Gen. Math. Notes, Vol. 26, No. 1, January 2015, pp. 53-60
ISSN 2219-7184; *Copyright © ICSRS Publication, 2015*
www.i-csrs.org
Available free online at http://www.geman.in

A Note on Fuzzy Almost Resolvable Spaces

G. Thangaraj[1] and D. Vijayan[2]

[1]Department of Mathematics, Thiruvalluvar University
Vellore- 632 115, Tamilnadu, India
E-mail: g.thangaraj@rediffmail.com
[2]Department of Mathematics
Muthurangam Government Arts College (Autonomous)
Vellore- 632002, Tamilnadu, India
E-mail: jyoshijeev@gmail.com

(Received: 12-9-14 / Accepted: 10-11-14)

Abstract

In this paper we study the conditions under which a fuzzy topological space becomes a fuzzy almost resolvable space and the inter-relations between fuzzy almost resolvable, fuzzy almost irresolvable spaces, fuzzy submaximal spaces, fuzzy first category spaces, fuzzy Baire spaces, fuzzy weakly Volterra spaces are also investigated.

Keywords: *Fuzzy almost resolvable, fuzzy almost irresolvable, fuzzy submaximal, fuzzy first category, fuzzy second category, fuzzy weakly Volterra, fuzzy Baire.*

1 Introduction

In order to deal with uncertainties, the idea of fuzzy sets fuzzy set operations was introduced by L.A. Zadeh [16] in his classical paper in the year 1965, describing fuzziness mathematically for the first time. Among the first fields of Mathematics to be considered in the content of fuzzy sets was general topology. The concept of fuzzy topology was defined by C.L. Chang [3] in the year 1968. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. Today fuzzy topology has been firmly established as one of the basic disciplines of fuzzy mathematics. E. Hewitt [6] introduced the concepts of resolvability and irresolvability in topological spaces. A.G. El'kin [4] introduced open hereditarily irresolvable spaces in the classical topology. The concept of almost resolvable spaces was introduced by Richard Bolstein [7] as a generalization of resolvable spaces of E. Hewitt [6]. The concept of almost resolvable spaces in fuzzy setting was introduced and studied by G. Thangaraj and D. Vijayan [15]. In this paper several characterizations of fuzzy almost resolvable spaces are studiedand the inter-relations between fuzzy almost resolvable, fuzzy almost irresolvable spaces, fuzzy submaximal spaces, fuzzy first category spaces, fuzzy Baire spaces, fuzzy weakly Volterra spaces are also investigated.

2 Preliminaries

By a fuzzy topological space we shall mean a non-empty set X together with a fuzzy topology T (in the sense of Chang) and denote it by (X,T).

Definition 2.1: *Let λ and μ be any two fuzzy sets in (X,T). Then we define $\lambda \vee \mu$: $X \to [0, 1]$ as follows: $(\lambda \vee \mu)(x) = Max\{\lambda(x), \mu(x)\}$. Also we define $\lambda \wedge \mu$: $X \to [0, 1]$ as follows: $(\lambda \wedge \mu)(x) = Min\{\lambda(x), \mu(x)\}$.*

For a family $\{\lambda_i / i \in \mathrm{I}\}$ of fuzzy sets in (X,T), the union $= \vee_i (\lambda_i)$ and the intersection $\delta = \wedge_i (\lambda_i)$ are defined respectively as $\psi(x) = sup_i \{\lambda_i(x), x \in \mathrm{X}\}and\delta(x) = inf_i \{\lambda_i(x), x \in \mathrm{X}\}$.

Definition 2.2: *Let (X,T) be a fuzzy topological space andλ be any fuzzy set in (X,T). We define the closure and the interior of λ as follows:*

(i) $\mathrm{Int}(\lambda) = \vee \{\mu / \mu \leq \lambda, \mu \in \mathrm{T}\}$,
(ii) $\mathrm{Cl}(\lambda) = \wedge \{/ \lambda \leq \mu, 1-\mu \in \mathrm{T}\}$.

