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Weighted Composition Operators Induced by Operator Valued Maps On Spaces of Orlicz-Functions

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Abstract

In this paper we characterize the weighted composition operators induced by operator valued maps on Orlicz function spaces and effort is made to characterize compactness, invertibility, dense range and closed range of these operators.

Keywords: *Weighted composition operator, vector valued Orlicz space, invertible operator and compact operator.*

1 Introduction

Suppose H is a Hilbert space and (Ω, S, μ) be a σ-finite measure space. Let $L^{\phi}(\Omega, H) = \Big\{ f | f : \Omega \to H$ is measurable such that $\int_{\Omega} \phi\Big(\frac{||f(w)||}{\alpha}\Big) d\mu(w) < \infty$, for some $\alpha > 0 \Big\}$. Then $L^{\phi}(\Omega, H)$ is a Banach space under the norm,

$$||f||_{\phi} = \inf\{\alpha > 0 : \int_{\Omega} \phi(\frac{||f(w)||}{\alpha}) d\mu(w) \leq 1\}$$

where $\phi : [0, \infty) \to [0, \infty)$ be a continuous convex function which satisfy the following:
(i) $\phi(x) = 0$ if and only if x = 0,
(ii) $\lim\limits_{x \to \infty} \phi(x) = \infty$.

Such a function ϕ is known as a Young's function.
A mapping $u : \Omega \to B(H)$ is said to strongly measurable if $||u(.)x|| : \Omega \to C$ is measurable for each $x \in H$. Let $u : \Omega \to B(H)$ be an operator valued measurable function and $v : \Omega \to \Omega$ be a non-singular measurable transformation. Then a bounded linear transformation, $m_{u,v} : L^{\phi}(\Omega, H) \to L^{\phi}(\Omega, H)$ defined by

$$(m_{u,v}f)(x) = u(x).f(v(x))$$

is called a weighted composition operator induced by the pair (u, v). If we take $u(x) = 1$, the constant one function on Ω, we write $m_{u,v}$ as T_v and call it a composition operator induced by v. In case $v(x) = x$ for some $x \in \Omega$, we write $m_{u,v}$ as m_u and call it a multiplication operator induced by u. By $B(L^{\phi}(\Omega, H))$ we mean the set of all bounded linear operators from $L^{\phi}(\Omega, H)$ into itself.
If v is a non-singular measurable transformation, then the measure μv^{-1} is absolutely continuous with respect to the measure μ. Hence by Radon Nikodym derivative theorem there exists a positive measurable function w such that $\mu(v^{-1}(E)) = \int w d\mu$ for some $E \in S$. The function w is called the Radon Nikodym derivative of the measure μv^{-1} with respect to the measure μ. It is denoted by $w = \frac{d\mu v^{-1}}{d\mu}$.
If (Ω, S, μ) be a σ-finite measure space and $S_0 \subset S$ be a σ-finite subalgebra. Then the conditional expectation $E(.|S_0)$ is defined as a linear transformation from certain S-measurable function spaces ($i.e. L^1, L^2$ etc) into their S_o - measurable counterparts. In particular the conditional expectation with respect to the σ-algebra $v^{-1}(S)$ is a bounded projection from $L^p(\Omega, S, \mu)$ on $L^p(\Omega, v^{-1}(S), \mu)$. We denote this transformation by E. The transformation E has the following properties :
(i) $E(f.gov) = E(f).(gov)$
(ii) If $f \geq g$ almost everywhere, then $E(f) \geq E(g)$ almost everywhere
(iii) $E(1) = 1$
(iv) $E(f)$ has the form $E(f) = gov$ for exactly one σ- measurable function g.
(v) $|E(fg)|^2 \leq (E|f|^2)(E|g|^2)$
(vi) For $f > 0$ almost everywhere, $E(f) > 0$ almost everywhere.
(vi) If ϕ is a convex function, then $\phi(E(f)) \leq E(\phi(f))$.

For more details on Orlicz spaces one can refer to ([2], [8], [9], [10]), where as the classes of weighted composition operators on some function spaces are considered by ([1], [3], [4], [5], [6], [7], [11]). In this paper we plan to study the weighted composition operators induced by operator valued maps on Orlicz function spaces.

2 Weighted Composition Operators Induced by Operator Valued Maps

Theorem 2.1 *Let $u : \Omega \to B(H)$ be a strongly measurable operator valued map and let $v : \Omega \to \Omega$ be a non-singular measurable transformation. Then $m_{u,v} : L^{\phi}(\Omega, H) \to L^{\phi}(\Omega, H)$ is a bounded operator if and only if there exists a constant $M > 0$ such that*

$$w(x)E(\phi(||\ (u(.)ov^{-1})(x)|y|\ ||) \leq \phi(M||y||) \qquad ...(i)$$

for μ-almost all $x \in \Omega$ and $y \in H$.

