Gen. Math. Notes, Vol. 29, No. 2, August 2015, pp.55-66
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

On Some Qualitative Properties of a Non-Autonomous Lienard Equation

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(Received: 12-6-15 / Accepted: 29-7-15)

Abstract

In this paper we consider the problem about the conditions on f(x), a(t) and g(x) to ensure that all solutions of (1) are bounded or oscillatory using a non usual Lyapunov Function and two equivalent systems.

Keywords: *Boundedness, oscillation, asymptotic behavior, Liénard equation.*

1 Introduction

We consider the equation:

$$x'' + f(x)x' + a(t)g(x) = 0, \tag{1}$$

where a, f and g are continuous functions satisfying the following condition:

a) $xg(x) > 0$ for $x \neq 0$,
b) ${}_0^{\pm\infty} g(s)ds = +\infty$,

c) $a \in C^1([0,+\infty))$, satisfying $0 < a \leq a(t) \leq A < +\infty$ for $t \in [0,+\infty)$.

Various questions on the stability, oscillation and periodicity of solutions of (1) have received a considerable amount of attention in the last years (one can consult the references for a more complete picture) under condition $f(x)>0$ for all $x \in \mathbb{R}$. In this paper we study the asymptotic behaviour of solutions of (1) without making use of this condition and using a new method in which the usual Lyapunov function is not used (cf. [2-4]).

To apply Lyapunov's direct method to the equation (1), we usually define a Lyapunov function $V(t,x,y)$ by:

$$V(t,x,y) = b(t)W(t,x,y), \tag{2}$$

where:

$$W(t,x,y) = G(x) + \frac{y^2}{2\,[a(t)]} \tag{3}$$

$G(x) = \int_0^x g(t)dt$ and $b(t) = exp\left(-\int_0^t \frac{a'(s)^-}{a(s)} ds\right)$ with $a'(t)_- = max(-a'(t), 0)$. Let $V'_{(1)}(t,x,y)$ be the total derivative along the solutions of (1). If $V'_{(1)}(t,x,y)$ is non-positive in a suitable neighbourhood of the $(0,0)$, then the stability of the zero solution of (1) follows. For the non-positivity of $V'_{(1)}(t,x,y)$ we need that $F(x)$ satisfies:

$$F(-x) \leq 0 \leq F(x) \text{ somewhere in } x \geq 0, \tag{4}$$

since $V'_{(1)}(t,x,y) = -\frac{b(t)}{a(t)}\left[a'(t)^- G(x) + \frac{y^2 a'(t)^+}{2a^2(t)} + a(t)g(x)F(x)\right]$. In other point of view, the non-positivity of $V'_{(1)}(t,x,y)$ implies that every solutions of (1) departing from a bounded region by a closed curve, remains in this region as t increases. This fact plays an essential role in our work where the assumptions (4) is not used. So, we need alternative assumptions on $F(x)$ and $g(x)$ under which the last remark is still valid.

The equation (1) is equivalent to the system:

$$\begin{aligned} x' &= y, \\ y' &= f(x)y - a(t)g(x). \end{aligned} \tag{5}$$

The regularity of functions involved in this system ensures existence and uniqueness of solutions of (5). The condition a) shows that $(0,0)$ is the only point of equilibrium for system (5) and the condition b) ensures that results obtained are in global sense. From [10], obtain that condition c) is consistent with common sense.

2 Problem Formulations

Let α be a given real. We indicate by Ω_α the following open set:

$\Omega_\alpha \equiv \mathbb{R}^2$ if $\alpha \equiv 0$;

$\Omega_\alpha = \{(x, y) : y > -\alpha^{-1}\}$ if $\alpha>0$;

$\Omega_\alpha = \{(x, y) : y < -\alpha^{-1}\}$ if $\alpha<0$.

And let $F_g(\mathbb{R}) = \{f \in C(\mathbb{R}) : for\ x \geq 0, f(x) - \alpha Ag(x) > 0$ and for $x \leq 0$, $f(x) - \alpha Ag(x) < 0\}$.

