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On a Subclass of Univalent Functions Defined by Multiplier Transformations

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Abstract

In the present paper, we introduce a subclass of univalent functions with positive coefficients defined by multiplier transformations in the open unit disk $U=\{z \in \mathbb{C}: |z| < 1\}$. *We obtain some geometric properties, like coefficient inequality, closure theorem, neighborhoods for the subclass* $WA(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda)$, *radii of starlikeness, convexity and close-to-convexity, weighted mean, arithmetic mean, linear combination and integral representation.*

Keywords: *Univalent function, Multiplier transformations, Neighborhoods, Radius of starlikeness, Weighted mean, Arithmetic mean, Linear combination, Closure Theorem, Integral representation.*

1 Introduction

Let S be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \tag{1}$$

which are analytic and univalent in the unit disk U = {z ∈ ℂ: |z| < 1}.

A function $f \in$ S is said to be starlike of order β $(0 \leq \beta < 1)$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \ , (z \in U). \tag{2}$$

Denote the class of all starlike functions of order β in U by $S^*(\beta)$. A function $f \in S$ is said to be convex of order β if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \ , \ (0 \leq \beta < 1, z \in U) \tag{3}$$

Denote the class of all convex functions of order β in U by $C(\beta)$.

A function $f \in S$ is said to be close – to – convex of order β if and only if

$$Re\{f'(z)\} > \beta \ , \ (0 \leq \beta < 1, \ z \in U) \tag{4}$$

Let TH be subclass of S consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \ (a_n \geq 0) \tag{5}$$

For the functions $f \in TH$ given by (5)and $g \in TH$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \ (b_n \geq 0) \tag{6}$$

Define the convolution (or Hadamard product) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \ . \tag{7}$$

For any integer m, we define the multiplier transformations I_m^{ℓ}(see [4, 5] of functions $f \in TH$ by

$$I_m^{\ell} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+\ell}{\alpha+\ell+n-1}\right)^m a_n z^n$$

$$= z + \sum_{n=2}^{\infty} \theta(n,\alpha,\ell) a_n z^n, \quad (\ell \geq 0, \alpha > 0, z \in U), \tag{8}$$

Where

$\theta(n, \alpha, \ell) = ((\alpha+\ell)/(a+\ell+n-1))^m$.

Definition 1: *Let g be a fixed function defined by (6). The function $f \in TH$ given by (5) is said to be in the new class $WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$ if and only if*

$$\left| \frac{\dfrac{z(I_m^{\ell}(f*g(z)))''}{(I_m^{\ell}(f*g(z)))'}}{\eta\dfrac{z(I_m^{\ell}(f*g(z)))''}{(I_m^{\ell}f*g(z)))'} + (\gamma_1+\gamma_2)} \right| < \lambda, \tag{9}$$

Where $\ell \geq 0$, $\alpha > 0$, $m \in Z$, $0 < \eta < 1$, $0 < \gamma_1 < 1$, $0 \leq \gamma_1 < 1$ and $0 < \lambda < 1$. The following interesting geometric properties of this function subclass were studied by several authors for another classes , like, Altintas et al. [1] Atshan and Buti [2], Atshan and Kulkarni [3], Kanas et al. [7,8], Murugusundaramoorthy and Magesh [9,10] and Murugusundaramoorthy and Srivastava [11].

2 Coefficient Inequality

We obtain the necessary and sufficient condition for a function f to be in the class $WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$.

Theorem 1: *Let $f \in TH$. Then $f \in WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$ if and only if*

$$\sum_{n=2}^{\infty} n\big((n-1)(1-\lambda\eta) - \lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell) a_n b_n \leq \lambda(\gamma_1+\gamma_2), \tag{10}$$

where $\ell \geq 0$, $\alpha > 0, m \in Z, 0 < \eta < 1, 0 < \gamma_1 < 1, 0 \leq \gamma_2 < 1$ *and* $0 < \lambda < 1$.

