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Graphs Associated with Finite Zero Ring

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Abstract

This paper introduces the notion of zero ring graphs. Basic properties of the zero ring graphs are investigated and characterization results regarding connectedness and planarity are given. Further, we determine the chromatic number for $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$*, where* p *is a prime,* $k > 1$ *and it is shown that an isomorphism exists among the zero ring graphs* $\Gamma M_2^0(\mathbb{Z}_{p^k})$, $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ *and* $\Gamma M_2^0(\mathbb{F}_{p^k})$.

Keywords: *Clique, Hamiltonian, Isomorphism, Planarity, Zero ring.*

1 Introduction

For all terminology and notation in graph theory and abstract algebra, not specifically defined in this paper, we refer the reader to the text-book by West [6] and Jacobson [4] respectively. Unless mentioned otherwise, all graphs considered in the paper are finite, connected and simple.
In this paper, our aim is to introduce the notion of *zero ring graph* (the graph whose vertices are the elements of a zero ring) and study on its basic properties such as degree, planarity, chromatic number and perfectness, for this first we should be familiar with the *zero ring* and its related results. In this regard, we state some results on zero ring which make the paper self contained.

A ring in which the product of any two elements is 0 is called a *zero ring* and denoted by R^0, where '$0'$ is the additive identity of *zero ring*. One of the standard example of *zero ring* is the set of 2×2 matrices defined as

$$\left\{\begin{bmatrix} r & -r \\ r & -r \end{bmatrix}\right\},$$

where $r \in R$ and R is a commutative ring. Throughout the paper we shall denote the above set by $M_2^0(R)$. The above example implies the validity of following result.

Theorem 1.1 *For every positive integer n, there exists a zero ring of order n.*

There are many interesting properties of zero ring available in the literature (e.g., see [1, 7]). Some elementary ones are listed below.

Proposition 1.2 *Every zero ring is commutative.*

Proof. Proof is immediately follows from the definition of zero ring. □

Given a ring R, an element $z \in R$ is called a *left zero-divisor* (respectively *right zero-divisor*) of R if there exists a non zero element $a \in R$ (respectively, a non zero element $b \in R$) such that $z.a := za = 0$ (respectively, $b.z := bz = 0$); in particular, if z is both a left zero-divisor and a right zero-divisor of R, then z is simply called a *zero-divisor* of R. The set of all zero-divisors of R is denoted by $Z(R)$.

Proposition 1.3 *For each prime p, the set of all zero-divisors of ring $\mathbb{Z}_{p^2}$ is a zero ring of order p.*

Proof. Consider the ring $\mathbb{Z}_{p^2}$, clearly all the multiples of p are zero-divisors i.e.,

$$Z(\mathbb{Z}_{p^2}) = \{0, p, 2p, \cdots p(p-1)\}.$$

Now, our aim is to show that the set $Z(\mathbb{Z}_{p^2})$ is a zero ring, towards this, firstly, we need to check whether it is a ring or not. It is easy to verify that $(Z(\mathbb{Z}_{p^2}), +_{p^2})$ is an abelian group with additive identity 0 and additive inverse of any element mp is $p^2 - mp$ and $(Z(\mathbb{Z}_{p^2}), \times_{p^2})$ is closed as well as associative. To examine the distributive property choose $0 \leq r_1, r_2, r_3 \leq p-1$ then

$$r_1p(r_2p +_{p^2} r_3p) = r_1(r_2 +_{p^2} r_3)p^2 = 0 = (r_1r_2p^2 +_{p^2} r_1r_3p^2).$$

We, next proceed to show that it is a zero ring, for every pair of two elements $a_1, a_2 \in Z(\mathbb{Z}_{p^2})$, one may have

$$a_1 \cdot a_2 = 0(\text{mod } p^2).$$

Hence the result. □

Motivated from the definition of zero ring, here we are defining the *proper zero ring.*
A ring in which the product of two distinct elements is 0 called a *proper zero ring* denoted by R^*, mathematically, $a \cdot b = 0 \ \forall \ a, b \in R^*$ and $a \neq b$, where '$0'$ is the additive identity of *proper zero ring.*

Proposition 1.4 *Every zero ring is a proper zero ring but the converse is not true.*

The following example illustrates Proposition 1.4.

Example 1.5 Consider the subring S of the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ defined as follows:

$$S = \{(0,0), (0,1)\}.$$

In subring S, one can easily see that

$$(0,1) \times (0,1) \neq (0,0) \text{ and } (0,0) \times (0,1) = (0,0),$$

which shows that S is a proper zero ring but not a zero ring.

Proposition 1.6 [7] *If* $n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdots p_k^{a_k}$, *then the number of zero rings of order* n *are* $p(a_1) \cdot p(a_2) \cdots p(a_k)$, *where* $p(a_i)$ *denotes the partition of a natural number.*

The following corollary is an immediate consequence of Proposition 1.6.

Corollary 1.7 *If* n *is the product of distinct primes, then there exists only one zero ring of order* n.

The following is a formal definition of the new notion to start with.

Definition 1.8 Let $(R^0, +, \cdot)$ be a finite zero ring then the *zero ring graph*, denoted as ΓR^0, is a graph whose vertices are the elements of zero ring R^0 and two distinct vertices x and y are adjacent if and only if $x + y \neq 0$, where '$0'$ is the additive identity of R^0.