Consider the following function V_α given by:

$$V_\alpha(t, x, y) = \frac{1}{a(t)} W_\alpha(y) + G(x),\ (x, y) \in \Omega_\alpha. \tag{6}$$

with $G(x)$ as above and $W_\alpha(y) ={}_0^y \frac{sds}{\alpha s+1}$.

Now we present some auxiliary results.

Lemma 2.1. *Under assumptions a)-c) and $f \in F_g$, $V_\alpha(t, x, y)$ is a definite positive function.*

Proof: Consider the following case.

Case $\alpha \equiv 0$

In this case we have that $V_\alpha(t, x, y)$ becomes in

$$V_0(t, x, y) = \frac{y^2}{2a(t)} + G(x)$$

From this we have $V_0(t, 0, 0) \equiv 0$ and $V_0(t, x, y) > 0$ for all $(x, y) \neq (0, 0)$.

Case $\alpha > 0$

It is clear that $V_\alpha(t, 0, 0) \equiv 0$ and

$${}_0^{+\infty} \frac{sds}{\alpha s+1} = +\infty ={}_0^{-\frac{1}{\alpha}} \frac{sds}{\alpha s+1}. \tag{7}$$

From this and definition of function $G(x)$ we have that $V_\alpha(t, x, y) > 0$ for all $(x, y) \neq (0, 0)$.

Case $\alpha < 0$

This case can be analysed in a similar way. End of proof.

Lemma 2.2. *The solutions of system (5), and equation (1), do not admit vertical asymptotes.*

Proof: It is enough, to this end, to show that all solutions of the equation

$$\frac{dy}{dx} = -f(x) - \frac{a(t)g(x)}{y}, y \neq 0 \tag{8}$$

do not admit vertical asymptotes.
Let us assume that (8) has a solution

$$y = y(x),\ a \leq x < b$$

such that

$$\underset{x \to b^-}{Lim}\ y(x) = +\infty. \tag{9}$$

We can assume with no loss of generality, that $0 < y(a) \leq y(x)$ for $a \leq x < b$. Let

$$F \geq \max_{a \leq x < b} |f(x)|\,,\ G \geq \max_{a \leq x < b} |g(x)|\,.$$

It follows from the mean value theorem that, for $a < x < b$,

$$y(x) - y(a) \leq \left[F + \frac{AG}{y(a)}\right](b - a)$$

which contradicts to (9). The other situations can be analysed in a similar way. This completes the proof.

Remark 2.3. *This is equivalent to proved the continuation of the solutions of system (5) and therefore, of equation (1).*

It can be immediately verified that the derivative of V relative to system (5) verified:

$$V'_\alpha(t, x, y) \leq -\frac{a'(t)_+}{a^2(t)} W_\alpha(y) - \frac{1}{a(t)} \frac{(f(x) - \alpha a(t)g(x))}{[\alpha(y - F(x)) + 1]} y^2 \tag{10}$$

Because $\frac{a'(t)}{a^2(t)} W_\alpha(x, y)$, $\alpha(y - F(x)) + 1$ and $\frac{y^2}{a(t)}$ they are positive for all $(x, y) \in \Omega_\alpha$, it follows that the non positivity of $V'_\alpha(t, x, y)$ depends only of $(f(x) - \alpha a(t)g(x))$.
From (6) we can define the function:

$$\overline{V_\alpha}(x, y) = \frac{1}{a} W_\alpha(y) + G(x), (x, y) \in \Omega_\alpha.$$

Lemma 2.4. *Assume there are $\alpha > 0$ and $b > 0$ such that for all $x \geq b$, $f(x) \geq \alpha A g(x)$. Let $y_0 > 0$, $L = \overline{V_\alpha}(b, y_0)$ and*

$$M = \left\{(x, y) \in \Omega_\alpha : x \geq b,\ \overline{V_\alpha}(x, y) \leq L\right\}$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (5) so that $\gamma(t_0) = (b, y_1)$, with $0 < y_1 < y_0$. Then, there is $t_1 > t_0$ such that

$$\gamma(t) \in M,\ t_0 \leq t \leq t_1$$

and $\gamma(t_1) = (b, y_2)$, with $-\frac{1}{\alpha} < y_2 < 0$.

