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On Some Ideals of Fuzzy Points Semigroups

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Abstract

Kim [Int. J. Math. & Math. Sc. ***26:11*** *(2001), 707-712.] Considered the semigroup $\underline{S}$ of the fuzzy points of a semigroup S. In this paper, we discuss the relation between some ideals A of S and the subset $\underline{C_A}$ of $\underline{S}$.*

Keywords: *Fuzzy set; Semigroup; Fuzzy point; Minimal ideal.*

1 Introduction

Zadeh [9] introduced the concept of a fuzzy set for the first time and this concept was applied by Rosenfeld [8] to define fuzzy subgroups and fuzzy ideals. Based on this crucial work, Kuroki [3, 4, 5, 6] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them. Authors in [1] investigated the existence of a fuzzy kernel and minimal fuzzy ideals in semigroups. They showed that a subset A of a semigroup S is minimal ideal if and only if the characteristic function of A , C_A, is minimal fuzzy ideal of S. In [2], Kim considered the semigroup $\underline{S}$ of the fuzzy points of a semigroup S, and discussed the relation between the fuzzy interior ideals and the subsets of $\underline{S}$. In this paper, we discuss the relation between some ideals A of S and the subset $\underline{C_A}$ of $\underline{S}$.

2 Basic Definitions and Results

Let S be a semigroup. A nonempty subset A of S is called a *left (resp., right) ideal* of S if $SA \subseteq A(resp., AS \subseteq A)$, and *a two-sided ideal* (or simply *ideal*) of S if A is both a left and a right ideal of S. A nonempty subset A of S is called an interior ideal of S if $SAS \subseteq A$. An ideal A of S is called *minimal* ideal of S if A does not properly contains any other ideal of S. If the intersection K of all the ideals of a semigroup S is nonempty then we shall call K *the kernel* of S. A subsemigroup A of S is called a bi-ideal of S if $ASA \subseteq A$ [7]. A function f from S to the closed interval [0, 1] is called a *fuzzy set* in S. The semigroup S itself is a fuzzy set in S such that $S(x) = 1$ for all $x \in S$, denoted also by S. Let A and B be two fuzzy sets in$S$. Then the inclusion relation $A \subseteq B$ is defined $A(x) \leq B(x)$ for all $x \in S$. $A \cap B$ and $A \cup B$ are fuzzy sets in S defined by $(A \cap B)(x) = min\,\{A(x), B(x)\}$, $(A \cup B)(x) = max\,\{A(x), B(x)\}$, for all $x \in S$. For any $\alpha \in (0,1]$ and $x \in S$, a fuzzy set x_α in S is called a *fuzzy point* in S if

$$x_\alpha(y) = \begin{cases} \alpha & if\ x = y, \\ 0 & otherwise, \end{cases}$$

for all $x \in S$[9]. The fuzzy point x_α is said to be contained in a fuzzy set A, denoted by $x_\alpha \in A$, iff $\alpha \leq A(x)$. The characteristic mapping of a subset A of a semigroup S is

$$\mathrm{C_A}(x) = \begin{cases} 1 & if\ x \in A, \\ 0 & otherwise, \end{cases}$$

for all $x \in S$.

Lemma 2.1 (see [1, Lemma 3.]): *For any nonempty subsets A and B of a semigroup* S, *we have* $A \subseteq B$ *if and only if* $\mathrm{C_A} \subseteq \mathrm{C}_B$.

Lemma 2.2 (see [1, Lemma 4.]): *Let A be a nonempty subset of a semigroup* S, *then* A *is an ideal of* S *if and only if* $\mathrm{C_A}$ *is a fuzzy ideal of* S.

Let $\mathcal{F}(S)$ be the set of all fuzzy sets in a semigroup S. For each $A, B \in \mathcal{F}(S)$, the product of A *and* B is a fuzzy set $A \circ B$ defined as follows:

$$(A \circ B)(x) = \begin{cases} sup_{x=ab}\ A(a) \wedge B(b) & if\ ab = x \\ 0 & otherwise. \end{cases}$$

for each $x \in S$. If S is a semigroup, then $\mathcal{F}(S)$ is a semigroup with the product "$\circ$"[2]. Let $\underline{S}$ be the set of all fuzzy points in a semigroup S. Then $x_\alpha \circ y_\beta = (xy)_{\alpha\beta} \in \underline{S}$ for $x_\alpha, y_\beta \in \underline{S}$ [2]. For any $A \in \mathcal{F}(S)$, $\underline{A}$ denotes the set of all fuzzy points contained in A, that is, $\underline{A} = \{\ x_\alpha \in \underline{S}: A(x) \geq \alpha\}$. for any $A, B \subseteq \underline{S}$, we define the product of A and B as $A \circ B = \{x_\alpha \circ y_\beta : x_\alpha \in A, y_\beta \in B\}$.

