Gen. Math. Notes, Vol. 24, No. 1, September 2014, pp.58-73
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

r-(τ_i, τ_j)-θ-Generalized Fuzzy Closed Sets in Smooth Bitopological Spaces

O.A. Tantawy[1], **S.A. El-Sheikh**[2] **and R.N. Majeed**[3,4]

[1]Department of Mathematics, Faculty of Science
Zagazig University, Cairo, Egypt
E-mail: drosamat@yahoo.com
[2]Department of Mathematics, Faculty of Education
Ain Shams University, Cairo, Egypt
E-mail: elsheikh33@hotmail.com
[3]Department of Mathematics, Faculty of Science
Ain Shams University, Abbassia, Cairo, Egypt
[4]Department of Mathematics, College of Education
Ibn-Al-Haitham, Baghdad University, Baghdad, Iraq
E-mail: rashanm6@gmail.com

(Received: 27-4-14 / Accepted: 5-6-14)

Abstract

In this paper a new class of fuzzy sets, namely r-(τ_i, τ_j)-θ-generalized fuzzy closed sets is introduced for smooth bitopological spaces. This class falls strictly in between the class of r-(τ_j, τ_i)fuzzy θ-closed sets and the class of r-(τ_i, τ_j)-generalized fuzzy closed sets. Furthermore, by using the class of r-(τ_i, τ_j)-θ-generalized fuzzy closed sets we establish a new fuzzy closure operator which is generate a smooth topology. Finally, (i, j) strongly-θ-fuzzy continuous, (i, j)-θ-generalized fuzzy continuous and (i, j)-θ-generalized fuzzy irresolute mappings are introduce, we show that (i, j)-θ-generalized fuzzy continuous properly fits in between (j, i) strongly-θ-fuzzy continuous and (i, j)-generalized fuzzy continuous.

Keywords: *Smooth topology, fuzzy closure operator, θ-generalized closed fuzzy set, θ-generalized fuzzy continuous mapping, θ-generalized fuzzy irresolute mapping, strongly θ-fuzzy continuous mapping.*

1 Introduction

Šostak [19], introduced the fundamental concept of a 'fuzzy topological structure', as an extension of both crisp topology and Chang's fuzzy topology [3], in the sense that not only the object were fuzzified, but also the axiomatics. Subsequently, Badard [1], introduced the concept of 'smooth topological space'. Chattopadhyay et al. [4] and Chattopadhyay and Samanta [5] re-introduced the same concept, under name 'gradation of openness'. Ramadan [18] and his colleagues introduced a similar definition, namely, smooth topological space for lattice $L = [0, 1]$. Following Ramadan, several authors have re-introduced and further studied smooth topological space (cf. [4, 5, 10, 20]). Thus, the terms **'fuzzy topology' in Šostak sense**, **'gradation of openness'** and **'smooth topology'** are essentially referring to the same concept. In our paper, we adopt the term smooth topology. Lee et al. [15] introduced the concept of smooth bitopological space (smooth bts, for short) as a generalization of smooth topological space and Kandil's fuzzy bitopological space [11].

The first step of generalizing closed sets was done by Levine [16]. He defined a set A to be generalized closed if its closure belongs to every open superset of A. Subsequently, Fukutake [9], generalized this notion to bitopological space and he defined a set A of a bitopological space X to be ij-generalized closed set if τ_j-$cl(A) \subset U$ whenever $A \subset U$ and U is τ_i-open in X. Since then many concepts related to generalized closed sets were defined and investigated. Balasubramanian and Sundaram [2] gave the concept of generalized fuzzy closed sets in Chang's fuzzy topology as an extension of generalized closed sets of Levine. Kim and Ko [14] defined generalized fuzzy closed sets in smooth topological spaces. Recently, we [21, 22] generalized this notion to smooth bts. Noiri in [17] and Dontchev and Maki in [6] gave another new generalization of Livine generalized closed set by utilizing the θ-closure operator. The concept of θ-generalized closed sets was applied to the digital line [7]. Khedr and Al-Saadi [12] generalized the notion of θ-generalized sets to bitopological space. El-Shafei and Zakari [8] introduced the concept of θ-generalized fuzzy closed sets in Chang's fuzzy topology.

The aim of this paper is to continue the study of generalized fuzzy closed sets in smooth bts, this time via the $(\tau_i, \tau_j)\theta$-fuzzy closure $T_{\tau_j}^{\tau_i}$ defined in [13] and study its basic properties. Moreover, we define a new fuzzy closure operator by using this class of θ-generalized fuzzy closed sets, which is induced a smooth topology. Finally, we introduce and study the concept of a new class of fuzzy mappings, namely (i, j) strongly-θ-fuzzy continuous, (i, j)-θ-generalized fuzzy continuous and (i, j)-θ-generalized fuzzy irresolute mappings and give the relations between them.