Lemma 2.1[1]: *For a fuzzy setλ of a fuzzy topological space X,*

(i) $1 - \mathrm{Int}(\lambda) = \mathrm{Cl}(1-\lambda)$,
(ii) $1 - \mathrm{Cl}(\lambda) = \mathrm{Int}(1-\lambda)$.

Definition 2.3 [8]: *A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy dense if there exists no fuzzy closed setμin (X,T) such that $\lambda < \mu < 1$.*

Definition 2.4 [8]: *A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X,T) such that $\mu < cl(\lambda)$. That is, int cl(λ) = 0.*

Definition 2.5 [2]: *A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy F_σ-set in (X,T) if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where $(1 - \lambda_i) \in T$ for $i \in I$.*

Definition 2.6 [2]: *A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy G_δ-set in (X, T) if $\lambda = \bigwedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in T$ for $i \in I$.*

Definition 2.7 [16]: *A fuzzy topological space (X,T) is called an fuzzy open hereditarily irresolvable space if int cl (λ) $\neq 0$, then int (λ) $\neq 0$, for any non-zero fuzzy set λ in (X,T).*

Definition 2.8 [2]: *A fuzzy topological space (X,T) is called a fuzzy submaximal space if cl(λ)=1 for any non-zero fuzzy set λ in (X,T), then $\lambda \in T$.*

Definition 2.9 [9]: *A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy first category if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where are fuzzy nowhere densesetsin (X,T). Any other fuzzy set in (X,T) is said to be of second category.*

Definition 2.10 [8]: *A fuzzy topological space (X,T) is called fuzzy first category if $\bigvee_{i=1}^{\infty}(\lambda_i) = 1$ where (λ_i)'s are fuzzy nowhere dense sets in (X,T). A topological space which is not of fuzzy first category, is said to be of fuzzy second category.*

Lemma 2.2 [1]: *For a family of $\mathscr{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy topological space (X,T), $\vee$ cl (λ_α) $\leq$ cl ($\vee(\lambda_\alpha)$). In case $\mathscr{A}$ is a finite set, $\vee$ cl (λ_α) = cl ($\vee(\lambda_\alpha)$). Also $\vee$int(λ_α) $\leq$ int ($\vee(\lambda_\alpha)$).*

Definition 2.11 [12]: *Let (X,T) be a fuzzy topological space. A fuzzy set λ in (X,T) is called a fuzzy σ-nowhere dense set, if λ is a fuzzy F_σ- set in (X,T) such that int(λ) = 0.*

Definition 2.12 [12]: *A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy σ- first category set if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where the fuzzy sets ($\lambda_i$)'s are fuzzy σ – nowhere dense sets in (X,T). Any other fuzzy set in (X,T) is said to be of fuzzy σ –second category.*

3 Fuzzy Almost Resolvable Spaces

Definition 3.1 [15]: *A fuzzy topological space (X,T) is called a fuzzy almost resolvable space if $\bigvee_{i=1}^{\infty}(\lambda_i) = 1$, where the fuzzy sets ($\lambda_i$)'s in (X,T) are such that int (λ_i) = 0. Otherwise (X, T) is called a fuzzy almost irresolvable space.*

Proposition 3.1: *If* $\wedge_{i=1}^{\infty}(\mu_i) = 0$, *where the fuzzy sets* $(\mu_i)'s$ *are fuzzy dense sets in a fuzzy topological space (X,T), then (X,T) is a fuzzy almost resolvable space.*

Proof: Suppose that $\wedge_{i=1}^{\infty}(\mu_i)) = 0$, where $cl(\mu_i) = 1$ in (X,T). Then we have $1-(\wedge_{i=1}^{\infty}(\mu_i)) = 1 - 0 = 1$, where $1-cl(\mu_i) = 0$. This implies that $\vee_{i=1}^{\infty}(1 - \mu_i) = 1$, where $int(1 - \mu_i) = 0$. Let $(1 - \mu_i) = \lambda_i$. Then, we have $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where $int(\lambda_i) = 0$, in (X,T). Hence (X,T) is a fuzzy almost resolvable space.

