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Strongly Regular Graphs Arising From Balanced Incomplete Block Design With $\lambda = 1$

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Abstract

In [M. Klin, A. Munemusa, M. Muzychuk, P.-H. Zieschang Directed strongly regular graphs obtained from coherent algebras. Linear Algebra and its Applications 337, (2004) 83-109] the flag algebra of a given balanced incomplete block design with parameters (ν, b, r, k, λ) where $\lambda = 1$, has been constructed. In this paper, we consider the association scheme which is related to this flag algebra. By quotient scheme of this association scheme, we construct a strongly regular graph which its parameters are related to the parameters of given balanced incomplete block design. The parameters of this strongly regular graph are

$$(\frac{kr^2 - r^2 + r}{k}, k(r-1), r - 2 + (k-1)^2, k^2).$$

Keywords: *Association scheme, strongly regular graph, balanced incomplete block design.*

1 Introduction

A balanced incomplete block design [6] with parameter (ν, b, r, k, λ) denoted by (ν, b, r, k, λ)-BIBD is an incidence structure $S = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ and $\mathcal{B}$ are called the set of points and blocks, respectively, with the following properties: $|\mathcal{P}| = \nu$ and $|\mathcal{B}| = b$; each block contains exactly k points; every pair of distinct points is contained in exactly λ blocks. It is well known that in a (ν, b, r, k, λ)-BIBD every point occurs in exactly $r = \lambda(\nu - 1)/(k - 1)$ blocks and it has exactly $b = \nu r/k = \lambda(\nu^2 - \nu)/(k^2 - k)$ blocks.

Let $S = (\mathcal{P}, \mathcal{B})$ be a (ν, b, r, k, λ)-BIBD with $\lambda = 1$. Set $x = k-1, y = r-1$. A straightforward computation shows that

$$\begin{cases} \nu = 1 + x + xy, & b = \frac{(1+x+xy)(y+1)}{x+1}, \\ r = y + 1, & k = x + 1. \end{cases} \tag{1}$$

A strongly regular graph [1] with parameters (n, m, a, c) is a m-regular graph with n vertices in which two adjacent vertices have a common neighbours, and two non-adjacent vertices have c common neighbours. This graph is denoted by $\mathrm{srg}(n, m, a, c)$. The parameters of a strongly regular graph satisfy the equation

$$m(m - a - 1) = (n - m - 1)c. \tag{2}$$

A complete characterization of the parameter sets of strongly regular graphs is not known. Note that the complement of a strongly regular graph is also a strongly regular graph.

1.1 Association Scheme

We prepare some notation and results in association schemes which will be used through the paper and we refer the reader to [4, 7] for more details.

Given a finite and non-empty set V, a d-class association scheme (briefly d-class scheme) on V is a pair $\mathcal{C} = (V, \mathcal{R})$, where $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ is a set of non-empty binary relations on V, which satisfies the following conditions.

(1) $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ is a partition of $V \times V$;

(2) The subset $1_V = \{(v, v) : v \in V\}$ is an element of $\mathcal{R}$, say R_0;

(3) For each $R_i \in \mathcal{R}$, the set $R_i^{\mathrm{t}} := \{(v, u) : (u, v) \in R\}$ is in $\mathcal{R}$, denote R_i^{t} by $R_{i'}$;

(4) For each triple $R_i, R_j, R_k \in \mathcal{R}$ there exists an *intersection number* p_{ij}^k such that $p_{ij}^k = |R_i(u) \cap R_{j'}(v)|$ for all $(u, v) \in R_k$, where $R(u)$ is the set of all elements $v \in V$ with $(u, v) \in R$ for each $R \in \mathcal{R}$.

The elements of V are called *points* and those of $\mathcal{R}$ are called *basis relations* of $\mathcal{C}$. The numbers $|V|$ and $|\mathcal{R}|$ are called the *degree* and the *rank* of $\mathcal{C}$, and are denoted by $\deg(\mathcal{C})$ and $\mathrm{rk}(\mathcal{C})$, respectively.

Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme. An equivalence relation E on V is called an *equivalence* of $\mathcal{C}$ if E is a union of some basis relations of $\mathcal{C}$. Denote by $\mathcal{E}(\mathcal{C})$ the set of all equivalences of $\mathcal{C}$. For each $E \in \mathcal{E}(\mathcal{C})$ we define degree of E, $d(E)$, the sum of valency of all basis relations of $\mathcal{C}$ which lie in E.

Let X be a non-empty subset of V and let $E \in \mathcal{E}(\mathcal{C})$. Denote by X/E the set of classes of the equivalence relation $E \cap (X \times X)$, and $R_{X/E}$ the set of pairs (Y, Z) in $(X/E) \times (X/E)$ such that $R_{Y,Z} \neq \emptyset$, where $R_{Y,Z} = R \cap (Y \times Z)$. Also denote by $\mathcal{R}_{X/E}$ the set of all non-empty relations $R_{X/E}$ on X/E where $R \in \mathcal{R}$. Then the pair

$$\mathcal{C}_{X/E} = (X/E, \mathcal{R}_{X/E})$$

is a scheme, called the *quotient scheme*. If $X \in V/E$, then the pair $\mathcal{C}_X = (X, \mathcal{R}_X)$ is a scheme.

In fact any 2-class association scheme is equivalent to a pair of complementary strongly regular graphs. The relation between association scheme and block design have been studied in [2, 3, 5].

1.2 The Flag Algebra of BIBD with $\lambda = 1$

In [3] the flag algebra of a BIBD with $\lambda = 1$ has been studied. Equivalently a 5-class association scheme constructed from a given BIBD.

For a BIBD, $S = (\mathcal{P}, \mathcal{B})$ with $\lambda = 1$, ν and b can be computed from k and r. As defined in [3], let $\mathcal{F}$ denote the set of incident point-block pairs. The elements of $\mathcal{F}$ are called flags. Set $n = |\mathcal{F}|$. Then

$$n = (1 + x + xy)(1 + y). \tag{3}$$

Let Consider the following binary relations on $\mathcal{F}$,

$$\begin{aligned}
&R_0 = \{(f, f) : f \in \mathcal{F}\},\\
&R_1 = \{(f, g) : p \neq q, C = D\},\\
&R_2 = \{(f, g) : p = q, C \neq D\},\\
&R_3 = \{(f, g) : p \neq q, C \neq D, (q, C) \in \mathcal{F}\},\\
&R_4 = \{(f, g) : p \neq q, C \neq D, (p, D) \in \mathcal{F}\},\\
&R_5 = \{(f, g) : p \neq q, C \neq D, (q, C) \notin \mathcal{F}, (p, D) \notin \mathcal{F}, C \cap D \neq \emptyset\},\\
&R_6 = \{(f, g) : p \neq q, C \neq D, (q, C) \notin \mathcal{F}, (p, D) \notin \mathcal{F}, C \cap D = \emptyset\}.
\end{aligned} \tag{4}$$

Where $f = (p, C)$ and $g = (q, D)$.

Lemma 1.1 *[3] Assume that S is a BIBD with $\lambda = 1$. Then, for each $0 \le i \le 6$, R_i defines a regular graph $\Gamma_i = (\mathcal{F}, R_i)$. Moreover, the valencies n_i of Γ_i are as follows:*

$$\begin{aligned} &n_0 = 1, \quad n_1 = x, \quad n_2 = y, \quad n_3 = n_4 = xy, \\ &n_5 = x^2 y, \quad n_6 = xy(y - x). \end{aligned} \tag{5}$$

The relations R_1, R_2, R_5, and R_6 are symmetric relations and the relations R_3 and R_4 are paired antisymmetric.