2 Preliminaries

Throughout this paper, let X be a non-empty set, $I = [0,1]$, $I_0 = (0,1]$. A fuzzy set μ of X is a mapping $\mu : X \longrightarrow I$, and I^X be the family of all fuzzy sets of X. For any $\mu_1, \mu_2 \in I^X$, $\mu_1 \wedge \mu_2 = \min\{\mu_1(x), \mu_2(x) : x \in X\}$, $\mu_1 \vee \mu_2 = \max\{\mu_1(x), \mu_2(x) : x \in X\}$. The complement of a fuzzy set λ is denoted by $\bar{1} - \lambda$. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha \ \forall x \in X$. By $\bar{0}$ and $\bar{1}$, we denote constant maps on X with value 0 and 1, respectively. For each $x \in X$ and $t \in I_0$, the fuzzy set x_t of X whose value t at x and 0 otherwise is called the fuzzy point in X. Let $Pt(X)$ be a family of all fuzzy points in X. $x_t \in \lambda$ if and only if $\lambda(x) \geq t$ and x_t is said to be quasi-coincident (q-coincident, for short) with λ, denoted by $x_t \, q \, \lambda$ if and only if $\bar{1} - \lambda(x) < t$. For $\mu, \lambda \in I^X$, μ is called q-coincident with λ, denoted by $\mu \, q \, \lambda$, if $\mu(x) + \lambda(x) > 1$ for some $x \in X$, otherwise we write $\mu \, \bar{q} \, \lambda$. Also, for two fuzzy sets λ_1 and $\lambda_2 \in I^X$, $\lambda_1 \leq \lambda_2$ if and only if $\lambda_1 \, \bar{q} \, \bar{1} - \lambda_2$. The indices are $i, j \in \{1, 2\}$ and $i \neq j$.

Definition 2.1. *[1, 4, 18, 19] A smooth topology on X is a mapping $\tau : I^X \to I$ which satisfies the following properties:*

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,

(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, $\forall \ \mu_1, \mu_2 \in I^X$,

(3) $\tau(\bigvee_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i)$, *for any* $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called a smooth topological space. The value of $\tau(\mu)$ is interpreted as the degree of openness of fuzzy set μ, that is mean for $r \in I_0$, we say μ is an r-open fuzzy set of X if $\tau(\mu) \geq r$, and μ is an r-closed fuzzy set of X if $\tau(\bar{1} - \mu) \geq r$. Note, Šostak [19] used the term **'fuzzy topology'** *and Chattopadhyay [4], used the term* **'gradation of openness'** *for a smooth topology τ.*

Definition 2.2. *[5] A mapping $C : I^X \times I_0 \to I^X$ is called a fuzzy closure operator if, for $\lambda, \mu \in I^X$ and $r, s \in I_0$, the mapping C satisfies the following conditions:*

(C1) $C(\bar{0}, r) = \bar{0}$,

(C2) $\lambda \leq C(\lambda, r)$,

(C3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$,

(C4) $C(\lambda, r) \leq C(\lambda, s)$ *if* $r \leq s$,

(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

The fuzzy closure operator C generates a smooth topology $\tau_C : I^X \longrightarrow I$ defined by

$$\tau_C(\lambda) = \bigvee \{r \in I | \ C(\bar{1} - \lambda, r) = \bar{1} - \lambda\}.$$

Theorem 2.1. *[5, 13] Let (X, τ_1, τ_2) be a smooth bts. For $\lambda \in I^X$ and $r \in I_0$, a τ_i-fuzzy closure of λ is a mapping $C_{\tau_i} : I^X \times I_0 \longrightarrow I^X$, defined as*

$$C_{\tau_i}(\lambda, r) = \bigwedge\{\mu \in I^X |\ \mu \geq \lambda \text{ and } \tau_i(\bar{1} - \mu) \geq r\}.$$

And, a τ_i-fuzzy interior of λ is a mapping $I_{\tau_i} : I^X \times I_0 \longrightarrow I^X$ defined as

$$I_{\tau_i}(\lambda, r) = \bigvee\{\mu \in I^X |\ \mu \leq \lambda \text{ and } \tau_i(\mu) \geq r\}.$$

Then:

(1) C_{τ_i} (resp., I_{τ_i}) is a fuzzy closure (resp., interior) operator.
(2) $\tau_{C_{\tau_i}} = \tau_{I_{\tau_i}} = \tau_i$.
(3) $I_{\tau_i}(\bar{1} - \lambda, r) = \bar{1} - C_{\tau_i}(\lambda, r)$, $\forall\ r \in I_0, \lambda \in I^X$.

Recall next the definitions of open Q-nbd, θ-cluster point and θ-fuzzy closure operator in smooth bts (X, τ_1, τ_2).

Definition 2.3. *[13] Let (X, τ_1, τ_2) be a smooth bts, $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. μ is called an r-open Q_{τ_i}-nighborhood of x_t if $x_t\ q\ \mu$ with $\tau_i(\mu) \geq r$, we denote*

$$Q_{\tau_i}(x_t, r) = \{\mu \in I^X |\ x_t\ q\ \mu, \tau_i(\mu) \geq r\}.$$

Definition 2.4. *[13] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:*

(1) A fuzzy point $x_t \in Pt(X)$ is called an r-$(\tau_i, \tau_j)\theta$-cluster point of λ if for every $\mu \in Q_{\tau_i}(x_t, r)$, $C_{\tau_j}(\mu, r)\ q\ \lambda$.

