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On Power Γ-Semigroups

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Abstract

Let S_1 and S_2 be Γ-semigroups. In this note, we determine when S_1 and S_2 are isomorphic if the power semigroup of S_1 and the power semigroup of S_2 are o-isomorphic.

Keywords: *Γ-semihypergroups, Power Semigroups.*

1 Prelimminaries

In [1], by a *po-semigroup* is a semigroup S such that S is a partially ordered set under a relation $\leq$ and

$$a \leq b \text{ implies } ca \leq cb \text{ and } ac \leq bc$$

for all $a, b, c \in S$.

Let S_1 and S_2 be *po*-semigroups. A map $\varphi : S_1 \to S_2$ is called an *o-isomorphism* if φ is one to one and onto such that

(i) $\varphi(ab) = \varphi(a)\varphi(b)$;

(ii) $a \leq b \Leftrightarrow \varphi(a) \leq \varphi(b)$

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for all $a, b \in S_1$. In this case, we say that S_1 and S_2 are *o-isomorphic*.

Let S be a (*po*-semigroup) semigroup. Define a binary operation on the set of all nonempty subsets of S, denoted by $P(S)$, by

$$AB = \{ab : a \in A, b \in B\}.$$

for $A, B \in P(S)$. Then $P(S)$ forms a *po*-semigroup (under the inclusion) which is called the *power semigroup* of S.

For any semigroups S_1 and S_2, it is known that if S_1 and S_2 are isomorphic then $P(S_1)$ and $P(S_2)$ are *o*-isomorphic. This leads to the question: if $P(S_1)$ and $P(S_2)$ are *o*-isomorphic, must S_1 and S_2 be isomorphic ?

For a semigroup S, let $I(S)$ and $PI(S)$ be the set of all ideals of S and the set of all principle ideals of S, respectively. We say that S is an *IO-semigroup* if

$$a \in S^1bS^1, b \in S^1aS^1 \Rightarrow a = b$$

for all $a, b \in S$. In [3], the author proved that:

Theorem 1.1 *Let S_1 and S_2 be commutative OI-semigroups such that $I(S_1) = PI(S_1)$ and $I(S_2) = PI(S_2)$. If $P(S_1)$ and $P(S_2)$ are o-isomorphic, then S_1 and S_2 are isomorphic.*

In this note, we generalize this result using the concept of Γ-semigroup.

Let S and Γ be nonempty sets. Then S is called a Γ-*semigroup* if there exists a mapping $S \times \Gamma \times S \to S; (a, \gamma, b) \mapsto a\gamma b$ such that $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$. It is known that the concept of a Γ-semigroup, defined by Sen and Saha in 1986, is a generalization of a semigroup. A nonempty subset T of S is called a Γ-*subsetmigroup* of S if $a\alpha b \in T$ for all $a, b \in T$ and all $\alpha \in \Gamma$.

A bijection map $\varphi : S_1 \to S_2$ from a Γ-semigroup S_1 into a Γ-semigroup S_2 is called an *isomorphism* if $\varphi(a\alpha b) = \varphi(a)\alpha\varphi(b)$ for all $a, b \in S_1$ and all $\alpha \in \Gamma$. In this case, we say that S_1 and S_2 are *isomorphic*.

A Γ-semigroup S is called a *po*-Γ-*semigroup* if S is a partially ordered set under a relation $\leq$ and

$$a \leq b \text{ implies } c\alpha a \leq c\alpha b \text{ and } a\alpha c \leq b\alpha c$$

for all $a, b, c \in S$ and all $\alpha \in \Gamma$.

Let S be a Γ-semigroup. For nonempty subsets A and B of S and $\gamma \in \Gamma$, let

$$A\gamma B = \{a\gamma b : a \in A, b \in B\}.$$

Then $P(S)$ forms a Γ-semigroup, called the *power* Γ-*semigroup* of S. Moreover, under the usual inclusion it is a *po*-Γ-semigroup.