Theorem 1.2 *[3] Let $\mathcal{R} = \{R_0, \ldots, R_6\}$ as defined in (4), then $\mathcal{C} = (\mathcal{F}, \mathcal{R})$ is an association scheme.*

In this paper we consider an equivalence of this scheme, and show that its quotient scheme is a 2-class association scheme. As a result we construct a strongly regular graph whose parameters are related to BIBD's parameters.

For example , it is well-known that the existence of srg$(69, 48, 32, 36)$ and srg$(85, 54, 33, 36)$ are unknown. By our results, the existence of $(46, 69, 9, 6, 1)$-BIBD and $(51, 85, 10, 6, 1)$-BIBD imply the existence of srg$(69, 48, 32, 36)$ and srg$(85, 54, 33, 36)$, respectively.

2 Main Results

In this section we suppose that $\mathcal{C} = (\mathcal{F}, \mathcal{R})$ is the scheme which defined in Theorem 1.2 from a $(\nu, b, r, k, 1)$-BIBD.

Lemma 2.1 *Let $E = R_0 \cup R_1$, then $E \in \mathcal{E}(\mathcal{C})$ and $d(E) = k$.*

Proof. From the definition of R_0 and R_1 in (4), it is easy to see that $E \in \mathcal{E}(\mathcal{C})$. The degree of E is the sum of valency of its basis relations, and by Lemma 1.1, we have $n_0 = 1$ and $n_1 = x$ where $x = k - 1$. Thus, $d(E) = k$. □

Proposition 2.2 *Let $X, Y \in \mathcal{F}/E$ such that $(X, Y) \in (R_2)_{\mathcal{F}/E}$, where $E = R_0 \cup R_1$. Then there is a unique $f \in X$ and a unique $g \in Y$, such that $(f, g) \in R_2$.*

Proof. By Lemma 2.1, $E \in \mathcal{E}(\mathcal{C})$ and $d(E) = k$, so each of its classes have the same size k. Suppose that $X = \{f_1, \ldots, f_k\}$ and $Y = \{g_1, \ldots, g_k\}$, where $f_i = (p_i, C_i)$ and $g_i = (q_i, D_i)$ for each $1 \le i \le k$. Since $(X, Y) \in (R_2)_{\mathcal{F}/E}$, hence there exist $1 \le t, t' \le k$ such that $(f_t, g_{t'}) \in R_2$.

We now prove uniqueness. By definition,

$$R_2 = \{(f, g) : p = q, C \ne D\}, \tag{6}$$

thus $p_t = q_{t'}$ and $C_t \neq D_{t'}$. It follows that

$$p_t \in C_t \cap D_{t'}. \tag{7}$$

Moreover, for each $i \neq t$, we have $(f_t, f_i) \in X^2$. Thus by definition of E, $(f_t, f_i) \in R_1$. Since

$$R_1 = \{(f, g) : p \neq q, C = D\},$$

we have $p_i \neq p_t$ and $C_i = C_t$ for each $i \neq t$. In the same way, $q_i \neq q_{t'}$ and $D_i = D_{t'}$ for each $i \neq t'$. It follows that $p_t \neq q_i$ and $C_t \neq D_i$ for each $i \neq t'$, and so by (6), we have

$$(f_t, g_i) \notin R_2 \qquad \text{for} \qquad i \neq t'.$$

Also we see that $q_{t'} \neq p_i$ and $D_{t'} \neq C_i$ for each $i \neq t$, and so by (6), we have

$$(f_i, g_{t'}) \notin R_2 \qquad \text{for} \qquad i \neq t.$$

It is sufficient to show that $(f_i, g_j) \notin R_2$ for $1 \leq i, j \leq k$, $i \neq t$ and $j \neq t'$. It is easy to see that $C_i \neq D_j$, thus by (6), it is equivalent to show that $p_i \neq q_j$.