The result is sharp for the function

$$f(z) = z + \frac{\lambda(\gamma_1+\gamma_2)}{n((n-1)(1-\lambda\eta) - \lambda(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)} z^n . \tag{11}$$

Proof: Suppose that (10) is true for $z \in U$ and $|z| = 1$. Then , we have

$$\left| z\,(I_m^{\ell}\ (f*g\,(z)))'' \right| - \lambda \left| \eta z \left(I_m^{\ell}(f*g(z))\right)'' + (\gamma_1+\gamma_2)\left(I_m^{\ell}(f*g(z))\right)' \right|$$

$$= \left| \sum_{n=2}^{\infty} n(n-1)\theta(n,\alpha,\ell)a_n b_n z^{n-1} \right| - \lambda \left| \eta \sum_{n=2}^{\infty} n(n-1)\theta(n,\alpha,\ell)a_n b_n z^{n-1} \right.$$

$$\left. +(\gamma_1+\gamma_2)(1+\sum_{n=2}^{\infty} n\theta(n,\alpha,\ell)a_n b_n z^{n-1}) \right|$$

$$= \left| \sum_{n=2}^{\infty} n(n-1)\theta(n,\alpha,\ell)a_n b_n z^{n-1} \right| - \lambda \left| (\gamma_1+\gamma_2) \right.$$

$$\left. + \sum_{n=2}^{\infty} n(\eta(n-1)+(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)a_n b_n z^{n-1} \right|$$

$$\leq \sum_{n=2}^{\infty} n(n-1)\theta(n,\alpha,\ell)a_n b_n |z|^{n-1}$$

$$- \sum_{n=2}^{\infty} n\lambda(\eta(n-1)+(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)a_n b_n |z|^{n-1} - \lambda((\gamma_1+\gamma_2)$$

$$= \sum_{n=2}^{\infty} n((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\,\theta(n,\alpha,\ell)a_n b_n - \lambda(\gamma_1+\gamma_2) \leq 0,$$

by hypothesis.

Hence, by maximum modulus principle, $f \in WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$.

Conversely, assume that $f \in WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$. Then from (9), we have

$$\left| \frac{\dfrac{z(I_m^{\ell}(f*g(z)))''}{(I_m^{\ell}(f*g(z)))'}}{\eta\, \dfrac{z(I_m^{\ell}(f*g(z)))''}{(I_m^{\ell}f*g(z)))'} + (\gamma_1+\gamma_2)} \right|$$

$$= \left| \frac{\sum_{n=2}^{\infty} n(n-1)\theta(n,\alpha,\ell)\; a_n b_n z^{n-1}}{\sum_{n=2}^{\infty} n\big(\eta(n-1)+(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)\; a_n b_n z^{n-1} + (\gamma_1+\gamma_2)} \right| < \lambda.$$

Since Re (z) $\leq |z|$ for all z $(z \in U)$, we get

$$Re\left\{ \frac{\sum_{n=2}^{\infty} n(n-1)\theta(n,\alpha,\ell)\; a_n b_n z^{n-1}}{\sum_{n=2}^{\infty} n\big(\eta(n-1)+(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)\; a_n b_n z^{n-1} + (\gamma_1+\gamma_2)} \right\} < \lambda. \quad (12)$$

We choose the value of z on the real axis so that

$$\frac{Z(I_m^{\ell}(f*g(z))''}{(I_m^{\ell}(f*g)(z)))'} \qquad is\ real$$

Letting $z \rightarrow 1^-$ through real values, we obtain inequality (10).

Finally, sharpness follows if we take

$$f(z) = z + \frac{\lambda(\gamma_1+\gamma_2)}{n((n-1)(1-\eta\lambda) - \lambda(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)b_n} z^n, \tag{13}$$

n=2, 3….

The proof is complete.

Corollary 1: *Let* $f \in WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$. *Then*

$$a_n \leq \frac{\lambda(\gamma_1+\gamma_2)}{n((n-1)(1-\eta\lambda)-\lambda(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)b_n}, n = 2, 3, ... \tag{14}$$

3 Closure Theorem

Theorem 2: *Let the functions* f_k *defined by*

$$f_k(z) = z + \sum_{n=2}^{\infty} a_{n,k} z^n, \left(a_{n,k} \geq 0, k = 1,2, ..., \mu\right),$$

be in the class $WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$ for every $k = 1,2, ... \mu$. Then the function h defined by

$$h(z) = z + \sum_{n=2}^{\infty} e_n z^n, (e_n \geq 0),$$

also belongs to the class $WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$, where

$$e_n = \frac{1}{\mu}\sum_{k=1}^{\mu} a_{n,k}, \ (n \geq 2).$$

since $f_k \in WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$, then by Theorem 1, we have