The graphs shown in Figure 1 are the zero ring graphs, where

$$v_i = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}; \quad i \in R$$

and R is a commutative ring, viz., $\mathbb{Z}_6, \mathbb{Z}_3$ and $\mathbb{Z}_2[x]/\langle x^2 \rangle$
The Proposition 1.9 is a consequence of Definition 1.8

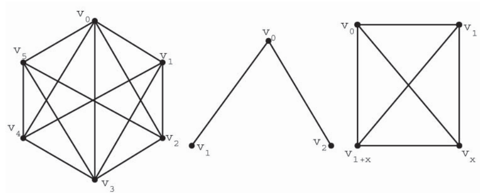

Figure 1: $\Gamma M_2^0(\mathbb{Z}_6)$, $\Gamma M_2^0(\mathbb{Z}_3)$ and $\Gamma M_2^0(\mathbb{Z}_2[x]/\langle x^2\rangle)$

Proposition 1.9 *For each positive integer $n > 1$, there exists a zero ring graph ΓR^0 of order n.*

Further, one may naturally ask: Does there exist a graph which is not a zero ring graph? The answer is in affirmative, for example the complete graph K_6. In fact the above question raises the following research problem:

Problem 1.10 *Characterize those graphs which are not the zero ring graphs.*

2 Zero Ring Graph $\Gamma M_2^0(\mathbb{Z}_n), n > 1$

In this section, we shall investigate some important concrete properties and establish theorems which we required in the subsequent sections.
We start with the graph theoretical properties of $\Gamma M_2^0(\mathbb{Z}_n), n > 1$.

Theorem 2.1 *The total number of edges in a zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$, is given by*

$$\begin{cases} \frac{(n-1)^2+1}{2}, & if\ n\ is\ even \\ \frac{(n-1)^2}{2}, & if\ n\ is\ odd \end{cases}$$

Proof. For a positive integer $n > 1$, $M_2^0(\mathbb{Z}_n)$ is a zero ring of order n associated with $\mathbb{Z}_n$, consisting the elements

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \ldots, \begin{bmatrix} n-1 & -(n-1) \\ n-1 & -(n-1) \end{bmatrix} \right\}$$

Clearly, this is the vertex set of $\Gamma M_2^0(\mathbb{Z}_n)$. In order to calculate the total number of edges, it is suffices to calculate, the non adjacent pairs in $\Gamma M_2^0(\mathbb{Z}_n)$. Any two vertices are non adjacent in $\Gamma M_2^0(\mathbb{Z}_n)$, if they produces a zero sum modulo n. Note that

$$\begin{bmatrix} i & -i \\ i & -i \end{bmatrix} + \begin{bmatrix} j & -j \\ j & -j \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

if and only if $i + j = 0(\text{mod } n)\ \forall\ i, j,$ where $1 \le i, j \le n-1$. Now there are two cases for n.

Firstly, let n be even, then there are exactly $\frac{n}{2}-1$ pairs of distinct vertices each of which produces a zero sum modulo n, viz., $(1,n-1),(2,n-2),\cdots(\frac{n}{2}-1,\frac{n}{2}+1)$. Therefore, $\frac{n}{2}-1$ edges will not appear in $\Gamma M_2^0(\mathbb{Z}_n)$. It is clear that maximum number of edges in a graph of order n is $\frac{n}{2}(n-1)$.
Thus $\Gamma M_2^0(\mathbb{Z}_n)$ would have $\frac{n}{2}(n-1)-(\frac{n}{2}-1)=\frac{n^2}{2}-n+1$ edges.

Next, let n be odd then by the previous argument, the pairs which are non adjacent in $\Gamma M_2^0(\mathbb{Z}_n)$ are precisely $(1,n{-}1),(2,n{-}2),(\frac{n}{2}{-}1,\frac{n}{2}{+}1),\cdots(\frac{n-1}{2},\frac{n+1}{2})$. Thus the total number of edges in $\Gamma M_2^0(\mathbb{Z}_n)$ is

$$\frac{n}{2}(n-1)-(\frac{n-1}{2})=\frac{(n-1)^2}{2}.$$

Hence the proof. □

Corollary 2.2 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n), n>2$ is never complete.*

Proof. Suppose on contrary that $\Gamma M_2^0(\mathbb{Z}_n), n>2$ is complete, this implies that total number of edges in $\Gamma M_2^0(\mathbb{Z}_n), n>2$ are $\frac{n}{2}(n-1)$ but in view of Theorem 2.1, we arrived at a contradiction to the completeness of $\Gamma M_2^0(\mathbb{Z}_n)$. □

Remark 2.3 *If a finite graph G have number of edges as in Theorem 2.1, then G need not be a zero ring graph of $\Gamma M_2^0(\mathbb{Z}_n)$.*

The graph shown in Figure 2 illustrates the Remark 2.3 in which both the graphs have same number of vertices and edges. However, the graph in (a) is the zero ring graph of $\Gamma M_2^0(\mathbb{Z}_5)$ but the other one is not.