(2) An $(\tau_i, \tau_j)\theta$-closure is a mapping $T^{\tau_i}_{\tau_j} : I^X \times I_0 \longrightarrow I^X$ defined as follows:

$$T^{\tau_i}_{\tau_j}(\lambda, r) = \bigvee\{x_t \in Pt(X)|\ x_t \text{ is } r\text{-}(\tau_i, \tau_j)\theta\text{-cluster point of } \lambda\}.$$

(3) λ is called an r-(τ_i, τ_j) fuzzy θ-closed iff $\lambda = T^{\tau_i}_{\tau_j}(\lambda, r)$. The complement of an r-(τ_i, τ_j) fuzzy θ-closed is called an r-(τ_i, τ_j) fuzzy θ-open.

(4) $\Theta^{\tau_i}_{\tau_j}(\lambda, r) = \bigwedge\{\mu \in I^X |\ \mu \geq \lambda,\ \mu = T^{\tau_i}_{\tau_j}(\mu, r)\}$, which is a fuzzy closure operator.

Theorem 2.2. *[13] Let (X, τ_1, τ_2) be a smooth bts, λ and $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. Then:*

(1) $T^{\tau_i}_{\tau_j}(\lambda, r) = \bigwedge\{\mu \in I^X |\ I_{\tau_j}(\mu, r) \geq \lambda,\ \tau_i(\bar{1} - \mu) \geq r\}$, i.e., $T^{\tau_i}_{\tau_j}(\lambda, r)$ is an r-τ_i-closed fuzzy set.

(2) x_t is an r-$(\tau_i, \tau_j)\theta$-cluster point of λ iff $x_t \in T^{\tau_i}_{\tau_j}(\lambda, r)$.

(3) $\lambda \leq C_{\tau_i}(\lambda, r) \leq T^{\tau_i}_{\tau_j}(\lambda, r)$.
(4) If $\tau_j(\lambda) \geq r$, *then* $C_{\tau_i}(\lambda, r) = T^{\tau_i}_{\tau_j}(\lambda, r)$.
(5) $\Theta^{\tau_i}_{\tau_j}(\lambda, r) = T^{\tau_i}_{\tau_j}(\Theta^{\tau_i}_{\tau_j}(\lambda, r), r)$, *i.e.,* $\Theta^{\tau_i}_{\tau_j}(\lambda, r)$ *is an* r-(τ_i, τ_j) *fuzzy* θ*-closed.*
(6) $T^{\tau_i}_{\tau_j}(\lambda, r) \leq \Theta^{\tau_i}_{\tau_j}(\lambda, r)$.

Definition 2.5. *[22] Let* (X, τ_1, τ_2) *be a smooth bts,* $\lambda \in I^X$ *and* $r \in I_0$*. A fuzzy set* λ *is called an* r-(τ_i, τ_j)*-generalized fuzzy closed (*r-(τ_i, τ_j)*-gfc set, for short), if* $C_{\tau_j}(\lambda, s) \leq \mu$*, whenever* $\lambda \leq \mu$ *such that* $\tau_i(\mu) \geq s \ \forall\ 0 < s \leq r$*. The complement of* r-(τ_i, τ_j)*-gfc set is an* r-(τ_i, τ_j)*-generalized fuzzy open (*r-(τ_i, τ_j)*-gfo set, for short).*

Definition 2.6. *[22] A mapping* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *is called* (i, j)*-generalized fuzzy continuous ((*i, j*)-*GF*-continuous, for short) if* $f^{-1}(\mu)$ *is an* r-(τ_i, τ_j)*-gfc set in* X *for each* $\mu \in I^Y$, $\sigma_j(\bar{1} - \mu) \geq r$.

3 r-(τ_i, τ_j)-θ-Generalized Fuzzy Closed Sets

This section is devoted to introduce the concept of r-(τ_i, τ_j)-θ-generalized fuzzy closed sets in smooth bts (X, τ_1, τ_2), and study its fundamental basic properties.

Definition 3.1. *Let* (X, τ_1, τ_2) *be a smooth bts,* $\lambda \in I^X$ *and* $r \in I_0$*. A fuzzy set* λ *is called:*

(1) an r-(τ_i, τ_j)-θ*-generalized fuzzy closed (*r-(τ_i, τ_j)-θ*-gfc set, for short) if* $T^{\tau_j}_{\tau_i}(\lambda, s) \leq \mu$ *whenever* $\lambda \leq \mu$ *such that* $\tau_i(\mu) \geq s \ \forall\ 0 < s \leq r$.