Definition 3.2 [5]: *A fuzzy topological space (X,T) is called a fuzzy hyper connected space if every fuzzy open set is fuzzy dense in (X,T). That is, cl* $(\mu_i) = 1$, *for all* $\mu_i \in T$.

Proposition 3.2: *If* $\wedge_{i=1}^{\infty}(\mu_i) = 0$, *where the fuzzy sets* $(\mu_i)'s$ *are fuzzy open sets in a fuzzy hyper connected space (X,T), then (X,T) is a fuzzy almost resolvable space.*

Proof: Suppose that $(\wedge_{i=1}^{\infty}(\mu_i)) = 0$, where $\mu_i \in T$. Since the fuzzy topological space (X,T) is a fuzzy hyper connected space, the fuzzy open set μ_i is a fuzzy dense set in (X,T) for each i. Hence we have $\wedge_{i=1}^{\infty}(\mu_i)) = 0$, where $cl(\mu_i) = 1$ in (X, T). Then by proposition 3.1, (X,T) is a fuzzy almost resolvable space.

Proposition 3.3: *If* $\vee_{i=1}^{\infty}(\lambda_i) = 1$, *where the fuzzy sets* $(\lambda_i)'s$ *are fuzzy* σ*-nowhere dense sets in a fuzzy topological space (X,T), then (X,T) is a fuzzy almost resolvable space.*

Proof: Let $(\lambda_i)'s$ (i = 1 to ∞) be fuzzy σ-nowhere dense sets in (X,T). Then $(\lambda_i)'s$ are fuzzy F_σ-sets with $int(\lambda_i) = 0$. Now $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where $int(\lambda_i) = 0$, implies that (X,T) is a fuzzy almost resolvable space.

Definition 3.3 [14]: *A fuzzy topological space (X,T) is called fuzzy P-space, if countable intersection of fuzzy open sets in (X,T) is fuzzy open. That is, every non-zero fuzzy* G_δ*-set in (X,T), is fuzzy open in (X,T).*

Proposition 3.4: *If* $\wedge_{i=1}^{\infty}(\mu_i) = 0$, *where the fuzzy sets* $(\mu_i)'s$ *are fuzzy* G_δ*- sets in a fuzzy hyper connected and P-space, then (X,T) is a fuzzy almost resolvable space.*

Proof: Let $(\mu_i)'s$ (i = 1 to ∞) be fuzzy G_δ-sets in the fuzzy P-space (X,T). Then $(\mu_i)'s$ are fuzzy open sets in (X,T). Hence, we have $\wedge_{i=1}^{\infty}(\mu_i)) = 0$, where the fuzzy sets $(\mu_i)'s$ are fuzzy open sets in a fuzzy hyper connected space (X,T). Therefore, by proposition 3.2, (X,T) is a fuzzy almost resolvable space.

Proposition 3.5: *In a fuzzy almost resolvable space (X,T), if (λ_i)' s are fuzzy F_σ – sets, then (X,T) is a fuzzy σ- first category space.*

Proof: Let (X,T) be a fuzzy almost resolvable space. Then we have $\bigvee_{i=1}^{\infty}(\lambda_i) = 1$, where $\text{int}(\lambda_i) = 0$. Since (λ_i)'s are fuzzy F_σ-sets in (X,T) and $\text{int}(\lambda_i) = 0$, we have (X,T) is fuzzy σ-first category space.

Definition 3.4 [11]: *A fuzzy topological space (X, T) is called a fuzzy nodec space, if every non-zero fuzzy nowhere dense set in (X,T), is a fuzzy closed set in (X,T).*

Proposition 3.6: *If (X,T) is a fuzzy first category space and fuzzy nodec space, then (X,T) is fuzzy almost resolvable space.*

Proof: Let (X,T) be a fuzzy first category space. Then we have $\bigvee_{i=1}^{\infty}(\lambda_i) = 1$, where the fuzzy sets (λ_i)' s are fuzzy nowhere dense sets in (X,T). Since (X,T) is a fuzzy nodec space, the fuzzy nowhere dense sets are fuzzy closed sets in (X,T). Hence (λ_i)' s are fuzzy closed sets in (X,T). That is, $\text{cl}(\lambda_i) = \lambda_i$. Now $\text{intcl}(\lambda_i) = 0$, implies that $\text{int}(\lambda_i) = 0$. Hence we have $\bigvee_{i=1}^{\infty}(\lambda_i) = 1$, where the fuzzy sets (λ_i)'s in (X,T) are such that $\text{int}(\lambda_i) = 0$. Hence (X,T) is a fuzzy almost resolvable space.