Proof: From $x'(t_0) = y_1 > 0$, it follows there is $t_2 > t_0$ so that $\gamma(t) \in M,\ t_0 \leq t \leq t_2$. On the other hand, being $x'(t) > 0$ on the half plane $y > 0$, $x'(t) < 0$ on the half plane $y < 0$, $y'(t) < 0$ on the positive half-axis x and $(0, 0)$ the only point of equilibrium, there must exist $t_3 > t_2$ such that $\gamma(t_3) \notin M$.

Let t1=$\{\tau > t_0 :\ \gamma(t) \in M,\ t_0 \leq t \leq \tau\}$. From the hypothesis $f(x) \geq \alpha A g(x)$, $x \geq b$, and from (11) it follows that $\overline{V_\alpha}'(x, y) \leq 0,\ t_0 \leq t \leq t_1$. Since $\overline{V_\alpha}(\gamma(t)) = \overline{V_\alpha}(b, y_1) < L$.

Because $x'(t) > 0$ on the $y > 0$ half-plane, it follows that $\gamma(t_1) = (b, y_2)$, with $-\frac{1}{\alpha} < y_2 < 0$.

In a similar way, we can demonstrate the following lemmas:

Lemma 2.5. *Assume there are $\alpha < 0$ and $c < 0$ such that for all $x \leq c$, $f(x) \geq \alpha A g(x)$. Let $y_0 < 0$, $L = \overline{V_\alpha}(c, y_0)$ and*

$$M = \left\{(x, y) \in \Omega_\alpha : x \leq c,\ \overline{V_\alpha}(x, y) \leq L\right\}$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (5) so that $\gamma(t_0) = (c, y_1)$, with $y_0 < y_1 < 0$. Then, there is $t_1 > t_0$ such that

$$\gamma(t) \in M,\ t_0 \leq t \leq t_1$$

and $\gamma(t_1) = (c, y_2)$, with $0 < y_2 < -\frac{1}{\alpha}$.

Lemma 2.6. *Assume there are $\alpha < 0$ such that for all $x < c$, $f(x) \leq 0$. Let $y_0 < 0$, $L = \overline{V_0}(c, y_0)$ and*

$$M = \left\{(x, y) \in \mathbb{R}^2 : x \leq c,\ \overline{V_0}(x, y) \leq L\right\}$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (5) so that $\gamma(t_0) = (c, y_1)$, with $y_0 < y_1 < 0$. Then, there is $t_1 > t_0$ such that

$$\gamma(t) \in M,\ t_0 \leq t \leq t_1$$

and $\gamma(t_1) = (c, y_2)$, with $0 < y_2 < |y_0|$.

Lemma 2.7. *Assume there are* $b > 0$ *such that* $f(x) \geq 0$, $x \geq 0$. *Let* $y_0 > 0$, $L = \overline{V_0}(b, y_0)$ *and*

$$M = \left\{(x, y) \in \mathbb{R}^2 : x \geq b,\ \overline{V_0}(x, y) \leq L\right\}$$

Let $\gamma(t) = (x(t), y(t))$ *be the solution of (5) so that* $\gamma(t_0) = (b, y_1)$, *with* $0 < y_1 < y_0$. *Then, there is* $t_1 > t_0$ *such that*

$$\gamma(t) \in M,\ \ t_0 \leq t \leq t_1$$

and $\gamma(t_1) = (b, y_2)$, *with* $-y_0 < y_2 < 0$.

Remark 2.8. *When* $a \equiv 1$, *our results are consistent with those obtained in [1], [5] and [13].*

Remark 2.9. *In the general case* $a(t) > 0$ *our results are non contradicts with the obtained in [9] and [14].*

Remark 2.10. *The results obtained in Lemmas 3-6 completes those obtained in [11], about the construction of a stability region for the equation (1).*

2.1 Oscillatory and Boundedness Results

We know that all solutions of (1) are continuable to the future, now consider instead the system (5) the following equivalent system to equation (1):

$$\begin{aligned} x' &= y - F(x), \\ y' &= -a(t)g(x). \end{aligned} \tag{11}$$

Now we will establish various results on the oscillatory character of this system. So, we have:

Theorem 2.11. *Under conditions a)-c) if*
1) $\int_0^{+\infty} \frac{a'(t)^-}{a(t)} dt$¡$\infty$, *and*
2) there is N¿0 such that $|F(x)| \leq N$ *for* $x \in \mathbb{R}$,
then all solutions of the system are oscillatory if and only if:

$$\int_{t_0}^{+\infty} a(t) g\left[\pm k(t - t_0)\right] dt = \pm\infty, \tag{12}$$

for all k>0 and all $t_0 \geq 0$.