(2) an r-(τ_i, τ_j)-θ*-generalized fuzzy open (*r-(τ_i, τ_j)-θ*-gfo set, for short) if* $\bar{1} - \lambda$ *is an* r-(τ_i, τ_j)-θ*-gfc set.*

If λ *is an* r-(τ_1, τ_2)-θ*-gfc set and an* r-(τ_2, τ_1)-θ*-gfc set, then it said to be pairwise* $r\theta$*-gfc set (P-*$r\theta$*-gfc set, for short).*

Some properties of $T^{\tau_i}_{\tau_j}$ fuzzy closure are state in the next proposition.

Proposition 3.1. *[13] Let* (X, τ_1, τ_2) *be a smooth bts,* $\lambda, \mu \in I^X$ *and* $r \in I_0$*. Then:*

(1) $T^{\tau_i}_{\tau_j}(\bar{0}, r) = \bar{0}$.

(2) $T^{\tau_i}_{\tau_j}(\lambda, r) \vee T^{\tau_i}_{\tau_j}(\mu, r) = T^{\tau_i}_{\tau_j}(\lambda \vee \mu, r)$.

(3) $T_{\tau_j}^{\tau_i}(\lambda, r) \leq T_{\tau_j}^{\tau_i}(\lambda, s)$, *if* $r \leq s$.

(4) $T_{\tau_j}^{\tau_i}(T_{\tau_j}^{\tau_i}(\lambda, r), r) \geq T_{\tau_j}^{\tau_i}(\lambda, r)$.

Proposition 3.2. *The union of any two r-(τ_i, τ_j)-θ-gfc sets in smooth bts (X, τ_1, τ_2) is an r-(τ_i, τ_j)-θ-gfc set.*

Proof. Let λ_1 and λ_2 are r-(τ_i, τ_j)-θ-gfc sets in X and $r \in I_0$. Let $\lambda_1 \vee \lambda_2 \leq \mu$ such that $\tau_i(\mu) \geq s$ for $0 < s \leq r$ this implies $\lambda_1 \leq \mu$ and $\lambda_2 \leq \mu$. Since λ_1 and λ_2 are r-(τ_i, τ_j)-θ-gfc sets, then from Proposition 3.1(2) and Definition 3.1(1), $T_{\tau_i}^{\tau_j}(\lambda_1 \vee \lambda_2, s) = T_{\tau_i}^{\tau_j}(\lambda_1, s) \vee T_{\tau_i}^{\tau_j}(\lambda_2, s) \leq \mu \vee \mu = \mu$. Hence, $\lambda_1 \vee \lambda_2$ is an r-(τ_i, τ_j)-θ-gfc set. □

The intersection of two r-(τ_i, τ_j)-θ-gfc sets need not to be an r-(τ_i, τ_j)-θ-gfc set as the following example show.

Example 3.1. *Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:*

$$\lambda_1 = a_{0.3} \vee b_{0.5}, \qquad \lambda_2 = a_{0.6} \vee b_{0.2}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \textit{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \textit{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \textit{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \textit{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then (X, τ_1, τ_2) is a smooth bts. Consider $\eta_1 = a_{0.1} \vee b_{0.6}$, $\eta_2 = a_{0.4} \vee b_{0.3} \in I^X$. It is easy to see that η_1 and η_2 are $\frac{1}{4}$-(τ_1, τ_2)-θ-gfc sets, but $\eta_1 \wedge \eta_2 = a_{0.1} \vee b_{0.3}$ is not a $\frac{1}{4}$-(τ_1, τ_2)-θ-gfc set.

The next results together with the example following them show that the class of r-(τ_i, τ_j)-θ-gfc sets is properly placed between the classes of r-(τ_j, τ_i)fuzzy-θ-closed sets and r-(τ_i, τ_j)-gfc sets.

Proposition 3.3. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-(τ_j, τ_i)fuzzy-θ-closed set, then λ is an r-(τ_i, τ_j)-θ-gfc set.*

Proof. Let λ be an r-(τ_j, τ_i)fuzzy-θ-closed set and let $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s$ for $0 < s \leq r$. In fact that λ is an r-(τ_j, τ_i)fuzzy-θ-closed set and by Proposition 3.1(3) we have, $T_{\tau_i}^{\tau_j}(\lambda, s) \leq T_{\tau_i}^{\tau_j}(\lambda, r) = \lambda$, and since $\lambda \leq \mu$, then we get, $T_{\tau_i}^{\tau_j}(\lambda, s) \leq \mu$. Hence, λ is an r-(τ_i, τ_j)-θ-gfc set. □

The converse of Proposition 3.3 is not true. In fact of Example 3.1, $\eta_1 = a_{0.1} \vee b_{0.6}$ is a $\frac{1}{4}$-(τ_1, τ_2)-θ-gfc set but it is not a $\frac{1}{4}$-(τ_2, τ_1)fuzzy θ-closed set because, $T^{\tau_2}_{\tau_1}(\eta_1, \frac{1}{4}) = \bar{1} \neq \eta_1$.

The next Proposition gives the sufficient condition of Proposition 3.3.