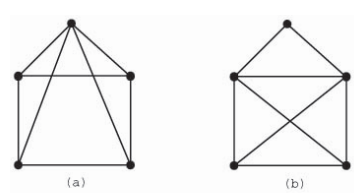

Figure 2:

The above discussion raises the following question:

Problem 2.4 *What additional conditions should be imposed on a finite graph G having number of edges as in Theorem 2.1, so that it becomes a zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$?*

Proposition 2.5 *For a zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$, the following statements hold:*

(i) for even n,

$$deg(v_i) = \begin{cases} n-1, & if\ i \in \{0, \frac{n}{2}\} \\ n-2, & otherwise \end{cases}$$

(ii) for odd n,

$$deg(v_i) = \begin{cases} n-1, & if\ i = 0 \\ n-2, & otherwise \end{cases}$$

Proof. (i) Trivially, the vertex $v_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in $\Gamma M_2^0(\mathbb{Z}_n)$ is adjacent to all other vertices, this implies that degree of v_0 is $n-1$. Any two distinct vertices v_i and v_j in $\Gamma M_2^0(\mathbb{Z}_n)$ are adjacent if and only if

$$\begin{bmatrix} i & -i \\ i & -i \end{bmatrix} + \begin{bmatrix} j & -j \\ j & -j \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

if and only if $i + j \neq 0(\text{mod } n)$, $\forall\ i \neq j$, where $1 \leq i, j \leq n-1$, therefore it is easy to derive that the degree of vertex $v_{\frac{n}{2}}$ is $n-1$. Due to Theorem 2.1, for even n, the pairs of distinct vertices each of which produces a zero sum modulo n, are precisely, $(v_1, v_{n-1}), (v_2, v_{n-2}), \cdots, (v_{\frac{n}{2}-1}, v_{\frac{n}{2}+1})$. This implies that all these vertices in $\Gamma M_2^0(\mathbb{Z}_n)$ have same degree, which is $< n-1$. However, each vertex in the above pairs is non adjacent with exactly one vertex. Hence, the remaining vertices have degree $n-2$.

(ii) Now if n is odd, then v_0 is the only vertex in $\Gamma M_2^0(\mathbb{Z}_n)$ which is adjacent to all other vertices, therefore the degree of vertex v_0 is $n-1$. The remaining part of the proof can be given by the arguments analogues to those used in (i). □

In light of Proposition 2.5, the following remark is obvious.

Remark 2.6 *The zero ring graph* $\Gamma M_2^0(\mathbb{Z}_n), n > 2$ *is never regular.*

A graph is called (r_1, r_2)-*semi regular* if its vertex set can be partitioned into two subsets V_1 and V_2 such that all the vertices in V_i are of degree r_i for $i = 1, 2$.

Corollary 2.7 *The zero ring graph* $\Gamma M_2^0(\mathbb{Z}_n)$ *is always* $(n-1, n-2)$*-semi regular.*

Proof. The proof of the result follows due to Proposition 2.5 together with the definition of semi-regularity. □

For distinct vertices x and y of a graph G, let $d(x, y)$ be the length of the shortest path from x to y, the *diameter* of a graph, denoted by diam(G), is given by $diam(G) = \max\{d(x, y) : \ x, y\ \in V(G)\}$.

Theorem 2.8 *The diameter of zero ring graph* $\Gamma M_2^0(\mathbb{Z}_n),\ n > 2$ *is* 2.

Proof. Consider the zero ring graph $\Gamma M_2^0(\mathbb{Z}_n), n > 2$ having the vertices of the form

$$v_i = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}; \quad i \in \mathbb{Z}_n \ .$$

Since the vertex v_0 is adjacent to every vertex of $\Gamma M_2^0(\mathbb{Z}_n),\ n > 2$, however v_j is not adjacent to v_{n-j} for all $1 \leq j < n-1$ (i.e., each vertex is non adjacent with exactly one vertex), so $d(v_j, v_{n-j}) > 1$, but in $\Gamma M_2^0(\mathbb{Z}_n),\ n > 2$, there always exists a path $v_{n-j} \to v_0 \to v_j$, which gives

$$d(v_{n-j}, v_j) = 2 \quad \forall\ j$$

It follows that

$$\text{diam}(\Gamma M_2^0(\mathbb{Z}_n) = 2. \quad \square$$

Theorem 2.9 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$, $n > 3$ is never a cycle graph. However, $\Gamma M_2^0(\mathbb{Z}_n)$ properly contains a cycle of length n.*

Proof. We know that a graph G is cycle if the degree of each vertex is 2. Now due to Proposition 2.5 each vertex in $\Gamma M_2^0(\mathbb{Z}_n)$ can not have degree 2 for any $n > 3$. We shall now show that $\Gamma M_2^0(\mathbb{Z}_n)$ always contains a cycle. To do this, let v_i be the i^{th} vertex in $\Gamma M_2^0(\mathbb{Z}_n)$ defined as $v_i = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, 0 \leq i \leq n-1$, and consider the following two cases:

Case 1: If n is even, then

$$v_0 \to v_1 \to v_2 \to v_3 \to \cdots \to v_n \to v_0$$

ensure the existence of a cycle which covers all the vertices (as sum of any two consecutive vertices is non zero).

Case 2: If n is odd, then we construct a cycle

$$v_0 \to v_1 \to v_2 \to v_3 \to \cdots \to v_{\frac{n-1}{2}} \to v_{\frac{n+3}{2}} \to v_{\frac{n+5}{2}} \to \cdots \to v_{n-1} \to v_{\frac{n+1}{2}} \to v_0,$$

which again covers all the vertices. Hence, in both the cases there exist a cycle of length n in $\Gamma M_2^0(\mathbb{Z}_n)$ covering all the vertices. $\square$

A graph is *Hamiltonian* if it has a cycle that visits every vertex exactly once; such a cycle is called a *Hamiltonian cycle*. In a graph to find the *Hamiltonian cycle* is an *NP-complete problem*. In fact, it is considered as a particular case of the traveling salesman problem and still there is not any result which gives the characterization of a *Hamiltonian graph*. An Interesting example of *NP-complete problem* is the graph isomorphism problem. For more details on *Hamiltonian cycle* the reader is referred to [3].