Proposition 3.7: *If the fuzzy topological space (X,T) is a fuzzy second category space, then (X,T) is a fuzzy almost irresolvable space.*

Proof: Let (X,T) be a fuzzy second category space. Then, $\bigvee_{i=1}^{\infty}(\lambda_i) \neq 1$, where the fuzzy sets (λ_i)'s are fuzzy nowhere dense sets in (X,T). That is, $\bigvee_{i=1}^{\infty}(\lambda_i) \neq 1$, where $\text{int cl}(\lambda_i) = 0$. Now $\text{int}(\lambda_i) \leq \text{intcl}(\lambda_i)$, implies that $\text{int}(\lambda_i) = 0$. Hence $\bigvee_{i=1}^{\infty}(\lambda_i) \neq 1$, where $\text{int}(\lambda_i) = 0$ and therefore (X,T) is a fuzzy almost irresolvable space.

Definition 3.5 [13]: *A fuzzy topological space (X,T) is called a fuzzy Volterra space if $cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1$, where (λ_i)' s are fuzzy dense and fuzzy G_δ-sets in (X,T).*

Definition 3.6 [13]: *A fuzzy topological space (X,T) is called a fuzzy weakly Volterra space if $cl(\bigwedge_{i=1}^{N}(\lambda_i)) \neq 0$, where (λ_i)' s are fuzzy dense and fuzzy $G_\delta$sets in (X,T).*

Proposition 3.8: *If a fuzzy topological space (X,T) is not a fuzzy weakly Volterra space, then (X,T) is a fuzzy almost resolvable space.*

Proof: Let (X,T) be a fuzzy non-weakly Volterra space. Then, we have $\text{cl}(\bigwedge_{i=1}^{N}(\lambda_i)) = 0$, where (λ_i)' s are fuzzy dense and fuzzy G_δ-sets in (X,T).

Now $cl(\wedge_{i=1}^{N} (\lambda_i)) = 0$, implies that int $(\vee_{i=1}^{N}(1-\lambda_i) = 1$ and cl $(\lambda_i) = 1$, implies that int $(1-\lambda_i) = 0$. Let (μ_j)'s (j = 1 to ∞) be fuzzy sets in (*X*,*T*) such that int $(\mu_j) = 0$ and take the first N (μ_j)'s as $(1-\lambda_i)'s$. Now $\vee_{i=1}^{N}(1-\lambda_i) \leq \vee_{j=1}^{\infty} (\mu_j)$, implies that $\text{int}(\vee_{i=1}^{N}(1-\lambda_i)) \leq \text{int}\,(\vee_{j=1}^{\infty} (\mu_j)) \leq \vee_{j=1}^{\infty} (\mu_j)$. Then, we have $1 \leq (\vee_{j=1}^{\infty} (\mu_j))$. That is, $\vee_{j=1}^{\infty} (\mu_j)) = 1$, where the fuzzy sets (μ_j) 's in (*X*,*T*) are such that int $((\mu_j)) = 0$. Hence the fuzzy topological space (*X*,*T*) is a fuzzy almost resolvable space.