Proposition 3.4. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is both an r-τ_i-open fuzzy set and an r-(τ_i, τ_j)-θ-gfc set, then λ is an r-(τ_j, τ_i)fuzzy θ-closed set.*

Proof. Since λ is an r-τ_i-open fuzzy set, i.e., $\tau_i(\lambda) \geq r$. Then, $\tau_i(\lambda) \geq s$ for $0 < s \leq r$. Since $\lambda \leq \lambda$ and λ is r-(τ_i, τ_j)-θ-gfc set. Then from Definition 3.1(1), $T^{\tau_j}_{\tau_i}(\lambda, s) \leq \lambda$ for $0 < s \leq r$. On the other hand clearly, $\lambda \leq T^{\tau_j}_{\tau_i}(\lambda, s)$. Thus, $T^{\tau_j}_{\tau_i}(\lambda, s) = \lambda$ for $0 < s \leq r$. Consequently, $T^{\tau_j}_{\tau_i}(\lambda, r) = \lambda$. Hence, λ is an r-(τ_j, τ_i)fuzzy θ-closed set. □

Proposition 3.5. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-(τ_i, τ_j)-θ-gfc set, then λ is an r-(τ_i, τ_j)-gfc set.*

Proof. Let λ is an r-(τ_i, τ_j)-θ-gfc set and let $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s$ for $0 < s \leq r$. To show $C_{\tau_j}(\lambda, s) \leq \mu$. By Theorem 2.2(3), $C_{\tau_j}(\lambda, s) \leq T^{\tau_j}_{\tau_i}(\lambda, s)$ and since λ is r-(τ_i, τ_j)-θ-gfc set we have, $T^{\tau_j}_{\tau_i}(\lambda, s) \leq \mu$. This implies, $C_{\tau_j}(\lambda, s) \leq \mu$. Hence, λ is an r-(τ_i, τ_j)-gfc set. □

The converse of Proposition 3.5 is not true as the following example show.

Example 3.2. *Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:*

$$\lambda_1 = a_{0.5} \vee b_{0.8}, \qquad \lambda_2 = a_{0.7} \vee b_{0.5}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise}; \end{cases} \quad \text{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise}. \end{cases}$$

Then (X, τ_1, τ_2) is a smooth bts. Consider the fuzzy set $\lambda = a_{0.3} \vee b_{0.5}$ is a $\frac{1}{2}$-(τ_1, τ_2)-gfc set but it is not a $\frac{1}{2}$-(τ_1, τ_2)-θ-gfc set.

Thus we have the following diagram

$$\begin{array}{ccc} r\text{-}(\tau_j, \tau_i)\textit{fuzzy } \theta\text{-closed} & \Longrightarrow & r\text{-}(\tau_i, \tau_j)\text{-}\theta\text{-}gfc \\ \Downarrow & & \Downarrow \\ r\text{-}\tau_j\text{-}\textit{fuzzy closed} & \Longrightarrow & r\text{-}(\tau_i, \tau_j)\text{-}gfc \end{array}$$

4 $\mathcal{G}\Theta_{\tau_i}^{\tau_j}$-Fuzzy Closure Operator

In this section we use the classes of r-(τ_i, τ_j)-θ-gfc sets to establish a new type of fuzzy closure operator call it generalized $\Theta_{\tau_i}^{\tau_j}$-fuzzy closure.

Definition 4.1. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. A generalized $\Theta_{\tau_i}^{\tau_j}$-fuzzy closure is a map $\mathcal{G}\Theta_{\tau_i}^{\tau_j} : I^X \times I_0 \longrightarrow I^X$ defined by*

$$\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda, r) = \bigwedge\{\rho \in I^X | \rho \geq \lambda,\ \rho \text{ is } r\text{-}(\tau_i, \tau_j)\text{-}\theta\text{-}gfc \text{ set}\}.$$

Theorem 4.1. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda, \lambda_1, \lambda_2 \in I^X$ and $r \in I_0$. Then:*

(1) If $\lambda_1 \leq \lambda_2$, then $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda_1, r) \leq \mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda_2, r)$.

(2) If λ is an r-(τ_i, τ_j)-θ-gfc set, then $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda, r) = \lambda$.

Proof. To prove (1), suppose $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda_1, r) \nleq \mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda_2, r)$. Then there exist $x \in X$ and $t \in (0, 1)$ such that

$$\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda_2, r)(x) < t < \mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda_1, r)(x). \tag{4.1}$$

Since $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda_2, r)(x) < t$, there exists an r-(τ_i, τ_j)-θ-gfc set ρ such that $\lambda_2 \leq \rho$ and $\rho(x) < t$. Since $\lambda_1 \leq \lambda_2$, then $\lambda_1 \leq \rho$ which implies $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda_1, r)(x) \leq \rho(x) < t$. This contradicts (4.1). The proof of (2), follows directly from the definition of $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda, r)$. □

The converse of Theorem 4.1(2) is not true, we show that in the next example. The example is inspired by the one introduced in [[14], p.333].