Proof:

$$\sum_{n=2}^{\infty} n\left((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2)\right)\theta(n,\alpha,\ell)\ a_{n,k}b_n \le \lambda(\gamma_1+\gamma_2), \tag{15}$$

for every k=1, 2,…..,μ. Hence

$$\sum_{n=2}^{\infty} n((n-1)\,(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)e_n b_n$$

$$=\sum_{n=2}^{\infty} n((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\ \theta(n,\alpha,\ell)b_n\ (\frac{1}{\mu}\sum_{k=1}^{\mu} a_{n,k})$$

$$=\frac{1}{\mu}\sum_{k=1}^{\mu}(\sum_{n=2}^{\infty} n\left((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)a_{n,k}\,b_n\right) \le \lambda(\gamma_1+\gamma_2).$$

By Theorem (1), it follows that $h \in WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$.

4 Neighborhoods for the Class $WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$

Definition 2: *For any function $f \in TH$ and $\delta \ge 0$, the δ −neighborhood of f is defined as:*

$$N_\delta(f) = \{g\,(z) = z + \sum_{n=2}^{\infty} b_n\ z^n\ \in TH\colon \sum_{n=2}^{\infty} n|a_n - b_n| \le \delta\}. \tag{16}$$

In particular, for the function e*(z)*= *z* ,we see that,

$$N_\delta\,(e)\ =\ \{g(z) =\ z\ + \sum_{n=2}^{\infty} b_n z^n\ \ \in TH\colon \sum_{n=2}^{\infty}\ n|b_n| \le \delta\}\,. \tag{17}$$

The concept of neighborhoods was first introduced by Goodman [6] and then general by Ruscheweyh [12].

Definition 3: *A function $f \in TH$ is said to be in the class $WA^\rho\,(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$ if there exists a function $g \in WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$, such that*

$$\left|\ \frac{f(z)}{g(z)} - 1\ \right| < 1-\rho \qquad (z \in U, 0 \le \rho < 1).$$

Theorem 3: *If* $g \in WA(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda)$ *and*

$$\rho = 1 - \frac{\delta((1-\lambda\eta) + \lambda(\gamma_1+\gamma_2))\theta(2,\alpha,\ell)a_2}{2((1-\lambda\eta) + \lambda(\gamma_1+\gamma_2))\theta(2,\alpha,\ell)a_2 - \lambda(\gamma_1+\gamma_2)}\,. \tag{18}$$

Then $N_\delta(g) \subset WA^\rho(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda)$.

Proof: Let $f \in N_\delta(g)$. We want to find from (16) that

$$\sum_{n=2}^{\infty} n\left| a_n - b_n \right| \le \delta,$$

which readily implies the following coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \le \frac{\delta}{2}\,.$$

Next, since $g \in WA(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda)$, we have from Theorem 1

$$\sum_{n=2}^{\infty} b_n \le \frac{\lambda(\gamma_1+\gamma_2)}{2((1-\lambda\eta) + \lambda(\gamma_1+\gamma_2))\theta(2,\alpha,\ell)a_2}\,.$$

So that

$$\left| \frac{f(z)}{g(z)} - 1 \right| \le \frac{\sum_{n=2}^{\infty}\left| a_n - b_n \right|}{1 - \sum_{n=2}^{\infty} b_n}$$

$$\le \delta\, \frac{((1-\lambda\eta) + \lambda(\gamma_1+\gamma_2))\theta(2,a,\ell)a_2}{2((1-\lambda\eta) + \lambda(\gamma_1+\gamma_2))\theta(2,a,\ell)a_2 - \lambda(\gamma_1+\gamma_2)} = 1 - \rho. \tag{19}$$

Thus by Definition (3), $f \in WA^\rho(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda)$ for ρ given by (18).

This completes the proof.

5 Radii of Starlikeness, Convexity and Close–to–Convexity

Using the inequalities (2), (3), (4) and Theorem 1, we can compute the radii starlikeness, convexity and close - to - convexity.