One of the interesting property of zero ring graphs is that they provides a class of *Hamiltonian graph,* (cf.: Theorem 2.10) thus study on the zero ring graphs seems to be an extremely useful tool for the above problem.

Theorem 2.10 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$, $n > 3$ is always Hamiltonian.*

Proof. The result is an immediate consequence of Theorem 2.9. □

A graph G is called *connected* if for any vertices x and y of G there is a path between x and y, otherwise, G is disconnected.

Theorem 2.11 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$ is always connected.*

Proof. In light of Theorem 2.10, one can easily see that $\Gamma M_2^0(\mathbb{Z}_n)$ is Hamiltonian which ensures that there exist a path between every pair of vertices. □

A connected graph without any cycle is called a *tree*.

Theorem 2.12 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$ is tree if and only if $n = 2$ or 3.*

Proof. It is easy to see from Theorem 2.9 that for $n > 3$, $\Gamma M_2^0(\mathbb{Z}_n)$ always contains a cycle, so can not be a tree. On the other hand, for $n = 2$ and 3, $\Gamma M_2^0(\mathbb{Z}_n)$ is K_2 and $K_{1,2}$ respectively. Thus the result follows. □

Theorem 2.13 [6] *A graph is Eularian if and only if degree of each vertex is even.*

Theorem 2.14 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$ is never Eularian.*

Proof. Suppose on contrary that $\Gamma M_2^0(\mathbb{Z}_n)$ is Eularian, which implies that degree of each vertex is even. By the Proposition 2.5, it is clear that the degree of each vertex is either $n-1$ or $n-2$. Now, there are two possibilities: if n is even, then $n-1$ is odd. On the other hand, if n is odd, then $n-2$ is odd. Hence in both the possibilities, we found that degree of each vertex can not be even. But, this contradicts that $\Gamma M_2^0(\mathbb{Z}_n)$ is Eularian. Thus, by contraposition, the result follows. □

Theorem 2.15 [6] *A simple graph G is bipartite if and only if G does not have any odd cycle.*

Theorem 2.16 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$ is bipartite if and only if $n \in \{2, 3\}$.*

Proof. Necessity: Suppose that G is bipartite. To show the result, let if possible $n \neq 2, 3$, firstly, let n be 4 and clearly the vertices $v_0, v_1, v_2 \in V(\Gamma M_2^0(\mathbb{Z}_4))$ are pairwise adjacent, where v_i is defined as in Theorem 2.8. Therefore,

$$v_0 \to v_1 \to v_2 \to v_0$$

would be a triangle. But, this contradicts our assumption that G is bipartite. It follows that n can not be 4. Further if $n > 4$, then clearly, vertices $v_0, v_1, v_2 \in V(\Gamma M_2^0(\mathbb{Z}_4))$ again form a 3-cycle. Therefore, if $n \geq 4$, then it is not difficult to see the existence of 3-cycle, by which we arrived at the contradiction. Thus n can not be ≥ 4. Hence n must be 2 or 3.

Sufficiency: The sufficiency is easy to see as for instance, $\Gamma M_2^0(\mathbb{Z}_2)$ and $\Gamma M_2^0(\mathbb{Z}_3)$ are isomorphic to K_2 and $K_{1,2}$ respectively. Clearly, both are bipartite. □

The following remark is an immediate consequence of Theorem 2.16.

Remark 2.17 *The zero ring graphs $\Gamma M_2^0(\mathbb{Z}_n), n > 3$, are not triangle free.*

Theorem 2.18 *The girth of zero ring graph $\Gamma M_2^0(\mathbb{Z}_n), n > 3$ is always* 3.

Proof. In view of Theorem 2.16, for $n \geq 4$, we always have a 3-cycle

$$v_0 \to v_1 \to v_2 \to v_0,$$

which is smallest. This completes the proof. □

For a nonnegative integer k, an *k–partite graph* is one whose vertex set is partitioned into k disjoint parts in such a way that the two end vertices for each edge belongs in distinct partitions.

Theorem 2.19 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n), n > 1$ is*

$$\begin{cases} \frac{n}{2} + 1 - partite, & if\ n\ is\ even \\ \frac{n+1}{2} - partite, & if\ n\ is\ odd \end{cases}$$

Proof. Consider the zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$ having the vertices

$$v_i = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}; \quad i \in \mathbb{Z}_n.$$

Due to the nature of vertices $\Gamma M_2^0(\mathbb{Z}_n)$, maximum two vertices can be in the independent set. Now there are two cases for n.