Example 4.1. *Let $X = \{a, b\}$. Define fuzzy topologies $\tau_1 = \tau_2 : I^X \longrightarrow I$ as follows:*

$$\tau_1(\lambda) = \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ 0.2 & \text{if } \lambda = a_{0.7}, \\ 0 & \text{otherwise.} \end{cases}$$

The fuzzy set $a_{0.7}$ is not a 1-(τ_1, τ_2)-θ-gfc set, but $\mathcal{G}\Theta_{\tau_1}^{\tau_2}(a_{0.7}, 1) = a_{0.7}$. Because, $a_{0.7} \vee b_s$ is a 1-(τ_1, τ_2)-θ-gfc set for $s \in I_0$. Therefore,

$\mathcal{G}\Theta_{\tau_1}^{\tau_2}(a_{0.7}, 1) = \bigwedge_{s \in I_0}(a_{0.7} \vee b_s) = a_{0.7} \vee \bigwedge_{s \in I_0} b_s = a_{0.7}$.

Next we prove $\mathcal{G}\Theta_{\tau_i}^{\tau_j}$ is a fuzzy closure operator.

Theorem 4.2. *Let (X, τ_1, τ_2) be a smooth bts. Then a mapping $\mathcal{G}\Theta_{\tau_i}^{\tau_j} : I^X \times I_0 \longrightarrow I^X$ is a fuzzy closure operator such that $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda, r) \leq \Theta_{\tau_i}^{\tau_j}(\lambda, r)$ for all $\lambda \in I^X$ and $r \in I_0$.*

Proof. To show $\mathcal{G}\Theta_{\tau_i}^{\tau_j}$ is a fuzzy closure operator, we need to satisfy conditions (C1)$-$(C5) in Definition 2.2.

(C1) Clearly $\bar{0}$ is an r-(τ_j,τ_i)fuzzy θ-closed set. Then, by Proposition 3.3, $\bar{0}$ is an r-(τ_i,τ_j)-θ-gfc set. In view of Theorem 4.1(2), we get $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\bar{0},r)=\bar{0}$.

(C2) Follows immediately from the definition of $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)$.

(C3) Since $\lambda\leq\lambda\vee\mu$ and $\mu\leq\lambda\vee\mu$, then from Theorem 4.1(1),

$$\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)\leq\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda\vee\mu,r)\ \text{ and }\ \mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mu,r)\leq\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda\vee\mu,r)$$

this implies, $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)\vee\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mu,r)\leq\mathcal{G}C_{12}(\lambda\vee\mu,r)$.

Suppose $\mathcal{G}C_{12}(\lambda\vee\mu,r)\not\leq\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)\vee\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mu,r)$. Consequently, $x\in X$ and $t\in(0,1)$ exist such that

$$\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)(x)\vee\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mu,r)(x)<t<\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda\vee\mu,r)(x). \tag{4.2}$$

Since $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)(x)<t$ and $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mu,r)(x)<t$, there exist r-(τ_i,τ_j)-θ-gfc sets ρ_1,ρ_2 with $\lambda\leq\rho_1$ and $\mu\leq\rho_2$ such that

$$\rho_1(x)<t,\rho_2(x)<t.$$

From Proposition 3.2, $\rho_1\vee\rho_2$ is an r-(τ_i,τ_j)-θ-gfc. Since $\lambda\vee\mu\leq\rho_1\vee\rho_2$, then we have $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda\vee\mu,r)(x)\leq(\rho_1\vee\rho_2)(x)<t$. This contradicts (4.2). Hence, $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)\vee\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mu,r)=\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda\vee\mu,r)$.

(C4) Let $r\leq s$, $r,s\in I_0$. Suppose $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)\not\leq\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,s)$. Consequently, $x\in X$ and $t\in(0,1)$ exist such that

$$\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,s)(x)<t<\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)(x). \tag{4.3}$$

Since $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,s)(x)<t$, there is an s-(τ_i,τ_j)-θ-gfc set ρ with $\lambda\leq\rho$ such that $\rho(x)<t$. This yields $T_{\tau_i}^{\tau_j}(\rho,s_1)\leq\mu$, whenever $\rho\leq\mu$ and $\tau_i(\mu)\geq s_1$, for $0<s_1\leq s$. Since $r\leq s$, then $T_{\tau_i}^{\tau_j}(\rho,r_1)\leq\mu$ whenever $\rho\leq\mu$ and $\tau_i(\mu)\geq r_1$, for $0<r_1\leq r\leq s_1\leq s$. This implies ρ is an r-(τ_i,τ_j)-θ-gfc. From Definition 4.1, we have $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)(x)\leq\rho(x)<t$. This contradicts (4.3). Hence, $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)\leq\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,s)$.