4 Inter-Relations between Fuzzy almost Resolvable, Fuzzy almost Irresolvable Spaces and Fuzzy First Category, Fuzzy Second Category Spaces, Fuzzy Baire Spaces

Proposition 4.1: *If the fuzzy almost resolvable space (X,T) is a fuzzy submaximal space, then (X,T) is a fuzzy first category space.*

Proof: Let (*X*,*T*) be a fuzzy almost resolvable space. Then $\vee_{i=1}^{\infty} (\lambda_i) = 1$, where the fuzzy sets (λ_i)'s in (*X*,*T*) are such that int $(\lambda_i) = 0$. Then we have $\wedge_{i=1}^{\infty} (1-\lambda_i) = 0$, where cl $(1-\lambda_i) = 1$. Since the space (*X*,*T*) is a fuzzy submaximal space, the fuzzy densesets $(1-\lambda_i)'s$ are fuzzy open sets in (*X*,*T*). Then (λ_i)'s are fuzzy closed sets in (*X*,*T*) and hence cl $(\lambda_i) = \lambda_i$. Now int $cl(\lambda_i) = \text{int}\,(\lambda_i) = 0$. Then (λ_i)'s are fuzzy nowhere dense sets in (*X*,*T*). Hence $\vee_{i=1}^{\infty} (\lambda_i) = 1$, where the fuzzy sets (λ_i)'s are fuzzy nowhere dense sets in (*X*,*T*) implies that (*X*,*T*) is a fuzzy first category space.

Remark: *In view of the above proposition, we have the following result. "If the fuzzy almost resolvable space (X,T) is a fuzzy submaximal space, then (X,T) is not a fuzzy second category space".*

Proposition 4.2: *If the fuzzy almost irresolvable space (X,T) is a fuzzy submaximal space, then (X,T) is a fuzzy second category space.*

Proof: Let (*X*,*T*) be a fuzzy almost irresolvable space. Then $\vee_{i=1}^{\infty} (\lambda_i) \neq 1$, where the fuzzy sets (λ_i)'s are such that $\text{int}(\lambda_i) = 0$. Now $\text{int}(\lambda_i) = 0$, implies that cl $(1-\lambda_i) = 1$. That is, $(1-\lambda_i)'s$ are fuzzy dense sets in (*X*,*T*). Since (*X*,*T*) is a fuzzy submaximal space, the fuzzy dense sets $(1-\lambda_i)'s$ are fuzzy open sets in (*X*,*T*). Then (λ_i)'s are fuzzy closed sets in (*X*,*T*). That is, cl $(\lambda_i) = \lambda_i$. Now $\text{int}(\lambda_i) = 0$, implies that int cl $(\lambda_i) = 0$. Then (λ_i)'s are fuzzy nowhere dense sets in (*X*,*T*). Hence we have $\vee_{i=1}^{\infty} (\lambda_i) \neq 1$, where the fuzzy sets (λ_i)'s are fuzzy nowhere dense sets in (*X*,*T*). Therefore (*X*,*T*) is a fuzzy second category space.

Definition 4.1 [10]: *A fuzzy topological space (X,T) is called a fuzzy Baire space if int* $[\vee_{i=1}^{\infty} (\lambda_i)] = 0$, *where* (λ_i)*'s are fuzzy nowhere dense sets in (X,T).*

Proposition 4.3: *If the fuzzy almost resolvable space (X,T) is a fuzzy submaximal space, then (X,T) is not a fuzzy Baire space.*

Proof: Let the fuzzy almost resolvable space (*X*,*T*) be a fuzzy submaximal space. Then, by proposition 4.1, (*X*,*T*) is a fuzzy first category space and hence $\vee_{i=1}^{\infty} (\lambda_i) = 1$, where the fuzzy sets (λ_i)' s are fuzzy nowhere dense sets in (*X*,*T*). Now $\mathrm{int}[\vee_{i=1}^{\infty} (\lambda_i)] = \mathrm{int}\,[1] = 1 \neq 0$. Hence (*X*,*T*) is not a fuzzy Baire space.