Assume, on the contrary, that $p_i = q_j$. It follows that $p_i \in C_i \cap D_j$. Moreover, we have $C_i = C_t$ and $D_j = D_{t'}$, thus $p_i \in C_t \cap D_{t'}$. On the other hand, by (7) we have $p_t, p_i \in C_t \cap D_{t'}$, which contradicts $\lambda = 1$ in BIBD, and the proof is completed. □

Theorem 2.3 *Let $E = R_0 \cup R_1$, then $\mathcal{C}_{\mathcal{F}/E} = (\mathcal{F}/E, \mathcal{R}_{\mathcal{F}/E})$ is a 2-class association scheme.*

Proof. Since $E = R_0 \cup R_1$, we have

$$(R_0)_{\mathcal{F}/E} = (R_1)_{\mathcal{F}/E} = 1_{\mathcal{F}/E}.$$

First, we show that

$$(R_2)_{\mathcal{F}/E} = (R_3)_{\mathcal{F}/E} = (R_4)_{\mathcal{F}/E} = (R_5)_{\mathcal{F}/E}, \tag{8}$$

and

$$(R_2)_{\mathcal{F}/E} \neq (R_6)_{\mathcal{F}/E}. \tag{9}$$

Let $(X, Y) \in (R_2)_{\mathcal{F}/E}$. Now suppose that $X = \{f_1, \ldots, f_k\}$ and $Y = \{g_1, \ldots, g_k\}$ where $f_i = (p_i, C_i)$ and $g_i = (q_i, D_i)$ for each $1 \leq i \leq k$.

Then by Proposition 2.2 there exist a unique $f_t \in X$ and a unique $g_{t'} \in Y$, $1 \leq t, t' \leq k$, such that $(f_t, g_{t'}) \in R_2$. The definition of R_2 shows that $p_t = q_{t'}$ and $C_t \neq D_{t'}$.

To prove (8), fix $j \neq t$, $1 \leq j \leq k$, then by definition of R_1 we have $p_j \neq p_t$ and $C_j = C_t$. It follows that $p_j \neq q_{t'}$ and $C_j \neq D_{t'}$. Moreover, $q_{t'} = p_t \in C_t$

and $C_t = C_j$, so $q_{t'} \in C_j$. Thus by definition of R_3 we have $(f_j, g_{t'}) \in R_3$. Therefore, $(X, Y) \in (R_3)_{\mathcal{F}/E}$ and so $(R_2)_{\mathcal{F}/E} = (R_3)_{\mathcal{F}/E}$. In the same way we prove that $(R_2)_{\mathcal{F}/E} = (R_4)_{\mathcal{F}/E}$ and also $(R_2)_{\mathcal{F}/E} = (R_5)_{\mathcal{F}/E}$. The proof of (8) is completed.

Now, to prove (9) it is sufficient to show that $(X \times Y) \cap R_6$ is an empty set. Equivalently it is sufficient to show that for each $1 \leq i, j \leq k$, we have $C_i \cap D_j \neq \emptyset$.

Since $f_i \in X$ by definition of R_1 we have

$$C_i = C_j \quad \text{and} \quad D_i = D_j, \quad \text{for each} \quad 1 \leq i, j \leq k. \tag{10}$$

On the other hand, $(f_t, g_{t'}) \in R_2$ follows that $p_t = q_{t'}$ belongs to the set $C_t \cap D_{t'}$. Thus $C_t \cap D_{t'} \neq \emptyset$ and so from (10) we have $C_i \cap D_j \neq \emptyset$.

Now, since $(R_0)_{\mathcal{F}/E}$, $(R_2)_{\mathcal{F}/E}$ and $(R_6)_{\mathcal{F}/E}$ are distinct. Thus $\mathcal{C}_{\mathcal{F}/E}$ is a 2-class association scheme and the proof is completed. $\square$

It is straightforward to check that any 2-class association scheme is corresponding to a strongly regular graph. Using notation of Theorem 2.3 we have the following theorem:

Theorem 2.4 *The graph $\Gamma = (\mathcal{F}/E, (R_2)_{\mathcal{F}/E})$ is a strongly regular graph with parameters*

$$\left(\frac{kr^2 - r^2 + r}{k}, k(r-1), r - 2 + (k-1)^2, k^2\right).$$

Proof. By Theorem 2.3, $\mathcal{C}_{\mathcal{F}/E} = (\mathcal{F}/E, \mathcal{R}_{\mathcal{F}/E})$ is a 2-class association scheme. Thus the graph $\Gamma = (\mathcal{F}/E, (R_2)_{\mathcal{F}/E})$ is a strongly regular graph.