Lemma 2.3 (see [2, Lemma 3.2]): *Let A and B be two fuzzy subsets of a semigroup S, then*

1) $\underline{A \cup B} = \underline{A} \cup \underline{B}$.
2) $\underline{A \cap B} = \underline{A} \cap \underline{B}$.
3) $\underline{A} \circ \underline{B} \subseteq \underline{A \circ B}$.

Lemma 2.4: *Let A be nonempty subset of a semigroup S, we have $x_\alpha \in \underline{C_A}$ if and only if $x \in A$.*

Proof: Suppose that $x_\alpha \in \underline{C_A}$ for any $x \in S$, then $C_A(x) \geq \alpha$. Hence $C_A(x) = 1$ for any $\alpha > 0$, which implies that $x \in A$. Conversely, Let $x \in A$, then $C_A(x) = 1 \geq \alpha$ for any $\alpha > 0$. This means that $x_\alpha \in \underline{C_A}$. ■

Lemma 2.5: *For any nonempty subsets A and B of a semigroup S, we have*

1) *$A \subseteq B$ if and only if $\underline{C_A} \subseteq \underline{C_B}$.*
2) *$C_A \subseteq C_B$ if and only if $\underline{C_A} \subseteq \underline{C_B}$.*

Proof: (1) Assume that $A \subseteq B$, and let $x_\alpha \in \underline{C_A}$. By lemma 2.4, $x \in A \subseteq B$ and $x_\alpha \in \underline{C_B}$, this implies that $\underline{C_A} \subseteq \underline{C_B}$. Conversely, suppose that $\underline{C_A} \subseteq \underline{C_B}$. Let $x \in A$, then by lemma 2.4, $x_\alpha \in \underline{C_A}$ for any $\alpha > 0$, $x_\alpha \in \underline{C_B}$ and hence $x \in B$. (2) Let $x_\alpha \in \underline{C_A} \subseteq \underline{C_B}$, then lemma 2.5 implies that $A \subseteq B$ and from lemma 2.1, we have $C_A \subseteq C_B$. This completes the proof. ■

3 Main Results

Lemma 3.1: *Let A be a nonempty subset of a semigroup S. Then A is an ideal of S if and only if $\underline{C_A}$ is an ideal of $\underline{S}$.*

Proof: By lemma 2.2, A is an ideal of S if and only if C_A is a fuzzy ideal of S, and from lemma 3.1[2], C_A is a fuzzy ideal of S if and only if $\underline{C_A}$ is an ideal of $\underline{S}$. ■

Theorem 3.2: *Let A be a nonempty subset of a semigroup$S$. Then A is a minimal ideal of S if and only if $\underline{C_A}$ is a minimal ideal of $\underline{S}$.*

Proof: By theorem 7[1], A is a minimal ideal of S if and only if C_A is a fuzzy minimal ideal of S. We only need to prove that, C_A is a minimal fuzzy ideal of S if and only if $\underline{C_A}$ is a minimal ideal of $\underline{S}$. Let C_A be a minimal fuzzy ideal of S, then by lemma 3.1[2], $\underline{C_A}$ is an ideal of $\underline{S}$. Suppose that $\underline{C_A}$ is not minimal, then there exists some ideals $\underline{C_B}$ of $\underline{S}$ such that $\underline{C_B} \subseteq \underline{C_A}$. Hence by lemma 2.5,

$C_B \subseteq \underline{C_A}$, where C_B is a fuzzy ideal of S. This is a contradiction to C_A is a minimal fuzzy ideal of S. Thus $\underline{C_A}$ is a minimal ideal of $\underline{S}$. Conversely, assume that $\underline{C_A}$ is a minimal ideal of $\underline{S}$ and that C_A is not a minimal fuzzy ideal of S. Then there exists a fuzzy ideal C_B of S such that $\underline{C_B} \subseteq \underline{C_A}$. Now, lemma 2.5 implies that $\underline{C_B} \subseteq \underline{C_A}$, where $\underline{C_B}$ is an ideal of $\underline{S}$. This contradicts that $\underline{C_A}$ is a minimal ideal of $\underline{S}$. This completes the proof of the theorem. ■

