrow 1-),$

which shows that the bound in (17) is the best possible for each$k \in \mathbb{N}$.
Similarly, if we put

$$g_2(z) = (1+d_n)\left(\frac{s_n(z)}{f(z)} - \frac{d_n}{1+d_n}\right) = 1 - \frac{(1+d_n)\sum_{k=n}^{\infty} a_k z^{k+1}}{1+\sum_{k=1}^{n-1} a_k z^{k+1}}$$

and make use of (18) we obtain

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_n)\sum_{k=n}^{\infty} a_k}{2-2\sum_{k=1}^{n-1} a_k + (1-d_n)\sum_{k=n}^{\infty} a_k} \leq 1 \qquad (21)$$

which leads us to the assertion (17).The bounds given in the right of (17)is sharp with the function given by (21).The proof of the theorem is complete.

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