Firstly, suppose that n is odd, then independent set are

$$\{v_1, v_{n-1}\}, \{v_2, v_{n-2}\}, \cdots, \{v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}\}.$$

Since v_0 is adjacent to all the vertices, it will be in a different part. Therefore, the number of independent set is equal to

$$\frac{n-1}{2} + 1 = \frac{n+1}{2} = \text{ cardinality of a part.}$$

On the other hand, when n is even then the independent sets are

$$\{v_1, v_{n-1}\}, \{v_2, v_{n-2}\}, \cdots, \{v_{\frac{n-2}{2}}, v_{\frac{n+2}{2}}\}.$$

Since vertices v_0 and $v_{\frac{n}{2}}$ are adjacent to all the vertices, both will be in different-different part. Following the above procedure, the number of independent set is equal to

$$\frac{n-2}{2} + 2 = \frac{n}{2} + 1$$

which is the cardinality of part. Hence the result. □

A *k-edge coloring* of a graph G is an assignment of k *labels*, also called colors, to the edges of G such that every pair of distinct edges sharing a common vertex are assigned two different colors. The chromatic index of G, denoted $\chi'(G)$, is the smallest positive integer k such that G has *k-edge coloring.* An important theorem for edge chromatic number given by Vizing is as follows:

Theorem 2.20 [2] *Let G be a graph whose maximum vertex degree is Δ, then $\Delta \leq \chi'(G) \leq \Delta + 1$.*

The graphs which attains lower bound of Vizing inequality called class 1 and which attains upper bound called class 2.

Theorem 2.21 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$ is of class 1.*

Now, we shall characterize the planarity and outerplanarity of zero ring graphs $\Gamma M_2^0(\mathbb{Z}_n)$.

A graph is said to be *planar* if it can be drawn in the plane in such a way that its edges intersect only at their ends. A remarkable simple characterization of *planar graphs* was given by Kuratowski [5] as follows:

Theorem 2.22 [6] *A graph is planar if and only if it does not have a subgraph homeomorphic to K_5 or $K_{3,3}$.*

Theorem 2.23 [6] *Let G be simple planar graph having $n(G) \geq 3$ vertices and $e(G)$ edges then*

$$e(G) \leq 3n(G) - 6.$$

Theorem 2.24 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$ is planar if and only if $n \in \{2, 3, 4, 5\}$.*

Proof. If $n \in \{2, 3, 4, 5\}$, then one can easily produce a planar representation of $\Gamma M_2^0(\mathbb{Z}_n)$. We shall now show that $\Gamma M_2^0(\mathbb{Z}_n)$ is non-planar for any other values of n. Since a simple planar connected graph has a vertex of degree less than six, by Proposition 2.5, all the vertices in $\Gamma M_2^0(\mathbb{Z}_n)$ have degree either

$n-1$ or $n-2$, therefore, if $n-2 \geq 6$, then $\Gamma M_2^0(\mathbb{Z}_n)$ can not be planar. This implies that $\Gamma M_2^0(\mathbb{Z}_n)$ is non-planar for all $n \geq 8$. Next, it remains to prove the result for $n = 6$, and 7, we shall show the desired result by contradiction, suppose if possible $\Gamma M_2^0(\mathbb{Z}_6)$ and $\Gamma M_2^0(\mathbb{Z}_7)$ are planar, then the number of edges $e(G)$ in zero ring graphs (cf.: Theorem 2.1) are 13 and 18 respectively. However, according to the assumption $\Gamma M_2^0(\mathbb{Z}_6)$ is planar then it must satisfy

$$e(G) \leq 3n(G) - 6.$$

$$13 \leq 3 \times 6 - 6$$

$$13 \leq 12,$$

which is a preposterous. Similarly for $n = 7$, the argument is same as given for $n = 6$. Therefore, our assumption is wrong. Hence $\Gamma M_2^0(\mathbb{Z}_n)$ is non planar for all $n \geq 6$. □

A planar graph is called *outerplanar* if it can be embedded in a plane in such a way that all of its vertices are in the same face.

Theorem 2.25 [2] *A graph G is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$ except $K_4 - x$.*

Theorem 2.26 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_n)$ is outerplanar if and only if $n \in \{2, 3, 4, 5\}$.*

Proof. Due to Theorem 2.24, the only possibility for $\Gamma M_2^0(\mathbb{Z}_n)$ to be outerplanar is that $n \in \{2, 3, 4, 5\}$. If $n \in \{2, 3, 4, 5\}$, then we can easily produce an outerplanar representation of $\Gamma M_2^0(\mathbb{Z}_n)$. Due to Theorem 2.25, $\Gamma M_2^0(\mathbb{Z}_n), n > 5$ is non-outerplanar. Hence the result. □

An *Isomorphism* from G to H is a bijection f that maps $V(G)$ to $V(H)$ and $E(G)$ to $E(H)$ such that each edge of G with end vertices u and v is mapped to an edge with end vertices $f(u)$ and $f(v)$.

Theorem 2.27 [*6*] *Two graphs G and H are isomorphic if and only if their complements are isomorphic, i.e., $G \cong H$ iff $G^c \cong H^c$.*

To prove the next result, first we need the following. For the quotient ring $Q = \mathbb{Z}_p[x]/\langle x^k \rangle$, where p is a prime and $k > 1$, the elements are congruence classes of polynomials modulo under the principal ideal $\langle x^k \rangle$. Any element in Q is of the form $q(x) = q_0 + q_1x + q_2x^2 + \cdots + q_{k-1}x^{k-1}$, where $q_i \in \mathbb{Z}_p$ for each i.

Note that if two zero rings are isomorphic, then obviously their zero ring graphs are also isomorphic, but the zero ring graphs may be isomorphic although rings are not. The next theorem provides such a class.