(C5) Let ρ be any r-(τ_i,τ_j)-θ-gfc set containing λ. Then, from Definition 4.1, we have $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)\leq\rho$. From Theorem 4.1(1), we obtain $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r),r)\leq\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\rho,r)=\rho$. This mean that $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r),r)$ is contained in every r-(τ_i,τ_j)-θ-gfc set containing λ. Hence, $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r),r)\leq\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)$. However, from (C2), $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)\leq\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r),r)$. Hence, $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r),r)=\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)$. Thus, $\mathcal{G}\Theta_{\tau_i}^{\tau_j}$ is a fuzzy closure operator. Since every r-(τ_j,τ_i)fuzzy θ-closed set is r-(τ_i,τ_j)-θ-gfc set, then $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\lambda,r)\leq\Theta_{\tau_i}^{\tau_j}(\lambda,r)$, for all $\lambda\in I^X$ *and* $r\in I_0$. □

After we show $\mathcal{G}\Theta_{\tau_i}^{\tau_j}$ is a fuzzy closure operator. The next theorem show $\mathcal{G}\Theta_{\tau_i}^{\tau_j}$ generate a smooth topology, $\tau_{\mathcal{G}\Theta_{\tau_i}^{\tau_j}}$ on X which is finer than $\tau_{\Theta_{\tau_i}^{\tau_j}}$.

Theorem 4.3. *Let (X, τ_1, τ_2) be a smooth bts. Define a mapping $\tau_{\mathcal{G}\Theta_{\tau_i}^{\tau_j}} : I^X \longrightarrow I$ by*

$$\tau_{\mathcal{G}\Theta_{\tau_i}^{\tau_j}}(\lambda) = \bigvee\{r \in I |\ \mathcal{G}\Theta_{\tau_i}^{\tau_j}(\bar{1} - \lambda, r) = \bar{1} - \lambda\}.$$

Then $\tau_{\mathcal{G}\Theta_{\tau_i}^{\tau_j}}$ is a smooth topology on X, for which $\tau_{\Theta_{\tau_i}^{\tau_j}}(\lambda) \leq \tau_{\mathcal{G}\Theta_{\tau_i}^{\tau_j}}(\lambda)$ for all $\lambda \in I^X$.

Proof. Since $\mathcal{G}\Theta_{\tau_i}^{\tau_j}$ is a fuzzy closure operator. Then by Definition 2.2, $\tau_{\mathcal{G}\Theta_{\tau_i}^{\tau_j}}$ is a smooth topology on X. By Proposition 3.3, $\Theta_{\tau_i}^{\tau_j}(\bar{1} - \lambda, r) = \bar{1} - \lambda$ which yields $\mathcal{G}\Theta_{\tau_i}^{\tau_j}(\bar{1} - \lambda, r) = \bar{1} - \lambda$. Thus, $\tau_{\Theta_{\tau_i}^{\tau_j}}(\lambda) \leq \tau_{\mathcal{G}\Theta_{\tau_i}^{\tau_j}}(\lambda)$ for all $\lambda \in I^X$. □

At the end of this section we state the following proposition which is description each r-(τ_i, τ_j)-θ-gfc set in smooth topological space $(X, \tau_{\mathcal{G}\Theta_{\tau_i}^{\tau_j}})$.

Proposition 4.1. *Let (X, τ_1, τ_2) be a smooth bts. $\lambda \in I^X$ and $r \in I_0$. If λ is an r-(τ_i, τ_j)-θ-gfc set, then λ is an r-$\tau_{\mathcal{G}\Theta_{\tau_i}^{\tau_j}}$-closed fuzzy set.*

Proof. The proof follows from Theorem 4.1(2) and Theorem 4.3. □

5 (i, j)-θ-GF-Continuous (Irresolute) and (i, j)-S-θ-Fuzzy Continuous Mappings

In this section we introduce the concepts of (i, j)-θ-GF-continuous, (i, j)-θ-GF-irresolute and (i, j)-strongly-θ-fuzzy continuous and investigate some of its properties. For a mapping f from (X, τ_1, τ_2) into (Y, σ_1, σ_2) we shall denote the fuzzy continuous (respectively, open) mapping from (X, τ_j) into (Y, σ_j), $j \in \{1, 2\}$ by j-fuzzy continuous (respectively, open) mapping (where a mapping f is called j-fuzzy continuous (respectively, open), if $\tau_j(f^{-1}(\mu)) \geq \sigma_j(\mu)$ for each $\mu \in I^Y$ (respectively, $\sigma_j(f(\lambda)) \geq r$ for each $\tau_j(\lambda) \geq r$)).

Definition 5.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be smooth bts's. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called:*

(1) (i, j)-θ-generalized fuzzy continuous ((i, j)-θ-GF-continuous, for short) if $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-θ-gfc set in X for each $\mu \in I^Y$, $\sigma_j(\bar{1} - \mu) \geq r$.

(2) (i, j)-θ-generalized fuzzy irresolute ((i, j)-θ-GF-irresolute, for short) if $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-θ-gfc set in X for each r-(σ_i, σ_j)-θ-gfc set $\mu \in I^Y$.

(3) (i,j)-strongly-θ-fuzzy continuous ((i,j)-S-θ-fuzzy continuous, for short) if for each $x_t \in Pt(X)$ and for each $\mu \in Q_{\sigma_i}(f(x_t), r)$, there exists $\lambda \in Q_{\tau_i}(x_t, r)$ such that $f(C_{\tau_j}(\lambda, r)) \leq \mu$.