Let S_1 and S_2 be *po*-Γ-semigroups. A bijection map $\varphi : S_1 \to S_2$ is called an *o-isomorphosm* if

(i) $\varphi(a\gamma b) = \varphi(a)\gamma\varphi(b)$;

(ii) $a \leq b \Leftrightarrow \varphi(a) \leq \varphi(b)$

for all $a, b \in S_1$ and all $\gamma \in \Gamma$. In this case, we say that S_1 and S_2 are *o-isomorphic*.

Theorem 1.2 *Let S_1 and S_2 be Γ-semigroups. If S_1 and S_2 are isomorphic, then $P(S_1)$ and $P(S_2)$ are o-isomorphic.*

Proof. Let $\varphi : S_1 \to S_2$ be an isomorphism. Define $\bar{\varphi} : P(S_1) \to P(S_2)$ by $\bar{\varphi}(X) = \varphi(X)$ for all $X \in P(S_1)$ ($\varphi(X) = \{\varphi(x) : x \in X\}$). Since φ is a bijection, so is $\bar{\varphi}$. Let $X, Y \in P(S_1)$ and $\alpha \in \Gamma$. We have

$$\bar{\varphi}(X\alpha Y) = \varphi(X\alpha Y) = \varphi(X)\alpha\varphi(Y) = \bar{\varphi}(X)\alpha\bar{\varphi}(Y).$$

If $X, Y \in P(S_1)$ such that $X \subseteq Y$, then

$$\bar{\varphi}(X) = \varphi(X) \subseteq \varphi(Y) = \bar{\varphi}(Y).$$

Thus $\bar{\varphi}$ is an o-isomorphism.

2 Main Results

Let S be a Γ-semigroup. For nonempty subsets A, B of S, let

$$A\Gamma B = \{a\gamma b : a \in A, b \in B, \gamma \in \Gamma\}.$$

A nonempty subset I of S is called an *ideal* of S if $S\Gamma I\Gamma S \subseteq I$, that is $a\alpha c\beta b \in I$ for all $a, b \in S, c \in I$ and all $\alpha, \beta \in \Gamma$. The set of all ideals of S will be denoted by $I(S)$. Note that $I(S)$ is a Γ-subsemigroup of $P(S)$. The intersection of all ideals containning an element a of S is an ideal of S and it is of the form $[a] = S\Gamma a\Gamma S \cup \{a\}$, this is called the *principal ideal* generated by a. The set of all principal ideals of S is denoted by $PI(S)$.

Lemma 2.1 *Let S_1 and S_2 be Γ-semigroups. If $\varphi : P(S_1) \to P(S_2)$ is an o-isomorphism , then $\varphi_{|_{I(S_1)}} : I(S_1) \to I(S_2)$ is onto. Moreover, $\varphi_{|_{I(S_1)}}$ is an o-isomorphism.*

Proof. We shall prove that $\varphi_{|_{I(S_1)}}(I(S_1)) = I(S_2)$. Since $S_2 \in P(S_2)$ and φ is an o-isomorphism we have that $\varphi(Y) = S_2$ for some $Y \in P(S_1)$. Let $X \in I(S_1)$ and $\alpha \in \Gamma$. Since $X\alpha Y \subseteq X$ and $Y\alpha X \subseteq X$, it follows that

$$\varphi(X\alpha Y) \subseteq \varphi(X) \text{ and } \varphi(Y\alpha X) \subseteq \varphi(X)$$

Let $\alpha \in \Gamma$. Then

$$\varphi_{|_{I(S_1)}}(X)\alpha S_2 = \varphi(X)\alpha\varphi(Y) = \varphi(X\alpha Y) \subseteq \varphi(X) = \varphi_{|_{I(S_1)}}(X)$$

and

$$S_2\alpha\varphi|_{|_{I(S_1)}}(X) = \varphi(Y)\alpha\varphi(X) = \varphi(Y\alpha X) \subseteq \varphi(X) = \varphi_{|_{I(S_1)}}(X).$$

Thus $\varphi_{|_{I(S_1)}}(X)$ is an ideal of S_2. Therefore, $\varphi_{|_{I(S_1)}}(I(S_1)) \subseteq I(S_2)$.