Theorem 2.28 *For each odd prime p, the zero ring graphs of order p^2 are isomorphic.*

Proof. In view of Proposition 1.6 there exist only two zero rings of order p^2, viz., $M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ and $M_2^0(\mathbb{Z}_{p^2})$, clearly both rings are non-isomorphic (under addition only one forms a cyclic group). Now, it remains to show that ΓR_1^0 and ΓR_2^0 are isomorphic, where $R_1^0 = M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ and $R_2^0 = M_2^0(\mathbb{Z}_{p^2})$. Since $\Gamma^c R_1^0 \cong \frac{(p^2-1)}{2} K_2 \cup K_1 \cong \Gamma^c R_2^0$, so, due to Theorem 2.27, we get the desired result. The complement of $\Gamma M_2^0(\mathbb{Z}_9)$ and $\Gamma M_2^0(\mathbb{Z}_3[x]/\langle x^2\rangle)$ are shown in Figure 3(a) and 3(b) respectively. □

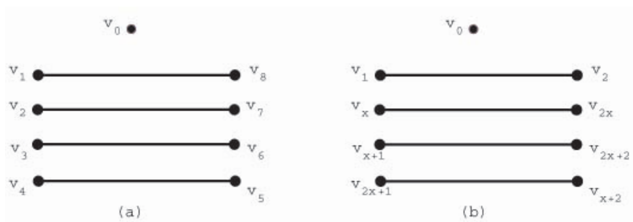

Figure 3: $(a) \cong \Gamma^c M_2^0(\mathbb{Z}_9)$ and $(b) \cong \Gamma^c M_2^0(\mathbb{Z}_3[x]/\langle x^2\rangle)$

Motivation for studying zero ring graph associated with polynomial quotient rings $\mathbb{Z}_p[x]/\langle x^k\rangle$, mainly stems from Theorem 2.28.

3 Zero Ring Graph $\Gamma(M_2^0(R))$, where R is a Polynomial Quotient Ring

In this section, we shall discuss the structure of $\Gamma(M_2^0(R))$, where R is a polynomial quotient ring $\mathbb{Z}_p[x]/\langle x^k\rangle$, and p is a prime with $k > 1$. Further, we determine the clique and chromatic number of $\Gamma(M_2^0(R))$, using this, we discover a connection between $\Gamma M_2^0(\mathbb{Z}_{p^k})$ and $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$, p is an odd prime.

A *complete graph* is a graph in which each pair of distinct vertices is joined by an edge, denoted as K_n. It is interesting to note that the following theorem asserts that $\Gamma M_2^0(\mathbb{Z}_2[x]/\langle x^k\rangle)$ is complete.

Theorem 3.1 *The zero ring graph $\Gamma M_2^0(\mathbb{Z}_2[x]/\langle x^k\rangle)$, $k > 1$ is complete.*

Proof. Consider the zero ring $M_2^0(\mathbb{Z}_2[x]/\langle x^k\rangle)$ having the elements

$$\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} x & -x \\ x & -x \end{bmatrix}$$

$$\begin{bmatrix} x+1 & -(x+1) \\ x+1 & -(x+1) \end{bmatrix}, \dots, \begin{bmatrix} x^{k-1} & -x^{k-1} \\ x^{k-1} & -x^{k-1} \end{bmatrix}, \begin{bmatrix} x^{k-1}+1 & -(x^{k-1}+1) \\ x^{k-1}+1 & -(x^{k-1}+1) \end{bmatrix}\}.$$

Clearly, it is the vertex set of zero ring graph and it is not difficult to verify that the sum of any two elements is non zero, therefore, every pair of distinct vertices are adjacent. Hence $\Gamma M_2^0(\mathbb{Z}_2[x]/\langle x^k\rangle)$ is complete. □

The *clique number* of a graph G denoted by $\omega(G)$, is the size of a maximum clique.

Theorem 3.2 *The clique number of* $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ *is* $\frac{p^k+1}{2}$, *where* p *is an odd prime and* $k>1$.

Proof. Consider $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$, where p is an odd prime and $k>1$.
Let $v_0, v_1, v_2, \cdots v_{p^k-1}$, be p^k vertices of zero ring graph $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$, as follows

$$\begin{bmatrix} a(x) & -a(x) \\ a(x) & -a(x) \end{bmatrix}: \quad a(x) \in \mathbb{Z}_p[x]/\langle x^k\rangle.$$

Clearly, every element $a(x) \in \mathbb{Z}_p[x]/\langle x^k\rangle$ is a polynomial in x with degree $< k$.
Let v_i and v_j be two vertices of $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ such that

$$v_i = \begin{bmatrix} a(x) & -a(x) \\ a(x) & -a(x) \end{bmatrix}; \quad \text{and} \quad v_j = \begin{bmatrix} b(x) & -b(x) \\ b(x) & -b(x) \end{bmatrix},$$

where

$$a(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{k-1}x^{k-1}, \ a_i \in \mathbb{Z}_p, \ 0 \le i \le k-1$$

and

$$b(x) = b_0 + b_1x + b_2x^2 + \cdots + b_{k-1}x^{k-1}, \ b_i \in \mathbb{Z}_p, \ 0 \le i \le k-1$$

then the sum $v_i + v_j = \begin{bmatrix} a(x)+b(x) & -(a(x)+b(x)) \\ a(x)+b(x) & -(a(x)+b(x)) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Leftrightarrow a(x) + b(x) \cong 0(\text{mod } p)$$

$$\Leftrightarrow a_i + b_i \cong 0(\text{mod } p)$$

$$\Leftrightarrow b_i = p - a_i \text{ for all } 0 \le i \le k-1 \text{ and for all non zero } a_i \text{ and } b_i. \quad (1)$$

Trivially, the vertex v_0 is adjacent with all other vertices in $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$, also any two distinct vertices v_i and v_i are non adjacent if and only if the coefficient of their associated polynomial satisfies (1) it follows that any vertex v_i is non adjacent with exactly one vertex v_j and hence, such vertices are $p^k - 1$ in number. Clearly, the pair of non adjacent vertices are $\frac{p^k-1}{2}$, and hence total number of mutually adjacent vertices are $p^k - (\frac{p^k-1}{2}) = \frac{p^k+1}{2}$, which is size of a maximum clique. □

An *independent set* of vertices (also called a *coclique*) in a graph is the set of pairwise non-adjacent vertices.