Proof. Suppose the condition (i) is true. Then for every $f \in L^{\phi}(\Omega, H)$, we have

$$\begin{aligned}
\int_{\Omega} \phi\Big(\frac{||m_{u,v}f(.)||}{M||f(\cdot)||_{\phi}}\Big)d\mu &= \int_{\Omega} \phi\Big(\frac{||u(.)fov(.)||}{M||f(.)||_{\phi}}\Big)d\mu \\
&= \int_{\Omega} E\Big(\phi\Big(\frac{||u(.)ov^{-1}(.)f(.)||}{M||f(.)||_{\phi}}\Big)\Big)d\mu v^{-1} \\
&\leq \int_{\Omega} w(.)E\Big(\phi\Big(\frac{||u(.)ov^{-1}(.)f(.)||}{M||f(.)||_{\phi}}\Big)\Big)d\mu \\
&= \int_{\Omega} \phi\Big(\frac{M||f(.)||}{M||f(.)||_{\phi}}\Big)d\mu \\
&\leq 1
\end{aligned}$$

This implies that $||m_{u,v}f(.)||_{\phi} \leq M||f(.)||_{\phi}$ for every $f \in L^{\phi}(\Omega, H)$. Hence $m_{u,v}$ is a bounded operator.

Conversely, If the condition (i) of the theorem is not satisfied, then for every positive integer n, there exists a measurable set G_n of Ω and some vector $y_n \in H$ such that
$G_n = \Big\{x \in \Omega : w(x)E(\phi(||\ (u(.)ov^{-1})(x)y_n||)) > (M||y_n||)\Big\}$.
Since μ is non atomic, for every positive integer n we can find a measurable subset H_n of G_n such that $\mu(H_n) < \infty$ and $w(x)E(\phi(||\ (u(.)ov^{-1})(x)y_n||)) \geq \phi(M||y_n||)$ for μ-almost all $x \in H_n$. Let

$$f_n = \frac{\chi_{H_n}C_{y_n}(.)}{||m_{u,v}\chi_{H_n}C_{y_n}(.)||_{\phi}},$$

where $C_{y_n}(.) : \Omega \to H$ is the constant function equal to y_n, so that

$$\int_{\Omega} \phi\Big(\frac{n||\chi_{H_n}C_{y_n}(.)||}{||m_{u,v}\chi_{H_n}C_{y_n}(.)||_{\phi}}\Big)d\mu = \int_{\Omega} \phi\Big(\frac{n||y_n||}{||m_{u,v}\chi_{H_n}C_{y_n}(.)||_{\phi}}\Big)d\mu$$

$$\begin{aligned} &\leq \int_{\Omega} w(.)E\Big(\phi\Big(\frac{||u(.)ov^{-1}(.)y_n||}{||m_{u,v}\chi_{H_n}C_{y_n}(.)||_\phi}\Big)\Big)d\mu \\ &= \int_{\Omega} \phi\Big(\frac{||m_{u,v}\chi_{H_n}C_{y_n}(.)||}{||m_{u,v}\chi_{H_n}C_{y_n}(.)||_\phi}\Big)d\mu \\ &\leq 1 \end{aligned}$$

Hence

$$n||\chi_{H_n}C_{y_n}(.)||_\phi \leq ||m_{u,v}\chi_{H_n}C_{y_n}(.)||_\phi$$

for every n = 1, 2, 3, . . . , which shows that $m_{u,v}$is an unbounded operator. Hence the condition (i) must be true.

Theorem 2.2 *Let $m_{u,v} \in B(L^{\phi}(\Omega, H))$. Then $m_{u,v}$ has closed range if and only if there exists $\delta > 0$ such that*

$$w(x)E(\phi(||u(.)ov^{-1})(x)y||)) \geq \phi(\delta||y||) \qquad ...(ii)$$

for every $y \in H$ and for every measurable subsets G of Ω.