Theorem 4: *If* $f \in WA(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda)$, *then f is univalent starlike of order* ψ $(0 \le \psi < 1)$ *in the disk* $|z| < r_1$, *where*

$$r_1(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda,\psi) = inf_n\{\frac{n(1-\psi)\big((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)b_n}{(n-\psi)\lambda(\gamma_1+\gamma_2)}\}^{\frac{1}{n-1}}, \qquad n\geq 2$$

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\psi \;, (0\leq\psi<1),$$

for $|z| < r_1\,(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda,\psi)$,

we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{\sum_{n=2}^{\infty}(n-1)a_n\, z^{n-1}}{1+\sum_{n=2}^{\infty}a_n\, z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n\,|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n\,|z|^{n-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\psi,$$

if

$$\sum_{n=2}^{\infty}\frac{(n-\psi)a_n|z|^{n-1}}{1-\psi} \leq 1\,. \tag{20}$$

Hence, by Theoremn1, (20) will be true if

$$\frac{(n-\psi)|z|^{n-1}}{1-\psi} \leq \frac{n\big((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)b_n}{\lambda(\gamma_1+\gamma_2)},$$

Equivalently if

$$|z| \leq \{\frac{n(1-\psi)((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)b_n}{(n-\psi)\lambda(\gamma_1+\gamma_2)}\}^{\frac{1}{n-1}}\;,\;\; n\geq 2$$

Setting $|z| = r_1$, we get the desired result.

Theorem 5: *If* $f\in WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$, *then f is univalent convex of order* ψ *(0 ≤* ψ *<1) in the disk* $|z| < r_2$, *where*

$$r_2\,(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda,\psi) = inf_n\,\{\frac{(1-\psi)((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)b_n}{(n-\psi)\lambda(\gamma_1+\gamma_2)}\}^{\frac{1}{n-1}},\; n\geq 2.$$

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\psi\ , \qquad (0 \leq \psi < 1),$$

for $\quad |z| < r_2(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda, \psi),$

We have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1+\sum_{n=2}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n |z|^{n-1}}.$$

Thus

$$\left|\frac{zf''(z)}{zf'(z)}\right| \leq 1-\psi\ ,$$

If

$$\sum_{n=2}^{\infty} \frac{n(n-\psi)\, a_n\, |z|^{n-1}}{(1-\psi)} \leq 1\,. \tag{21}$$

Hence by Theorem 1, (21) will be true if

$$\frac{n(n-\psi)\ |z|^{n-1}}{(1-\psi)} \leq \frac{n((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)b_n}{\lambda(\gamma_1+\gamma_2)}$$

Equivalently if

$$|z| \leq \left\{\frac{(1-\psi)((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)b_n}{(n-\psi)\lambda(\gamma_1+\gamma_2)}\right\}^{\frac{1}{n-1}}, \qquad n \geq 2.$$

Setting $|z| = r_2$, we get the desired result.

Theorem 6: *Let a function* $f \in WA(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda)$. *Then f is univalent close – to - convex of order* ψ $(0 \leq \psi < 1)$ *in the disk* $|z| < r_3$, *where*

$r_3(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda, \psi)$

$$= inf_n\left\{\frac{(1-\psi)((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\theta(n,\alpha,\ell)b_n}{\lambda(\gamma_1+\gamma_2)}\right\}^{\frac{1}{n-1}}, \qquad n \geq 2$$

Proof: It is sufficient to show that

$$|f'(z)-1| \leq 1-\psi \quad , \ (0 \leq \psi < 1),$$

for

$|z| < r_3\ (\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$,

We have

$$|f'(z)-1| = \Big|\sum_{n=2}^{\infty} n\, a_n\, z^{n-1}\Big| \le \sum_{n=2}^{\infty} n\, a_n\, |z|^{n-1}.$$

Thus

$$|f'(z)-1| \le 1-\psi\ ,$$

If

$$\sum_{n=2}^{\infty}\frac{n\, a_n |z|^{n-1}}{1-\psi} \le 1. \tag{22}$$

Hence, by Theorem 1, (22) will be true if

$$\frac{n|z|^{n-1}}{1-\psi} \le \frac{n((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\theta(n,\square,\ell)b_n}{\lambda(\gamma_1+\gamma_2)}\ ,$$

Equivalently if

$$|z| \le \left\{\frac{(1-\psi)((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2))\theta(n,\square,\ell)b_n}{\lambda(\gamma_1+\gamma_2)}\right\}^{\frac{1}{n-1}}\quad ,n\ge 2$$

Setting $|z| = r_3$, we get the desired result.