Proof: Necessity: We suppose that all solution of (11) are oscillatory, but condition (12) is not satisfy for some $k>0$. We shall construct a non-oscillatory solution of system (5), making in (12) $s = \pm k(t-t_0)$ we have:

$$\pm k \int_{t_0}^{+\infty} a(t)g[\pm k(t-t_0)]dt = \int_0^{\pm\infty} a(\pm\frac{s}{k}+t_0)g(s)ds,$$

thus:

$$\int_0^{\pm\infty} a(\pm\frac{s}{k}+t_0)g(s)ds = M < +\infty,$$

for some $k > 0$ and some $t_0 \geq 0$. We consider a solution of system (11), $(x(t), y(t))$ such that $x(t_0) = 0$, $y(t_0) = A$ with $A>k+N$. While that $y(t) > k+N$ we have $x'(t) \geq k>0$; from this inequality, after integration between t_0 and t we obtain $x(t) \geq k(t-t_0)$, then there is $x^{-1}(s)$ such that $x^{-1}(s) \leq \frac{s}{k}+t_0$. Consider the function $b(t) = exp\left(-\int_0^t \frac{a'(\tau)^-}{a(\tau)}d\tau\right)$, from condition 2.1 we have that $0<b_1 \leq b(t) \leq 1$ for $0 \leq t<+\infty$, for some b_1.

Since $a(t) = b(t)c(t)$, where $c(t) = a(0)exp\int_0^t \frac{a'(\tau)^+}{a(\tau)}d\tau$, we obtain:

$$\begin{aligned} M &= \int_{t_0}^{+\infty} a(t)g[k(t-t_0)]dt = \int_{t_0}^{+\infty} b(t)c(t)g[k(t-t_0)]dt \geq \\ &\geq b_1 \int_{t_0}^{+\infty} c(t)g[k(t-t_0)]dt \end{aligned}$$

and from here:

$$\int_{t_0}^{+\infty} c(t)g[k(t-t_0)]dt \leq \frac{M}{b_1} \equiv M_1.$$

From the second equation of system (11) we deduce that:

$$\frac{y'(t)}{b(t)} = -c(t)g(x(t)), \tag{13}$$

thus $y'(t) \geq \frac{y'(t)}{b(t)} = c(t)g(x(t))$, integrating (13) between t_0 and t we have

$$\begin{aligned} y(t) &\geq y(t_0) - \int_{t_0}^t c(s)g(x(s))dt \geq A - \frac{1}{k}\int_{t_0}^t c(s)g(x(s))x'(s)dt = \\ &= A - \frac{1}{k}\int_0^{x(t)} c(x^{-1}(s))g(s)dt. \end{aligned}$$

Since $x^{-1}(s) \leq \frac{s}{k}+t$ we have $c(x^{-1}(s)) \leq c(\frac{s}{k}+t_0)$ and from here we obtain

$$y(t) \geq A - \frac{1}{k}\int_0^{x(t)} c(\frac{s}{k}+t_0)g(s)dt \geq A - \frac{M_1}{k}.$$

Taking A such that $A - \frac{M_1}{k} \geq k+N$ for $t \geq t_0$ we have that $x(t) \geq k(t-t_0) \to +\infty$ as $t \to +\infty$. This is a contradictory with the initial supposition, so we have the necessity of condition (12). The case $x \leq 0$ can be proved in a similar way.