Theorem 4.1 [9]: *If the fuzzy topological space (X,T) is a fuzzy open hereditarily irresolvable space, then int(λ) = 0 for any non-zero fuzzy dense set λ in (X,T) implies that int cl(λ) = 0.*

Proposition 4.4: *If the fuzzy almost resolvable space (X,T) is a fuzzy open hereditarily irresolvable space, then (X,T) is not a fuzzy Baire space.*

Proof: Let (*X*,*T*) be a fuzzy almost resolvable space. Then, $\vee_{i=1}^{\infty} (\lambda_i) = 1$, where the fuzzy sets (λ_i)'s in (*X*,*T*) are such that int $(\lambda_i) = 0$. Since (*X*,*T*) is a fuzzy open hereditarily irresolvable space, int $(\lambda_i) = 0$, implies that intcl $(\lambda_i) = 0$. Now $\mathrm{int}[\vee_{i=1}^{\infty} (\lambda_i)] = \mathrm{int}\,[1] = 1 \neq 0$. Hence (*X*,*T*) is not a fuzzy Baire space.

Proposition 4.5: *If the fuzzy almost irresolvable space (X,T) is a fuzzy open hereditarily irresolvable space, then (X,T) is a fuzzy second category space.*

Proof: (*X*,*T*) is fuzzy almost irresolvable space. Then $\vee_{i=1}^{\infty} (\lambda_i) \neq 1$, where the fuzzy sets (λ_i)'s in (*X*,*T*) are such that int $(\lambda_i) = 0$. Since (*X*,*T*) is a fuzzy open hereditarily irresolvable space, int $(\lambda_i) = 0$, implies that intcl $(\lambda_i) = 0$. Then (λ_i)'s are fuzzy nowhere dense sets in (*X*,*T*). Hence$\vee_{i=1}^{\infty} (\lambda_i) \neq 1$, where the fuzzy sets (λ_i)'s fuzzy nowhere dense sets in (*X*,*T*), implies that (*X*,*T*) is a fuzzy second category space.

References

[1] K.K. Azad, On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity, *J. Math. Anal. Appl*, 82(1981), 14-32.

[2] G. Balasubramanian, Maximal fuzzy topologies, *Kybernetika*, 5(31) (1995), 459-464.

[3] C.L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, 24(1968), 182-190.

[4] A.G. El'kin, Ultra filters and undecomposable spaces, *Vestnik. Mosk. Univ. Mat.*, 24(5) (1969), 51-56.

[5] E. Ekici, Generalization of some fuzzy functions, *BullInsti. Math. Academia Sinica*, 33(3) (2005), 277-289.

[6] E. Hewitt, A problem in set theoretic topology, *Duke Math. J.*, 10(1943), 309-333.

[7] R. Bolstein, Sets of points of discontinuity, *Proc. Amer. Math. Society*, 38(1) (1973), 193-197.

[8] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, *J. Fuzzy Math.*, 11(2) (2003), 725-736.

[9] G. Thangaraj and G. Balasubramanian, On fuzzy resolvable and fuzzy irresolvable spaces, *Fuzzy Sets, Rough Sets and Multivalued Operations and Applications*, 1(2) (2009), 173-180.

[10] G. Thangaraj and S. Anjalmose, On fuzzy Baire spaces, *J. Fuzzy Math*, 21(3) (2013), 667-676.

[11] G. Thangaraj and S. Anjalmose, Some remarks on fuzzy Baire spaces, *Scientia Magna.*, 9(1) (2013), 1-7.

[12] G. Thangaraj and E. Poongothai, On fuzzy σ-Baire spaces, *Int. J. Fuzzy Math. Sys*, 3(4) (2013), 275-283.

[13] G. Thangaraj and S. Soundarajan, On fuzzy Volterra spaces, *J. Fuzzy Math.*, 21(4) (2013), 895-904.

[14] G. Thangaraj, C. Anbazhagan and P. Vivakanandan, On fuzzy P-spaces, weak fuzzy P-spaces and fuzzy almost P-spaces, *Gen. Math. Notes*, 18(2) (2013), 128-139.

[15] G. Thangaraj and D. Vijayan, On fuzzy almost resolvable and fuzzy almost irresolvable spaces, *Int. J. Stat. Math.*, 9(2) (2014), 61-65.

[16] L.A. Zadeh, Fuzzy sets, *Information and Control,* 8(1965), 338-353.