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Some Properties of $M-$Class Q and $M-$Class Q^* Operators

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Abstract

Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. In this paper we introduce a new class of operators: $M-$class Q^* operators. An operator $T \in \mathcal{L}(\mathcal{H})$ is of $M-$class Q^*, for a fixed real number $M \geq 1$, if T satisfies

$$\|T^*x\|^2 \leq \frac{1}{2}(M^2\|T^2x\|^2 + \|x\|^2),$$

for all $x \in \mathcal{H}$. We prove the basic properties of this class of operators and the $M-$class Q operators.

Keywords: *$M-$paranormal operator, operator of $M-$class Q, M^*-paranormal operator, operator of $M-$class Q^*.*

1 Introduction

Throughout this paper, let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{L}(\mathcal{H})$ denote the C^* algebra of all bounded operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by ker$T$ the null space, by $T(\mathcal{H})$ the range of T and by $\sigma(T)$ the spectrum of T. The null operator and the identity on $\mathcal{H}$ will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$.

We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all non-negative integers by $\mathbb{N}$ and the complex conjugate of a complex number λ by $\overline{\lambda}$. The closure of a set $\mathcal{M}$ will be denoted by $\overline{\mathcal{M}}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is a positive operator, $T \geq O$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. The operator T is an isometry if $\|Tx\| = \|x\|$, for all $x \in \mathcal{H}$. The operator T is called unitary operator if $T^*T = TT^* = I$.

An operator $T \in \mathcal{L}(\mathcal{H})$, is said to be paranormal [4], if

$$\|Tx\|^2 \leq \|T^2x\|$$

for any unit vector x in $\mathcal{H}$. Further, T is said to be $*-$paranormal [2], if

$$\|T^*x\|^2 \leq \|T^2x\|$$

for any unit vector x in $\mathcal{H}$.

Arora and Kumar in [1] introduced the new class of operators, the $M-$paranormal. An operator T is $M-$paranormal if and only if

$$M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq O$$

for all $\lambda > 0$ and for a fixed real number $M \geq 1$. Equivalently T is $M-$paranormal operator if and only if

$$\|Tx\|^2 \leq M\|T^2x\|,$$

for all $x \in \mathcal{H}$, where $\|x\| = 1$ and for a fixed real number $M \geq 1$.

An operator T is M^*-paranormal [5] if and only if

$$M^2T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq O$$

for all $\lambda > 0$ and for a fixed real number $M \geq 1$. Equivalently T is M^*-paranormal operator if and only if

$$\|T^*x\|^2 \leq M\|T^2x\|,$$

for all $x \in \mathcal{H}$, where $\|x\| = 1$ and for a fixed real number $M \geq 1$.

Duggal, Kubrusly, Levan in [3] introduced a new class of operators, the class Q. An operator $T \in \mathcal{L}(\mathcal{H})$ belongs to class Q if

$$T^{*2}T^2 - 2T^*T + I \geq O.$$

It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of class Q if

$$\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2),$$

for all $x \in \mathcal{H}$.

D. Senthilkumar and Prasad T.in [6] has defined the new class of operators, the $M-$class Q. An operator $T \in \mathcal{L}(\mathcal{H})$ is of $M-$class Q, for a fixed real number $M \geq 1$, if T satisfies

$$M^2T^{*2}T^2 - 2T^*T + I \geq O.$$

They proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of $M-$class Q if

$$\|Tx\|^2 \leq \frac{1}{2}(M^2\|T^2x\|^2 + \|x\|^2),$$

for all $x \in \mathcal{H}$ and for a fixed real number $M \geq 1$.

Youngoh Yang, Cheoul Jun Kim in [7] introduced a new class of operators, the class Q^*. An operator $T \in \mathcal{L}(\mathcal{H})$ belongs to class Q^* if

$$T^{*2}T^2 - 2TT^* + I \geq O.$$

They proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of class Q^* if

$$\|T^*x\|^2 \leq \frac{1}{2}\|T^2x\|^2 + \|x\|^2),$$

for all $x \in \mathcal{H}$.

Now we introduce the new class of operator $M-$class Q^* defined as follows:

Definition 1.1. *An operator T is said to be of the $M-$class Q^* if*

$$\|T^*x\|^2 \leq \frac{1}{2}(M^2\|T^2x\|^2 + \|x\|^2),$$

for all $x \in \mathcal{H}$ and for a fixed real number $M \geq 1$.

The concept of this class is motivated by Youngoh Yang, Cheoul Jun Kim [7] and by D. Senthilkumar and Prasad T.in [6].