Sufficiency: Let (x(t),y(t)) be the solution of (11) leaving a point $B(x_0, F(x_0))$, at $t=0$. Suppose that (x(t),y(t)) does not traverse the y-axis. Then (x(t),y(t)) stays in the region $R_2 = \{(x,y) : x \geq 0, y < F(x)\}$ as long as the solution is defined for $t \geq 0$, hence $x\prime(t) < 0$ and therefore $x(t) \leq x(t_0)$. Let $N_1 = \underset{0\leq x\leq x_0}{max} |F(x)|$, then the solution $(x(t), y(t))$ does not traverse the curve

$$V_\alpha(t, x(t), y(t)) = \overline{V_\alpha}(x_0, F(x_0)) = \frac{1}{A}\int_0^{F(x_0)+N_1} \frac{sds}{\alpha s+1} + G(x_0)$$

as t increases. Therefore the orbit (x(t),y(t)) traverses the y-axis at $C(0, y_C)$. Since $x'=0$ and $y'<0$ on the curve $y = F(x)$ in the region $x > 0, F(0) = 0$ implies $y_C \leq 0$. Thus the orbit traverses the negative y-axis at some finite time t_1. We choose $x(t_1) = 0$, $y(t_1) = y_C$. In the region $R_3 = \{(x,y) : x \leq 0, y < F(x)\}$, $x'(t) \leq y_C + N$, so we have $x(t) \leq (y_C+N)(t-t_0)$ from here $x^{-1}(s) \geq \frac{s}{y_C+N} + t_0$ and $\frac{y'}{b_1} \geq -c(t)g(x(t))$. It follows then, for all $t > t_1$, that:

$$y(t) \geq (y_C+N) - \frac{b_1}{y_C+N}\int_{t_1}^{t} c(s)g(x(s))x'(s)ds,$$

and hence:

$$y(t) \geq y_C - \frac{b_1}{y_C+N}\int_0^{x(t)} c(\frac{r}{y_0}+t_0)g(r)dr. \tag{14}$$

Since $y(t)<F(x(t))$ if $x(t) \to \pm\infty$ then from (14) we have that $y(t) \to +\infty$, and the orbit (x(t),y(t)) traverses the curve $y = F(\mathtt{x})$. Now consider the region $R_3 = \{(x,y) : x < 0, y > F(x)\}$, here $x'(t) > 0, y'(t) > 0$, the analysis of phases velocities show the existence of a point $D(0, y_D)$ on the y-axis positive. If x(t) is bounded, i.e., $x(t_1) \geq x(t) \geq M$ we have that $x(t) \to M^-$ while that $y(t)$ is increasing. Again an analysis of phases velocities show that there is a finite time t' such that $y(t') = F(x(t'))$. This completes the proof of theorem.

Remark 2.12. *The simple case $x'' - 2x' + x = 0$, with non-oscillatory solution $x(t) = e^t$, shows that positivity of f is probably necessary in some sense. This is an open problem.*

Theorem 2.13. *Under assumptions of Lemma 1 if the following conditions:*
1) $a\prime(t)>0$ *for* $t \geq 0$,
2) $|F(x)| \leq N$ *for some* $N>0$ *and* $x \in \mathbb{R}$,
3) $G(\infty) = \infty$,
hold. Then the solutions of the equation (1) are bounded if and only if the condition (12) is fulfilled.

Proof: We suppose that condition (12) is fulfilled. Then all solutions of are oscillatory. In this case $c(t) = a(t)$ for all $t \geq t_0 \geq 0$. We taking in account the function V_α defined in (6) and his total derivative (7) we have that:

$$V_\alpha(t, x(t), y(t)) \leq V_\alpha(t_0, x(t_0), y(t_0)).$$

From Theorem 2.7 there are $t_2 \geq t_1 \geq t_0$ such that $x(t_1) > 0$, $x(t_2) < 0$, and $y(t_1) = F(x(t_1))$, $y(t_2) = F(x(t_2))$. Also we obtain, from decreasing of functions V_α, that:

$$V_\alpha(t, x(t), y(t)) \leq V_\alpha(t_1, x(t_1), y(t_1)) = G(x(t_1))$$

and consequently:

$$G(x(t)) \leq G(x(t_1)).$$

From this we obtain that $x(t) \leq x(t_1)$. Similarly, we can obtain that $x(t_2) \leq x(t)$. So, putting $M = max(-x(t_2), x(t_1))$ we have $|x(t)| \leq M$ for $t \geq max\{t_2, t_1\}$. This prove the sufficiency. In the Theorem 7 we proved that if the condition is not true, there are unbounded solutions of equation (1). Thus the proof of theorem is finished.

