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New Theorems for Absolute Matrix Summability Factors

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Abstract

In this paper, we have given two theorems for $|A, p_n; \delta|_k$ summability which generalize recent theorems on $|A, p_n|_k$ summability. Study also reveals many factor theorems for other summability methods.

Keywords: *Absolute matrix summability, quasi power increasing sequences, infinite series.*

1 Introduction

A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that $Kn^\beta \gamma_n \geq m^\beta \gamma_m$ holds for all $n \geq m \geq 1$ (see [4]). A sequence (λ_n) is said to be of bounded variation, denote by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta \lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.
Let $\sum a_n$ be a given infinite series with the partial sums (s_n) and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\Delta} A_n(s)|^k < \infty, \tag{2}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{3}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\bar{\Delta} A_n(s)|^k < \infty, \tag{4}$$

The series $\sum a_n$ is said to be summable $|A, \delta|_k$, $k \geq 1$, if (see [7])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |\bar{\Delta} A_n(s)|^k < \infty \tag{5}$$

and it is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |\bar{\Delta} A_n(s)|^k < \infty. \tag{6}$$

In the special case when $p_n = 1$, $|A, p_n; \delta|_k$ summability is the same as $|A, \delta|_k$ summability. Also if we take $\delta = 0$, then $|A, p_n; \delta|_k$ summability is the same as $|A, p_n|_k$ summability. Finally, when $a_{nv} = \frac{p_v}{P_n}$ the method reduces to $\left|\bar{N}, p_n; \delta\right|_k$ summability method (see [3]) and when $a_{nv} = \frac{p_v}{P_n}, \quad \delta = 0$ it reduces to $\left|\bar{N}, p_n\right|_k$ summability method (see [1]).

Now, we will introduce some further notations necessary for our main theorems.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{7}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{8}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{9}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{10}$$

2 Main Result

In [6], Özarslan and Yavuz have proved two theorems for $|A, p_n|_k$ summability method by using quasi β-power increasing sequences. The aim of this paper is to generalize their theorems to $|A, p_n; \delta|_k$ summability. Now, we state our main theorems.

Theorem 2.1 *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, ... \tag{11}$$

$$a_{n-1,v} \geq a_{nv}, \quad for \quad n \geq v + 1, \tag{12}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{13}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v \hat{a}_{nv}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k-1}\right\}, \tag{14}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\right\} \tag{15}$$

and let there be sequences (β_n) and (λ_n) such that

$$(\lambda_n) \in \mathcal{BV}, \tag{16}$$

$$|\Delta \lambda_n| \leq \beta_n, \tag{17}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{18}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{19}$$

$$|\lambda_n|X_n = O(1) \quad as \quad n \to \infty, \tag{20}$$

where (X_n) *is a quasi* β*-power increasing sequence for some* $0 < \beta < 1$.
If

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|s_v|^k}{v} = O(X_n), \tag{21}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |s_n|^k = O(X_m), \quad m \to \infty, \tag{22}$$

then $\sum a_n\lambda_n$ *is summable* $\left|A, p_n; \delta\right|_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.

Theorem 2.2 *Let conditions (11)–(20) and (22) of Theorem 2.1 be satisfied. If*

$$\sum_{n=1}^{\infty} P_n|\Delta\beta_n|X_n < \infty, \tag{23}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{|s_n|^k}{P_n} = O(X_m), \tag{24}$$

then $\sum a_n\lambda_n$ *is summable* $\left|A, p_n; \delta\right|_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.

We need following lemmas for the proof of our theorems.

Lemma 2.3 *(see [4]). Let* (X_n) *be a quasi* β*-power increasing sequence for some* $0 < \beta < 1$. *If conditions (18) and (19) are satisfied, then*

$$nX_n\beta_n = O(1) \quad as \quad n \to \infty, \tag{25}$$

$$\sum_{n=1}^{\infty} X_n\beta_n < \infty. \tag{26}$$

Lemma 2.4 *Let* (X_n) *be a quasi* β*-power increasing sequence for some* $0 < \beta < 1$. *If conditions (18) and (23) are satisfied, then*

$$P_n\beta_nX_n = O(1), \tag{27}$$

$$\sum_{n=1}^{\infty} p_n\beta_nX_n < \infty. \tag{28}$$

The proof of Lemma 2.4 is similar to that of Bor in [2] and hence omitted.