Proof. Suppose the condition (ii) is true. Let $m_{u,v}g^{(n)} \to f$ for some sequence $\{g^{(n)}\} \subset L^{\phi}(\Omega, H)$ and for some $f \in L^{\phi}(\Omega, H)$. Consider

$$\begin{aligned} \int_{\Omega} \phi\Big(\frac{\delta||g^{(n)}(x) - g^{(m)}(x)||}{||m_{u,v}(g^{(n)} - g^{(m)})||_\phi}\Big)d\mu &\leq \int_{\Omega} w(x)E\Big(\phi\Big(\frac{||(u(.)ov^{-1})(x)(g^{(n)}(x) - g^{(m)}(x))||}{||m_{u,v}g^{(n)} - m_{u,v}g^{(m)}||_\phi}\Big)\Big)d\mu \\ &\leq \int_{\Omega} \phi\Big(\frac{||m_{u,v}(g^{(n)}(x) - g^{(m)}(x))||}{||m_{u,v}g^{(n)} - m_{u,v}g^{(m)}||_\phi}\Big)d\mu \\ &\leq 1 \end{aligned}$$

which shows that $||g^{(n)}(.) - g^{(m)}(.)||_\phi \leq \frac{1}{\delta}||m_{u,v}g^{(n)} - m_{u,v}g^{(m)}||_\phi \to 0$ as $m, n \to \infty$. We conclude that $\{g^{(n)}(.)\}$ is a Cauchy sequence, which in view of completeness of $L^{\phi}(\Omega, H)$ yields $g(.) \in L^{\phi}(\Omega, H)$ such that $g^{(n)}(.) \to g(.)$ in $L^{\phi}(\Omega, H)$. Hence $m_{u,v}g^{(n)} \to m_{u,v}g$. Thus $m_{u,v}g = f$.

Conversely, suppose $m_{u,v}$ has closed range. If the condition (ii) not satisfied, then for every positive integer n, we can find a measurable subset $G_n, 0 < \mu(G_n) < \infty$ and vector $y_n \in H$ such that

$$w(x)E(\phi(||(u(.)ov^{-1})(x)y_n||)) < \phi(\frac{1}{2^n}||y_n||).$$

Since μ is non-atomic, we can assume that the sequence $\{G_n\}$ is a sequence of pairwise disjoint measurable sets. Take

$$F = \sum_{n=1}^{\infty} \frac{w(x)\chi_{G_n}(.)E(u(.))ov^{-1}C_{y_n}(.)}{||\chi_{G_n}C_{y_n}(.)||_\phi}$$

Now

$$\begin{aligned}
\int \phi\Big(||F(.)||\Big)d\mu &= \sum_{n=1}^{\infty}\int_{G_n} w(x)E\Big(\phi\Big(\frac{||u(.)ov^{-1}\chi_{G_n}C_{y_n}(.)||}{||\chi_{G_n}C_{y_n}(.)||_\phi}\Big)\Big)d\mu \\
&< \sum_{n=1}^{\infty}\frac{1}{2^n}\int_{G_n}\phi\Big(\frac{||\chi_{G_n}C_{y_n}(.)||}{||\chi_{G_n}C_{y_n}(.)||_\phi}\Big)d\mu \\
&\leq \sum_{n=1}^{\infty}\frac{1}{2^n} \\
&< \infty.
\end{aligned}$$

Hence $F \in L^\phi(\Omega, H)$. Take

$$F_n = w(x)\sum_{k=1}^{n}\frac{\chi_{G_k}(.)C_{y_k}(.)}{||\chi_{G_k(.)}C_{y_k}(.)||_\phi}.$$

Consider

$$\begin{aligned}
||m_{u,v}F_n - F||_\phi &= \sum_{k>n}\int_{G_k} w(x)E\Big(\phi\Big(\frac{||u(.)ov^{-1}C_{y_k}(.)||}{||\chi_{G_k}(.)C_{y_k}(.)||_\phi}\Big)\Big)d\mu \\
&\leq \sum_{k>n}\int_{G_k} w(x)E\Big(\phi\Big(\frac{||u(.)ov^{-1}\chi_{G_k}C_{y_k}(.)||}{||\chi_{G_k}(.)C_{y_k}(.)||_\phi}\Big)\Big)d\mu \\
&= \sum_{k>n}\frac{1}{2^k} \\
&\to 0 \;\; as \;\; k \to \infty
\end{aligned}$$

Hence $F \in \overline{ranm_{u,v}}$. Therefore $F = m_{u,v}J$ for some $J \in L^\phi(\Omega, H)$. This implies that for every $x \in G_n$,

$$\frac{C_{y_k}(.)}{||\chi_{G_k}(.)C_{y_k}(.)||_\phi} = J(.)$$

This contradicts that $J \in L^\phi(\Omega, H)$. Hence the theorem.