Theorem 3.3 *The chromatic number of* $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ *is* $\frac{p^k+1}{2}$, *where* p *is an odd prime and* $k > 1$.

Proof. Using $\chi(G) \geq \omega(G)$ [6] together with the Theorem 3.2, we get

$$\chi(G) \geq \frac{p^k+1}{2}.$$

Now, we shall show the desired result with the help of coclique. Except the vertex v_0, every vertex in a zero ring graph is non adjacent with precisely one vertex. Therefore, maximum coclique is of size 2, moreover such coclique are $\frac{p^k-1}{2}$ in number. In fact each coclique is uniquely colorable, means that for all these vertices we need $\frac{p^k-1}{2}$ colours. However vertex v_0 is adjacent with all the vertices, thus we require one more colour distinct from these colours. Hence, minimum number of colours required to colour the zero ring graph are $\frac{p^k-1}{2}+1$, which is equal to $\frac{p^k+1}{2}$. Thus the result follows. □

A graph G is called *weakly perfect* if its chromatic number is equal to clique number. The following result ensures that the zero ring graph is weakly perfect. Moreover, it gives a class of weakly perfect graph.

Theorem 3.4 *For each odd prime* p, $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$, $k > 1$ *is weakly perfect.*

Proof. Invoking the Theorems 3.3 and 3.4, result follows. □

Theorem 3.5 *The zero ring graphs* $\Gamma M_2^0(\mathbb{Z}_{p^k})$ *and* $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ *are isomorphic, where* p *is an odd prime and* $k > 1$.

Proof. We shall establish the result with the help of Theorem 2.27, clearly the vertices which are non adjacent in graph will be adjacent in the complement. In view of Theorem 3.3, notice that a maximum coclique is of size two and number of such coclique are $\frac{p^k-1}{2}$, it follows that $\Gamma^c M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ have $\frac{p^k-1}{2}$ copies of K_2 with an isolated vertex as 'v_0' is adjacent with all the vertices in $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ that is

$$\Gamma^c M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle) \cong (\frac{p^k-1}{2})K_2 \cup K_1.$$

Similarly, to see the complement of $\Gamma M_2^0(\mathbb{Z}_{p^k})$, it is sufficient to find the non adjacent pair in $\Gamma M_2^0(\mathbb{Z}_{p^k})$ and they are precisely

$$(v_1, v_{p^k-1}), (v_2, v_{p^k-2}), \cdots, (v_{\frac{p^k-1}{2}}, v_{\frac{p^k+1}{2}}).$$

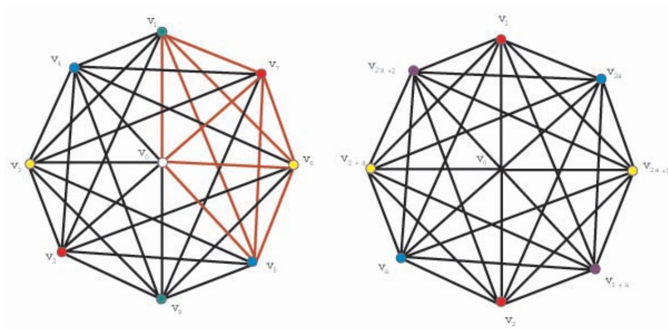

Figure 4: $\Gamma M_2^0(\mathbb{Z}_9)$ and $\Gamma M_2^0(\mathbb{Z}_3[x]/\langle x^2\rangle)$ containing K_5

Therefore, again all these form K_2 with a isolated vertex in complement. Thus $\Gamma^c M_2^0(\mathbb{Z}_{p^k}) \cong (\frac{p^k-1}{2})K_2 \cup K_1$. Hence the result. □

We conclude this section with two examples which illustrates Theorems 3.2 and 3.3, the chromatic number of $\Gamma M_2^0(\mathbb{Z}_9)$ is equal to five and the chromatic number of $\Gamma M_2^0(\mathbb{Z}_3[x]/\langle x^2\rangle)$ is also equal to five. The graph displayed in Figure 4. Here the different bullets indicates the presence of the different colors.

4 Zero Ring Graph $\Gamma(M_2^0(R))$, where R is a Finite Field

For the definition and basic properties of field, we refer the reader to Jacobson [4]. As treated in [4], we shall denote the finite field by $\mathbb{F}_q$, where $q = p^k$. Now to extend our study on a field, we need to consider two cases depending on the values of k, firstly, let $k = 1$ in this case q is just a prime, then $\mathbb{F}_p$ is isomorphic to $\mathbb{Z}_p$, then obviously their respective zero ring graphs will be isomorphic. Consequently, the study done in Section 2 for $\Gamma M_2^0(\mathbb{Z}_n)$, when n is a prime is similar to the study for $\Gamma M_2^0(\mathbb{F}_p)$. Next, it remains to study for $k > 1$.

The following theorem, which is one of the main result of this section contains the complete description for $k > 1$.