6 Weighted Mean and Arithmetic Mean

Definition 4: *Let f_1 and f_2 be in the class $WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$. Then the weighted mean V_j of f_1 and f_2 is given by:*

$$V_j(z) = \frac{1}{2}\,[(1-j)\, f_1(z) + (1+j)\, f_2(z)],\qquad 0< j <1$$

Theorem 7: *Let f_1 and f_2 be in the class $WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$. Then the weighted mean V_j of f_1 and f_2 is also in the class $WA\ (\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$.*

Proof: By Definition (4), we have

$$V_j(z) = \frac{1}{2}\,[\,(1-j)f_1\,(z) + (1+j)f_2(z)] \tag{23}$$

$$= \frac{1}{2}\left[(1-j)\left(z+\sum_{n=2}^{\infty} a_{n,1}\, z^n\right)+(1+j)\left(z+\sum_{n=2}^{\infty} a_{n,2}\, z^n\right)\right.$$

$$= z+\sum_{n=2}^{\infty}\frac{1}{2}\left[(1-j)a_{n,1}+(1+j)a_{n,2}\right]z^n.$$

Since f_1 and f_2 are in the class $WA\ (\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$ so by Theorem 1, we get

$$\sum_{n=2}^{\infty} n\big((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)a_{n,1}\, b_n \le \lambda(\gamma_1+\gamma_2),$$

and

$$\sum_{n=2}^{\infty} n\big((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)a_{n,2}\, b_n \le \lambda(\gamma_1+\gamma_2).$$

Hence

$$\sum_{n=2}^{\infty} n\big((n-1)(1-\lambda\eta)-(\gamma_1+\gamma_2)\lambda\big)\theta(n,\alpha,\ell)\ (\frac{1}{2}[\ (1-j)a_{n,1}z^n\ +(1+j)a_{n,2}])b_n$$

$$= \frac{1}{2}\ (1-j)\ \sum_{n=2}^{\infty} n\big((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)a_{n,1}\, b_n$$

$$+\frac{1}{2}\ (1+j)\sum_{n=2}^{\infty} n\big((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)a_{n,2}\, b_n$$

$$\le \frac{1}{2}(1-j)\lambda(\gamma_1+\gamma_2)+\frac{1}{2}\ (1+j)\lambda(\gamma_1+\gamma_2)=\lambda(\gamma_1+\gamma_2).$$

Therefore, $V_j \in WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$.

The proof is complete.

Theorem 8: *Let* $f_1(z)$, $f_2(z)$,…, $f_\mu(z)$ *defined by*

$$f_i(z)=z+\sum_{n=2}^{\infty} a_{n,i}\ z^n, \qquad \big(a_{n,i}\ \ge 0, i=1,2,\dots,\mu\ , n\ge 2\big) \tag{24}$$

be in the class $WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$. Then the arithmetic mean of $f_i(z)$ $(i = 1,2,\ldots,\mu)$defined by

$$\mathrm{h(z)} = \frac{1}{\mu}\sum_{i=1}^{\mu} f_i(z) \tag{25}$$

is also in the class $WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$.

Proof: By (24), (25), we can write

$$h(z) = \frac{1}{\mu}\sum_{i=1}^{\mu}(z+\sum_{n=2}^{\infty} a_{n,i}\ z^n) = z + \sum_{n=2}^{\infty}(\frac{1}{\mu}\sum_{i=1}^{\mu} a_{n,i})z^n.$$

Since $f_i \in WA(\ell,\alpha,m,\eta,\gamma_1,\gamma_2,\lambda)$ for every (i=1,2,…,μ) so by using Theorem 1,

We prove that

$$\sum_{n=2}^{\infty} n\big((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)(\ \frac{1}{\mu}\sum_{i=1}^{\mu} a_{n,i}\,)b_n$$

$$= \frac{1}{\mu}\sum_{i=1}^{\mu}(\sum_{n=2}^{\infty} n\big((n-1)(1-\lambda\eta)-\lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)\, a_{n,i}b_n)$$

$$\leq \frac{1}{\mu}\sum_{i=1}^{\mu}\lambda(\gamma_1+\gamma_2) = \lambda(\gamma_1+\gamma_2).$$ The proof is complete.