Theorem 2.3 *Let $u : \Omega \to B(H)$ be a strongly measurable operator valued map and let $v : \Omega \to \Omega$ be a non-singular measurable transformation. Then $m_{u,v} : L^\phi(\Omega, H) \to L^\phi(\Omega, H)$ is compact if and only if $m_{u,v}$ is the zero operator.*

Proof. If possible, suppose $m_{u,v}$ is a non-zero operator on $L^\phi(\Omega, H)$. Then there exists $\epsilon > 0$ such that the set

$$G = \Big\{x : w(x)E(\phi(||(u(.)ov^{-1})(x)f(.)||)) \geq \epsilon\phi(||f(.)||)\Big\}$$

has non-zero measure for every $f(.) \in H$. The restriction $m_{u,v}|_{L^\phi(G)} : L^\phi(G) \to L^\phi(G)$ is invertible, where

$$L^\phi(G) = \Big\{ g \in L^\phi(\Omega, H) : g(x) = 0 \ \ for \ \ \ every \ \ x \notin G \Big\}.$$

Also $m_{u,v}|_{L^\phi(G)}$ has closed range. Thus $L^\phi(G)$ is infinite dimensional, which is a contradiction. Hence $m_{u,v}$ must be the zero operator.

Corollary 2.4 *Let $m_{u,v} \in B(L^\phi(\mu))$. Then $m_{u,v}$ is bounded away from zero if and only if there exists a constant $\delta > 0$ such that*

$$w(x)E(\phi(||(u(.)ov^{-1})(x)y||)) > \phi(\delta||y||)$$

for every $y \in H$ and for μ-almost all $x \in \Omega$.

Corollary 2.5 *Let $m_{u,v} \in B(L^\phi(\mu))$. Then $m_{u,v}$ is injection if and only if u is non-zero a.e. and v is surjection.*

Corollary 2.6 *Let $m_{u,v} \in B(L^\phi(\mu))$. Then $m_{u,v}$ has dense range if and only if u is non-zero and $v^{-1}(S) = S$.*

Theorem 2.7 *Let $u : \Omega \to B(H)$ be a strongly measurable operator valued map and let $v : \Omega \to \Omega$ be a non singular measurable transformation. Then $m_{u,v} : L^\phi(\Omega, H) \to L^\phi(\Omega, H)$ is invertible if and only if*

1. *u is non zero, a.e.*
2. *$w(.)E(\phi(||(u(.)ov^{-1})(.)y||)) \geq \phi(\delta||y||)$ for every $y \in H$ and μ-almost all $x \in \Omega$.*
3. *v is invertible.*

Proof. Assume that the conditions of the theorem are true, then by Corollaries 2.4 and 2.6, $m_{u,v}$ is bounded away from zero and has dense range. Hence $m_{u,v}$ is invertible.

Conversely, suppose $m_{u,v}$ is invertible. Then clearly u is non-zero a.e. and $v^{-1}(S) = S$. Also in view of the corollary 2.4,

$$w(.)E(\phi(||(u(.)ov^{-1})(.)y||)) > \phi(\delta||y||)$$

for every $y \in H$ and for μ-almost all $x \in \Omega$ and for some $\delta > 0$. Since $v^{-1}(S) = S$. So v is injective. It follows from corollary 2.5 that $w \neq 0$, a.e. So that v is surjective. Thus v is invertible.

References

[1] J.W. Carlson, Weighted composition operators on ℓ^2, *Ph.D Thesis*, Purdue University, (1985).

[2] Y. Cui, H. Hudzik, R. Kumar and L. Maligranda, Composition operators in Orlicz spaces, *J. Austral. Math. Soc.*, 76 (2004), 189-206.

[3] W.E. Horner and J.E. Jamison, Weighted composition operators on $C(X, E)$, *J. Operator Theory*, 19 (1998), 307-317.

[4] H. Kamowitz, Compact weighted endomorphism of C(X), *Proc. Amer. Math. Soc.*, 83 (1981), 517-521.

[5] B.S. Komal, K. Raj and S. Gupta, On operators of weighted substitution on the generalized spaces of entire functions-I math today, *Math. Today*, XV (1997), 3-10.

[6] B.S. Komal, K. Raj and S. Gupta, Isometric and Fredholm substitution operators on the generalized spaces of entire functions, *Far East Journal of Mathematical Sciences*, Special Volume, Part-I (1998), 25-35.

[7] B.S. Komal and S. Gupta, Plication operators between Orlicz spaces, *Integr. Eqns. Oper. Theory*, 41(2001), 324-330.

[8] R. Kumar, Composition operators on Orlicz spaces, *Integral Equations and Operators Theory*, 29 (1997), 17-22.

[9] M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Marcel-Dekker Inc. New York, (1991).

[10] H. Takagi, Fredholm weighted composition operators, *Integral Equations and Operator Theory*, 16 (1993), 267-276.