Now, we compute the parameters of this strongly regular graph. By Lemma 2.1, $d(E) = k$. Thus using (3) we have

$$|\mathcal{F}/E| = \frac{kr^2 - r^2 + r}{k}.$$

Since, $d(R_2) = r - 1$ and by Proposition 2.2 the relation R_2 contains one element in each class of equivalence E and we know that $d(E) = k$, thus

$$d((R_2)_{\mathcal{F}/E}) = k(r-1).$$

Therefore, Γ is $k(r-1)$-regular graph with $\frac{kr^2-r^2+r}{k}$ vertices. Now, for this strongly regular graph we compute the number of common neighbors of two adjacent vertices and two non-adjacent vertices, a_Γ and c_Γ, respectively.

Let $X, Y \in \mathcal{F}/E$ such that $(X, Y) \in (R_2)_{\mathcal{F}/E}$. Let $X = \{f_1, \ldots, f_k\}$ and $Y = \{g_1, \ldots, g_k\}$ where $f_i = (p_i, C_i)$ and $g_i = (q_i, D_i)$ for each $1 \leq i \leq k$. Then by Proposition 2.2 there exist a unique $f_t \in X$ and a unique $g_{t'} \in Y$,

$1 \leq t, t' \leq k$, such that $(f_t, g_{t'}) \in R_2$. Since, $d(R_2) = r - 1$ thus there are the flags h_j, $1 \leq j \leq r-2$, such that $(f_t, h_j) \in R_2$ and each h_j is in different class. In this case it is easy to see that $(h_j, g_{t'}) \in R_2$. Moreover, for each p_i and for each q_j, $1 \leq t, t' \leq k$, $i \neq t$ and $j \neq t'$, by the definition of BIBD, since $\lambda = 1$ there is a unique block $T_{ij} \in \mathcal{B}$ such that p_i and q_j are points of the block T_{ij}. Thus, in this case the flags (p_i, T_{ij}) and (q_j, T_{ij}) are in the same class of the equivalence E. Also by definition of R_2 we have $(f_i, (p_i, T_{ij})) \in R_2$ and $((q_j, T_{ij}), g_j) \in R_2$. It follows that $(X, Z_{ij}) \in (R_2)_{\mathcal{F}/E}$ and $(Z_{ij}, Y) \in (R_2)_{\mathcal{F}/E}$, where Z_{ij} is the class of E which contains the flags of the block T_{ij}. Therefore, the class Z_{ij} is a common neighbor of X and Y.

On the other hand, if there is a class which is a common neighbor of X and Y then it contains the flags with point p_t or it contains flags of a point of block C_i and a point of block D_i. Thus we have

$$\begin{aligned} a_\Gamma &= |\{h_j : 1 \leq j \leq r-2\}| + |\{Z_{ij} : 1 \leq t, t' \leq k,\ i \neq t,\ j \neq t'\}| \\ &= (r-2) + (k-1)^2. \end{aligned}$$

By using (2), we have $c_\Gamma = k^2$, as desired. □

3 Conclusion

In this paper, we consider the association scheme which is related to the flag algebra of a BIBD, with $\lambda = 1$. By finding a suitable equivalence of this scheme, we construct a 2-class association scheme. Moreover, each 2-class association scheme is equivalent to a strongly regular graph. By these results existence of some unknown strongly regular graphs are equivalent to existence of some special BIBDs.

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