3 Proof of Theorem 2.1

Let (T_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, by (9), (10) and applying Abel's transformation we have

$$\begin{aligned}
\bar{\Delta}T_n &= \sum_{v=1}^{n} \hat{a}_{nv}a_v\lambda_v \\
&= \sum_{v=1}^{n-1} \Delta_v\left(\hat{a}_{nv}\lambda_v\right)\sum_{k=1}^{v} a_k + \hat{a}_{nn}\lambda_n\sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \left(\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1}\right)s_v + a_{nn}\lambda_n s_n \\
&= \sum_{v=1}^{n-1} \left(\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1} - \hat{a}_{n,v+1}\lambda_v + \hat{a}_{n,v+1}\lambda_v\right)s_v + a_{nn}\lambda_n s_n \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv})\lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v s_v + a_{nn}\lambda_n s_n \\
&= T_{n,1} + T_{n,2} + T_{n,3} \quad say.
\end{aligned}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3}|^k \leq 3^k\left(|T_{n,1}|^k + T_{n,2}|^k + T_{n,3}|^k\right),$$

to complete the proof of Theorem 2.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k+k-1}|T_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3. \tag{29}$$

Firstly, applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,1}|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v||s_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v|^k|s_v|^k\right) \\
&\quad \times\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v|^k|s_v|^k\right) \\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v\hat{a}_{nv}|
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_v||s_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k-1}|s_r|^k\\
&\quad + O(1)|\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|s_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)|\lambda_m|X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.3. Since $(\lambda_n)\in\mathcal{BV}$ by (16), applying Hölder's inequality with the same indices above, we have

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,2}|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta\lambda_v||\hat{a}_{n,v+1}||s_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta\lambda_v||\hat{a}_{n,v+1}||s_v|^k\right)\\
&\quad \times\left(\sum_{v=1}^{n-1}|\Delta\lambda_v||\hat{a}_{n,v+1}|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\sum_{v=1}^{n-1}\beta_v|\hat{a}_{n,v+1}||s_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta\lambda_v|\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\beta_v|s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|s_v|^k\beta_v\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|s_v|^k}{v}(v\beta_v)\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{|s_r|^k}{r}\\
&\quad + O(1)m\beta_m\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|s_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)X_v + O(1)m\beta_m X_m
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1} v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} \\
&\quad + O(1)m\beta_m X_m \\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.3.
Finally, by following the similar process as that in $T_{n,1}$ we have that

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,3}|^k &\leq \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|a_{nn}|^k|\lambda_n|^k|s_n|^k \\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n||s_n|^k = O(1) \quad as \quad m\to\infty.
\end{aligned}
$$

So, we get

$$
\sum_{n=1}^{\infty}(P_n/p_n)^{\delta k+k-1}|T_{n,r}|^k < \infty, \quad for \quad r=1,2,3.
$$

This completes the proof of Theorem 2.1.

4 Proof of Theorem 2.2

Using Lemma 2.4 and proceeding as that in the proof of Theorem 2.1, replacing $\sum_{v=1}^{m}(P_v/p_v)^{\delta k}|s_v|^k\beta_v$ by $\sum_{v=1}^{m}(P_v/p_v)^{\delta k}\frac{|s_v|^k}{P_v}(\beta_v P_v)$ we can easily prove Theorem 2.2.

5 Conclusion

We have proved theorems dealing with $|A,p_n;\delta|_k$ summability factors of infinite series. In these theorems, if we take $p_n=1$ then we have two new results dealing with $|A,\delta|_k$ summability factors of infinite series. Also, if we take $a_{nv}=\frac{p_v}{P_n}$, then we have another two new results concerning $|\bar{N},p_n;\delta|_k$ summability. Finally, when (X_n) is taken as almost increasing sequence, new factor theorems for $|A,p_n;\delta|_k$ summability are obtained.

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