Consider an o-isomorphism $\varphi^{-1} : P(S_2) \to P(S_1)$, we have that

$$\varphi^{-1}(I(S_2)) \subseteq I(S_1).$$

Since

$$\varphi|_{I(S_1)}\varphi^{-1}|_{I(S_2)}(I(S_2)) \subseteq \varphi|_{I(S_1)}(I(S_1))$$

we obtain

$$I(S_2) = \varphi|_{I(S_1)}\varphi^{-1}|_{I(S_2)}(I(S_2)) \subseteq \varphi|_{I(S_1)}(I(S_1)).$$

This completes the proof. That $\varphi_{|_{I(S_1)}}$ is an o-isomorphism is clear.

The next lemma is easy to see.

Lemma 2.2 *Let S be a commutative Γ-semigroup. For $a, b \in S$ and $\alpha \in \Gamma$,*

$$[a]\alpha[b] = [a\alpha b].$$

Let S be a Γ-semigroup. We say that S is an *IO-Γ-semigroup* if for all $a, b \in S$, $a \in [b], b \in [a] \Rightarrow a = b$.

Theorem 2.3 *Let S_1 and S_2 be commutative OI-Γ-semigroups such that $I(S_1) = PI(S_1)$ and $I(S_2) = PI(S_2)$. If $P(S_1)$ and $P(S_2)$ are o-isomorphic, then S_1 and S_2 are isomorphic.*

Proof. Assume that $P(S_1)$ and $P(S_2)$ are o-isomorphic. Let $\varphi : P(S_1) \to P(S_2)$ be an o-isomorphism. By Lemma 2.1, $\varphi|_{I(S_1)} : I(S_1) \to I(S_2)$ is an o-isomorphism. Let $\varphi_1 : S_1 \to I(S_1)$ by $a \mapsto [a]$ and $\varphi_2 : S_2 \to I(S_2)$ by $b \mapsto [b]$. Let $\alpha \in \Gamma$ and $a, a' \in S_1$. By Lemma 2.2 we have

$$\varphi_1(a\alpha a') = [a\alpha a'] = [a]\alpha[a'] = \varphi_1(a)\alpha\varphi_1(a').$$

Let $a, a' \in S_1$ be such that $\varphi_1(a) = \varphi_1(a')$, that is $[a] = [a']$. Since S_1 is an IO-semigroup, $a = a'$. Let $A \in I(S_1)$. Since $PI(S_1) = I(S_1)$, $A \in PI(S_1)$. Then $A = [a]$ for some $a \in S_1$. Since $\varphi_1(a) = [a] = A$, φ_1 is onto. Therefore φ_1 is an isomorphism. Similarly, φ_2 is an isomorphism. This implies that $\varphi_2^{-1}\varphi_{|_{I(X)}}\varphi_1 : S_1 \to S_2$ is an isomorphism. Thus S_1 and S_1 are isomorphic.

This follows from Theorem 3.2.

Corollary 2.4 *Let S_1 and S_2 be commutative OI-Γ-semgroups such that $I(S_1) = PI(S_1)$ and $I(S_2) = PI(S_2)$. If $I(S_1)$ and $I(S_2)$ are o-isomorphic, then S_1 and S_2 are isomorphic.*

3 Conclusion

These are the main results of the paper.

Theorem 3.1 *Let S_1 and S_2 be Γ-semigroups. If S_1 and S_2 are isomorphic, then $P(S_1)$ and $P(S_2)$ are o-isomorphic.*

Theorem 3.2 *Let S_1 and S_2 be commutative OI-Γ-semigroups such that $I(S_1) = PI(S_1)$ and $I(S_2) = PI(S_2)$. If $P(S_1)$ and $P(S_2)$ are o-isomorphic, then S_1 and S_2 are isomorphic.*

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