2 Main Results

In this section we prove some basic properties of $M-$class Q and this new class of operators $M-$class Q^*. Now, similary as D. Senthilkumar and Prasad T. in [6] we can prove the following propositions for a $M-$class Q^* operators.

Proposition 2.1. *An operator $T \in \mathcal{L}(\mathcal{H})$ is of the $M-$class Q^*, if and only if*

$$M^2T^{*2}T^2 - 2TT^* + I \geq O,$$

for a fixed real number $M \geq 1$.

Proof: Since T is of the $M-$class Q^*, then

$$\|T^*x\|^2 \leq \frac{1}{2}(M^2\|T^2x\|^2 + \|x\|^2),$$

for all $x \in \mathcal{H}$ and for a fixed real number $M \geq 1$. Then,

$$M^2\langle T^2x, T^2x\rangle + \langle x, x\rangle - 2\langle T^*x, T^*x\rangle \geq 0 \Rightarrow$$
$$\langle M^2T^{*2}T^2x, x\rangle + \langle x, x\rangle - 2\langle TT^*x, x\rangle \geq 0 \Rightarrow$$
$$\langle (M^2T^{*2}T^2 - 2TT^* + I)x, x\rangle \geq 0$$

for all $x \in \mathcal{H}$ and for a fixed real number $M \geq 1$.

The last relation is equivalent to

$$M^2T^{*2}T^2 - 2TT^* + I \geq O.$$

From the definition of M^*-paranormal operator, we have that every M^*-paranormal operator is operator of the $M-$class Q^* and it is clear that the following proposition is valid.

Proposition 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$. If $\lambda^{-\frac{1}{2}}T$ is an operator of the $M-$class Q^*, then T is a M^*-paranormal operator for all $\lambda > 0$.*

Proposition 2.3. *Let $\mathcal{M}$ be a closed $T-$invariant subspace of $\mathcal{H}$. Then, the restriction $T\mid_{\mathcal{M}}$ of a $M-$class Q^* operator T to $\mathcal{M}$ is operator of $M-$class Q^*.*

Proof: Let be $u \in \mathcal{M}$. Then,

$$\begin{aligned} &\|(T\mid_{\mathcal{M}})^*\|^2 \\ &= \|T^*u\|^2 \leq \frac{1}{2}\left(M^2\|T^2u\|^2 + \|u\|^2\right) \\ &= \frac{1}{2}\left(M^2\|(T\mid_{\mathcal{M}})^2u\|^2 + \|u\|^2\right) \end{aligned}$$

This implies that $T\mid_{\mathcal{M}}$ is an operator of the $M-$class Q^*.

In [6], author proved that a weighted shift operator T with decreasing weighted sequence (α_n) is an operator of the $M-$class Q if and only if

$$|\alpha_n|^2 \leq \frac{1}{2}(M^2|\alpha_n|^2|\alpha_{n+1}|^2 + 1)$$

for every n. In a following we proved the same results for a $M-$class Q^* operators.

Proposition 2.4. *A weighted shift T with decreasing weighted sequence (α_n) is an operator of the $M-$class Q^* if and only if*

$$|\alpha_{n-1}|^2 \leq \frac{1}{2}(M^2|\alpha_n|^2|\alpha_{n+1}|^2 + 1)$$

for every n.

Proof: Assume that the weighted shift T is an operator of the $M-$class Q^*, then

$$M^2T^{*2}T^2 - 2TT^* + I \geq O.$$

Since

$$Te_n = \alpha_n e_{n+1}, T^*e_n = \overline{\alpha_{n-1}}e_{n-1},$$
$$(T^{*2}T^2)e_n = |\alpha_n|^2|\alpha_{n+1}|^2 e_n,$$
$$TT^*e_n = |\alpha_{n-1}|^2 e_n,$$

Then we have

$$M^2T^{*2}T^2 - 2TT^* + I \geq O \Leftrightarrow$$
$$M^2|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_{n-1}|^2 + 1 \geq O \Leftrightarrow$$
$$|\alpha_{n-1}|^2 \leq \frac{1}{2}(M^2|\alpha_n|^2|\alpha_{n+1}|^2 + 1).$$

The following example show that the classes of $M-$class Q and $M-$class Q^* operators are independent

Example 2.5. *Let the operator T in l_2 defined by $T(x) = (0, \alpha_1 x_1, \alpha_2 x_2, ...)$ where: $\alpha_n = 1, \alpha_{n+1} = \frac{1}{2}, \alpha_{n+2} = 2, n \geq 1$. Clearly T is an operator of the $M-$class Q^* for all fixed real number $M \geq 1$ but T is an operator of the $M-$class Q for all fixed real number $M \geq 2$ so it is not $1-$class Q.*

In [3], authors proved that if $T \in \mathcal{L}(\mathcal{H})$ is operator of class Q and if operator T is invertible, then T^{-1} is operator of class Q. We give a similar result for an operator of $M-$class Q.