7 Linear Combination

In the following theorem, we prove a linear combination for the class

$WA(\ell,\alpha,\mathrm{m},\eta,\gamma_1,\gamma_2,\lambda)$.

Theorem 9: *Let*

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i}\, z^n, \qquad (a_{n,i} \geq 0, i = 1,2,\ldots,\mu, n \geq 2)$$

belong to the class $\mathrm{WA}(\ell,\alpha,\mathrm{m},\eta,\gamma_1,\gamma_2,\lambda)$. *Then*

$$F(z) = \sum_{i=1}^{\mu} c_i f_i(z) \in \; WA(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda),$$

where

$$\sum_{i=1}^{\mu} c_i = 1.$$

Proof: By Theorem (1), we can write for every $i \in \{1,2, \dots, \mu\}$

$$\sum_{n=2}^{\infty} \frac{n\big((n-1)(1-\lambda\eta) - \lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)}{\lambda(\gamma_1+\gamma_2)} \; a_{n,i} b_n \le 1.$$

Therefore

$$F(z) = \sum_{i=1}^{\mu} c_i \;(z + \sum_{n=2}^{\infty} a_{n,i}\, z^n)$$

$$= z + \sum_{n=2}^{\infty} (\sum_{i=1}^{\mu} c_i a_{n,i}) z^n \,.$$

However

$$\sum_{n=2}^{\infty} \frac{n\big((n-1)(1-\lambda\eta) - \lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)}{\lambda(\gamma_1+\gamma_2)} \; (\sum_{i=1}^{\mu} c_i a_{n,i}) b_n$$

$$= \sum_{i=1}^{\mu} c_i \, [\sum_{n=2}^{\infty} \frac{n\big((n-1)(1-\lambda\eta) - \lambda(\gamma_1+\gamma_2)\big)\theta(n,\alpha,\ell)}{\lambda(\gamma_1+\gamma_2)} \; a_{n,i} b_n] \; \le 1.$$

Then *F (z)* $\in WA(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda)$.

So the proof is complete.

8 Integral Representation

In the following theorem, we obtain integral representation for the function *f.*

Theorem 10: *Let f* $\in WA(\ell, \alpha, m, \eta, \gamma_1, \gamma_2, \lambda)$. *Then*

$$(I_m^{\ell}\,(f * g\,(z)\,))' = e^{\int_0^z \frac{(\gamma_1+\gamma_2)\theta(t)\lambda}{t(1-\eta\theta(t)\lambda)} dt},$$

where $|\;\theta(t)\;| < 1$, $z \in U$.

Proof: By putting

$$\frac{z\left(I_m^{\ell}\left(f * g(z)\right)\right)''}{\left(I_m^{\ell}\left(f * g(z)\right)\right)'} = M(z),$$

in (9), we have

$$\left| \frac{M(z)}{\eta M(z)+(\gamma_1+\gamma_2)} \right| < \lambda,$$

or equivalently

$$\frac{M(z)}{\eta M(z)+(\gamma_1+\gamma_2)} = \theta(z)\lambda\,, \left| \theta(z) \right| < 1, \qquad z \in U.$$

We get

$$\frac{\left(I_m^{\ell}\left(f * g(z)\right)\right)''}{\left(I_m^{\ell}\left(f * g(z)\right)\right)'} = \frac{(\gamma_1+\gamma_2)\theta(z)\lambda}{z(1-\eta\theta(z)\lambda)} \quad ,$$

After integration, we have

$$\log\left(\left(I_m^{\ell}\left(f * g(z)\right)\right)'\right) = \int_0^z \frac{(\gamma_1+\gamma_2)\theta(t)\lambda}{t(1-\eta\theta(z)\lambda)}\, dt.$$

Therefore,

$$\left(I_m^{\ell}\left(f * g(z)\right)\right)' = e^{\int_0^z \frac{(\gamma_1+\gamma_2)\theta(t)\lambda}{t(1-\eta\theta(t)\lambda)} dt}\ .$$

and this gives the required result.

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