Lemma 2.14. *If in addition to conditions a)-c) we have that g(x) is not increasing function and* $a(t) \to +\infty$ *as* $t \to +\infty$*, then condition (12) does not hold.*

Proof: If condition (12) is not valid, then there exits $k> 0$ and $t_0 \geq 0$ such that

$$\int_{t_0}^{+\infty} a(t)g[k(t-t_0)]dt = M < +\infty,$$

(the negative case is similar). From Theorem 2.7 the equation (1) have non-oscillatory solutions defined for $t \geq t_0 \geq 0$. We consider a solution $x = x(t)$ with this property, without loss of generality we can suppose that there exists $T_1 \geq t_0$ such that for some m, $a(t)>m$ if $t \geq T_1$ (the case $x(t)< -m<0$ is analogous). It is easy follow that for $m>0$ there exists $T_2 \geq t_0$ such that:

$$k(t-t_0)>m>0, t\geq T_2. \tag{15}$$

By use of (13) and definition of g we have:

$$g[k(t-t_0)]\geq g(m)>0, t\geq T_2.$$

Therefore we obtain:

$$a(t)g(m)\leq a(t)g[k(t-t_0)), t\geq T_2. \tag{16}$$

Let us consider $T=max\{T_1,T_2\}$ after integration of (16) between T and $+\infty$ we obtain:

$$g(m)\int_T^{+\infty}a(t)dt\leq\int_T^{+\infty}a(t)g[k(t-t_0)]dt=M^*<+\infty,$$

hence

$$\int_T^{+\infty}a(t)dt\leq\frac{M^*}{g(m)}<+\infty. \tag{17}$$

Since $a(t)\to+\infty$ as $t\to+\infty$ we have that:

$$\int_T^{+\infty}a(t)dt=+\infty,$$

which is a contradiction to (17). Hence the condition (12) holds. Thus the proof is now complete.

Corollary 2.15. *Under conditions of Lemma 9 all solutions of equation (1) are oscillatory if the following conditions:*
a) $\int_0^{+\infty}\frac{a'(t)^-}{a(t)}dt<+\infty$,
b) there exist $N>0$ *such that* $F(x)\leq N$ *for* $x\in\mathbb{R}$
hold.

Proof: It follows from Lemma 2.2, Lemma 2.1 and Theorem 2.7.

Theorem 2.16. *Under condition Lemma 1 if the condition:*
1)$\int_0^{+\infty}\frac{a'(t)^-}{a(t)}dt<+\infty$,
holds, then all solutions of equation (1) are bounded.

Proof: By similar arguments to sufficiency of Theorem 2.8 we obtain that there exists $R>0$ such that $|x(t)|\leq R$.

Corollary 2.17. *Under condition of Lemma 9 all solutions of equation (1) are bounded if the conditions:*
a) $a'(t)>0$ *for all* $t\geq 0$,

b) there exists $N > 0$ *such that* $F(x) \leq N$ *for* $x \in \mathbb{R}$
hold.

Proof: The proof follows immediately applying Lemma 2.9 and Theorem 2.8.

Finally we give examples of functions $f(x)$ which show that our results contains those in [15] and [16].

Example 1: $f(x) = \begin{cases} x, \ if \ |x| \leq 1, \\ x^{-1}, \ if \ |x| > 1. \end{cases}$

Example 2: $f(x) = \begin{cases} 1, \ if \ x \geq 1, \\ x, \ if \ |x| < 1, \\ -1, \ if \ x \leq -1. \end{cases}$

These examples do not satisfy the conditions of Repilado and Ruiz, but we can guarantee the boundedness of the solutions under Corollary 12.

Acknowledgements: The authors acknowledge the comments and suggestions of the referees and editor, who improved the quality of this article.

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