Theorem 4.1 *The zero ring graphs* $\Gamma M_2^0(\mathbb{F}_{p^k})$ *and* $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ *are isomorphic, where* p *is a prime.*

Proof. To show the desired result, we shall deal with two cases, viz., p is an odd prime and p is an even prime together with $k > 1$.

Case I: Firstly, let us suppose that p is an even prime. In light of Theorem 3.1, $\Gamma M_2^0(\mathbb{Z}_2[x]/\langle x^k\rangle)$ is isomorphic to complete graph. To show the isomorphism, it is enough to examine the structure of $\Gamma M_2^0(\mathbb{F}_{2^k})$. We know that $\mathbb{F}_{2^k}$ is isomorphic to $\mathbb{Z}_2[x]/\langle p(x)\rangle$, where $p(x)$ is an irreducible polynomial of degree k over $\mathbb{Z}_2$. Consider $p(x) = x^k + x + 1$ then the elements of zero ring $M_2^0(\mathbb{F}_{2^k})$ are

$$\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} x & -x \\ x & -x \end{bmatrix} \begin{bmatrix} x+1 & -(x+1) \\ x+1 & -(x+1) \end{bmatrix}$$
$$,\dots, \begin{bmatrix} x^{k-1} & -x^{k-1} \\ x^{k-1} & -x^{k-1} \end{bmatrix}, \begin{bmatrix} x^{k-1}+1 & -(x^{k-1}+1) \\ x^{k-1}+1 & -(x^{k-1}+1) \end{bmatrix}\}.$$

Which is the vertex set of $\Gamma M_2^0(\mathbb{F}_{2^k})$ and it is not difficult to verify that the sum of any two elements is non zero, therefore every pair of vertices are adjacent. Thus, $\Gamma M_2^0(\mathbb{F}_{2^k})$ is also isomorphic to the complete graph. Hence, in case of even prime we have shown the isomorphism between $\Gamma M_2^0(\mathbb{F}_{p^k})$ and $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$.

Case II: Next, let us assume that p is an odd prime. For this case we shall make use of Theorem 2.27, towards this first, we determine the complement of $\Gamma M_2^0(\mathbb{F}_{p^k})$. Since $\mathbb{F}_{p^k} \cong \mathbb{Z}_p[x]/\langle p(x)\rangle$, where $p(x)$ is an irreducible polynomial of degree k over $\mathbb{Z}_p$. Take $p(x) = x^k + x + d,\ d \in \mathbb{Z}_p$ then

$$V(\Gamma M_2^0(\mathbb{F}_{p^k})) = \left\{ \begin{bmatrix} a(x) & -a(x) \\ a(x) & -a(x) \end{bmatrix} : \quad a(x) \in \mathbb{Z}_p[x]/\langle x^k + x + d\rangle \right\}.$$

Clearly, $a(x) \in \mathbb{Z}_p[x]/\langle x^k + x + d\rangle$ is a polynomial in x with degree $< k$.

Choose arbitrary vertices v_i and $v_j \in V(\Gamma M_2^0(\mathbb{F}_{p^k}))$, then following the same procedure as done in Theorem 3.2, we again get the equation (1), by which we can say that vertex v_0 is a full degree vertex in $\Gamma M_2^0(\mathbb{F}_{p^k})$ and any two vertices v_i and v_i are non adjacent if and only if the coefficient of their associated polynomial satisfies equation (1). It follows that any vertex v_i is non adjacent with exactly one vertex v_j and hence the pair of non adjacent vertices are $\frac{p^k-1}{2}$ in number. Now, it is not difficult to determine the structure $\Gamma^c M_2^0(\mathbb{F}_{p^k})$, clearly, the non adjacent pair in $\Gamma M_2^0(\mathbb{F}_{p^k})$ will give K_2 and the full degree vertex v_0 will be isolated in the complement graph. Therefore, $\Gamma^c M_2^0(\mathbb{F}_{p^k}) \cong (\frac{p^k-1}{2})K_2 \cup K_1$. Moreover, in light of Theorem 3.5 we have $\Gamma^c M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle) \cong (\frac{p^k-1}{2})K_2 \cup K_1$. Hence, due to Theorem 2.27 the proof is seen to be complete. □

On combining Theorems 3.5 and 4.1 the following corollary is obtained, which establishes the relation of isomorphism among three class of graphs.

Corollary 4.2 *For an odd prime p, $\Gamma M_2^0(\mathbb{Z}_{p^k})$, $\Gamma M_2^0(\mathbb{F}_{p^k})$ and $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k\rangle)$ are all isomorphic.*

Theorems 3.4, 3.5 together with Theorem 4.1, ensures the validity of the following conjecture.

Every zero ring graph is weakly perfect.

5 Conclusions and Scope

An important out come of the paper is the relation of isomorphism among the zero ring graphs $\Gamma M_2^0(\mathbb{Z}_{p^k})$, $\Gamma M_2^0(\mathbb{Z}_p[x]/\langle x^k \rangle)$ and $\Gamma M_2^0(\mathbb{F}_{p^k})$. Moreover it is also shown that zero ring graphs gives a class of Hamiltonian graph and class of weakly perfect graph. We also conjectured that *every zero ring graph is tree complete for all n.*

Further, to establish deeper connections in fundamentally different directions one can settle out the problem suggested in Section 1 as follows: Characterize non zero ring graph? In fact exploration of study on arbitrary zero ring R^0 of the sort under taken in this paper will provide new avenues of research.

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