Proposition 2.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of $M-$class Q and if operator T is invertible, then T^{-1} is operator of $M-$class Q.*

Proof: If T is invertible, then

$$2\|x\|^2 = 2\|TT^{-1}x\|^2 \leq M^2\|T^2(T^{-1}x)\|^2 + \|T^{-1}x\|^2,$$

for all $x \in \mathcal{H}$ and $M \geq 1$.
Take any y in $\mathcal{H} = ran(T)$ so that $y = Tx, x = T^{-1}y$ and $T^{-1}x = T^{-2}y$ for some $x \in \mathcal{H}$. Thus

$$2\|T^{-1}y\|^2 \leq M^2\|y\|^2 + \|T^{-2}y\|^2,$$

by the above inequality, and so T^{-1} is operator of $M-$class Q.

This two classes of operator $M-$class Q and $M-$class Q^* we proved that they are different and independent of each other but the technique of the proofs of the some results for both classes are almost the same. So, in the following we omit the proofs of the results of $M-$class Q^* and introduce the proofs of the results of $M-$class Q operators only.

Proposition 2.7. *If T is an operator of the $M-$class Q ($M-$class Q^*) and if T double commutes with an isometric operator S, then TS is an operator of the $M-$class Q ($M-$class Q^*.)*

Proof: Let $A = TS$, $TS = ST$, $S^*T = TS^*$ and $S^*S = I$.

$$\begin{aligned} M^2A^{*2}A^2 - 2A^*A + I & \\ &= M^2(TS)^{*2}(TS)^2 - 2(TS)^*(TS) + I \\ &= M^2T^{*2}T^2 - 2T^*T + I \geq O, \end{aligned}$$

so TS is an operator of the $M-$class Q.

Proposition 2.8. *Let $T \in \mathcal{L}(\mathcal{H})$. If $\|T\| \leq \frac{1}{\sqrt{2}}$, ($\|T^*\| \leq \frac{1}{\sqrt{2}}$,) then T is operator of the $M-$class Q ($M-$class Q^*.)*

Proof: From $\|T\| \leq \frac{1}{\sqrt{2}}$, we have $\|Tx\|^2 \leq \frac{1}{2}$. Then,

$$I - 2T^*T \geq 0 \Rightarrow$$

$$M^2T^{*2}T^2 - 2T^*T + I \geq 0$$

so T is of the $M-$class Q.

Proposition 2.9. *If T is an operator of $M-$class Q ($M-$class Q^*) operator and if T^2 is an isometry, then T is $M-$paranormal operator (M^*-paranormal operator).*

Proof: Let T be a $M-$class Q operator. Then

$$2\|Tx\|^2 \leq (M\|T^2x\| - \|x\|)^2 + 2M\|T^2x\|\|x\|.$$

Hence T^2 is isometry, so $\|T^2x\| = \|x\|$, for all $x \in \mathcal{H}$. Then, we have

$$2\|Tx\|^2 \leq (M\|x\| - \|x\|)^2 + 2M\|T^2x\|\|x\| = (M-1)^2\|x\|^2 + 2M\|T^2x\|\|x\| \leq 2M\|T^2x\|\|x\|,$$

for all $x \in \mathcal{H}$ and $M \geq 1$,

$$\|Tx\|^2 \leq M\|T^2x\|\|x\|,$$

so T is $M-$paranormal operator.

Proposition 2.10. *If T is an operator of the $M-$class Q ($M-$class Q^*) and if T is unitarily equivalent to operator S, then S is an operator of the $M-$class Q ($M-$class Q^*).*

Proof: Since T is unitarily equivalent to operator S, there is an unitary operator U such that $S = U^*TU$. Since T is an operator of the $M-$class Q, then

$$M^2T^{*2}T^2 - 2T^*T + I \geq O.$$

Hence,

$$M^2S^{*2}S^2 - 2S^*S + I =$$
$$M^2(U^*TU)^{*2}(U^*TU)^2 - 2(U^*TU)^*(U^*TU) + I =$$
$$U^*(M^2T^{*2}T^2 - 2T^*T + I)U \geq O,$$

so S is an operator of the $M-$class Q.

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