Theorem 3.3: *Let A be a nonempty subset of a semigroup S. Then A is the kernel of S if and only if $\underline{C_A}$ is the kernel of $\underline{S}$.*

Proof: Suppose that A is the kernel of S, then $A = \bigcap_i I_i$, where I_i is an ideal of S. Let $\underline{C_B}$ be an ideal of $\underline{S}$, then by lemma 3.1, B is an ideal of S. Now we need to show that, $\underline{C_A} \subseteq \underline{C_B}$. Let $x_\alpha \in \underline{C_A}$, by lemma 2.4, $x \in A$ and also $x \in B$ since A is the kernel of S. This implies that $x_\alpha \in \underline{C_B}$ and hence, $\underline{C_A}$ is the kernel of $\underline{S}$. Conversely, Let $\underline{C_A}$ be the kernel of $\underline{S}$, then $\underline{C_A} \subseteq \underline{C_B}$, for every ideal $\underline{C_B}$ of $\underline{S}$. Thus $A \subseteq B$, that is, A is the kernel of S. ■

The following lemma weakens the condition of theorem 3.3.

Lemma 3.4: *Let A be a minimal ideal of a semigroup$S$, then $\underline{C_A}$ is the kernel of $\underline{S}$.*

Proof: Since A be a minimal ideal of S, then C_A is a minimal fuzzy ideal of S [1, theorem 7]. Also theorem 8 in [1] implies that C_A is the fuzzy kernel of S. Now, let C_B be a fuzzy ideal of S, then we have $C_A \subseteq C_B$. By lemma 2.5, $\underline{C_A} \subseteq \underline{C_B}$, so $\underline{C_A}$ is a minimal ideal contained in every ideal of $\underline{S}$. Thus $\underline{C_A}$ is the kernel of $\underline{S}$. ■

Lemma 3.5: *Let A be a nonempty subset of a semigroup S. Then A is an interior ideal of S if and only if $\underline{C_A}$ is an interior ideal of $\underline{S}$.*

Proof: Let A be an interior ideal of S, and let $y_\beta, z_\gamma \in \underline{S}$ and $x_\alpha \in \underline{C_A}$. Since $x \in A$, hence $y_\beta \circ x_\alpha \circ z_\gamma = (yxz)_{\beta\wedge\alpha\wedge\gamma} \in \underline{C_A}$. This implies that $\underline{S} \circ \underline{C_A} \circ \underline{S} \subseteq \underline{C_A}$, thus $\underline{C_A}$ is an interior ideal of $\underline{S}$. Conversely, suppose that $\underline{C_A}$ is an interior ideal of $\underline{S}$. Let $y, z \in S$ and $x \in A$, then $x_\alpha \in \underline{C_A}$. Assume that, $y_\alpha \circ x_\alpha \circ z_\alpha = (yxz)_\alpha \in \underline{S} \circ \underline{C_A} \circ \underline{S} \subseteq \underline{C_A}$, then $yxz \in A$. This implies that $SAS \subseteq A$, and hence A is an interior ideal of S. ■

Lemma 3.6: *Let A be a nonempty subset of a semigroup S. Then A is a bi- ideal of S if and only if $\underline{C_A}$ is a bi- ideal of $\underline{S}$.*

Proof: Let A be a bi- ideal of S, and let $y_\beta, z_\gamma \in \underline{C_A}$ and $x_\alpha \in \underline{S}$. Since $y, z \in A$ and $yxz \in A$ then $y_\beta \circ x_\alpha \circ z_\gamma = (yxz)_{\beta\wedge\alpha\wedge\gamma} \in \underline{C_A}$. This implies that $\underline{C_A} \circ \underline{S} \circ \underline{C_A} \subseteq \underline{C_A}$, thus $\underline{C_A}$ is a bi-ideal of $\underline{S}$. Conversely, suppose that $\underline{C_A}$ is a bi-ideal of $\underline{S}$. Let $y, z \in A$ and $x \in S$, then by lemma 2.4, $y_\alpha, z_\alpha \in \underline{C_A}$. Assume that, $y_\alpha \circ x_\alpha \circ z_\alpha = (yxz)_\alpha \in \underline{C_A} \circ \underline{S} \circ \underline{C_A} \subseteq \underline{C_A}$, then $yxz \in A$. This implies that $ASA \subseteq A$, and hence A is a bi- ideal of S . ■

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