Theorem 5.1. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (j,i)-S-θ-fuzzy continuous, then f is (i,j)-θ-GF-continuous.*

Proof. Let $\lambda \in I^Y$ such that $\sigma_j(\bar{1} - \lambda) \geq r$. Let $f^{-1}(\lambda) \leq \mu$ such that $\tau_i(\mu) \geq s$ for $0 < s \leq r$. To prove $f^{-1}(\lambda)$ is an r-(τ_i, τ_j)-θ-gfc set in X, we need to prove $T^{\tau_j}_{\tau_i}(f^{-1}(\lambda), s) \leq \mu$. Suppose there exists $x_t \in Pt(X)$ such that $x_t \notin \mu$, this mean $x_t \; q \; \bar{1} - \mu$. In fact that $f^{-1}(\lambda) \leq \mu$ which implies, $\bar{1} - \mu \leq \bar{1} - f^{-1}(\lambda)$. Therefore, $x_t \; q \; \bar{1} - f^{-1}(\lambda)$, consequently, $f(x_t) \; q \; \bar{1} - \lambda$ such that $\bar{1} - \lambda$ is an r-σ_j-open fuzzy set in Y. This yields, $\bar{1} - \lambda \in Q_{\sigma_j}(f(x_t), r)$. From f is (j,i)-S-θ-fuzzy continuous, there exists $\eta \in Q_{\tau_j}(x_t, r)$ such that $f(C_{\tau_i}(\eta, r)) \leq \bar{1} - \lambda$. By take the inverse image of the last inequality we get, $C_{\tau_i}(\eta, r) \leq \bar{1} - f^{-1}(\lambda)$ which implies, $f^{-1}(\lambda) \leq \bar{1} - C_{\tau_i}(\eta, r) = I_{\tau_i}(\bar{1} - \eta, r)$ such that $\bar{1} - \eta$ is an r-τ_j-closed fuzzy set in X, and from Theorem 2.2(1), if $x_t \in \bar{1} - \eta$ this implies, $\bar{1} - \eta(x) \geq t$ implies to, $\eta(x) + t \leq 1$, consequently, $x_t \; \bar{q} \; \eta$ which is a contradiction. Thus, $x_t \notin T^{\tau_j}_{\tau_i}(f^{-1}(\lambda), r)$. Since $s \leq r$ then from Proposition 3.1(3), $x_t \notin T^{\tau_j}_{\tau_i}(f^{-1}(\lambda), s)$. Therefore, $T^{\tau_j}_{\tau_i}(f^{-1}(\lambda), s) \leq \mu$. Hence, f is (i,j)-θ-GF-continuous. □

The converse of Theorem 5.1 is not true as seen in the following example.

Example 5.1. *Let $X = \{a, b\}$ and $Y = \{p, q\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:*

$$\lambda_1 = a_{\frac{1}{2}} \vee b_{\frac{1}{3}}, \qquad \lambda_2 = a_{\frac{1}{3}} \vee b_{\frac{1}{2}}, \qquad \mu_1 = p_{\frac{1}{2}} \vee q_{\frac{1}{4}}, \qquad \mu_2 = p_{\frac{1}{4}} \vee q_{\frac{1}{2}}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ and $\sigma_1, \sigma_2 : I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise}; \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise}; \end{cases}$$

$$\sigma_1(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise}; \end{cases} \quad \text{and} \quad \sigma_2(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise}. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *defined by* $f(a) = q,\ f(b) = p$. *Then* f *is* $(1,2)$-θ-GF-*continuous but not* $(2,1)$-S-θ-*fuzzy continuous.*

Theorem 5.2. *If* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *is* (i,j)-θ-GF-*continuous, then* f *is* (i,j)-GF-*continuous.*

Proof. Let $\mu \in I^Y$ such that μ is an r-σ_j-closed fuzzy set. Then, from f is (i,j)-θ-GF-continuous, $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-θ-gfc set in X. By Proposition 3.5, every r-(τ_i, τ_j)-θ-gfc set is an r-(τ_i, τ_j)-gfc set. Hence, f is (i,j)-GF-continuous. □

The next example show the converse of above theorem is not true in general.

Example 5.2. *Let* $X = \{a, b, c\}$ *and* $Y = \{p, q\}$. *Define* $\lambda_1, \lambda_2 \in I^X$ *and* $\mu_1, \mu_2 \in I^Y$ *as follows:*

$$\lambda_1 = a_{0.5} \vee b_{0.2} \vee c_{0.9}, \quad \lambda_2 = a_{0.5} \vee b_{0.8} \vee c_{0.2}, \quad \mu_1 = p_{0.7} \vee q_{0.4}, \quad \mu_2 = p_{0.9} \vee q_{0.2}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ *and* $\sigma_1, \sigma_2 : I^Y \longrightarrow I$ *as follows:*

$$\tau_1(\lambda) = \begin{cases} 1 & \textit{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \textit{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \textit{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \textit{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}; \end{cases}$$

$$\sigma_1(\mu) = \begin{cases} 1 & \textit{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \textit{if } \mu = \mu_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \sigma_2(\mu) = \begin{cases} 1 & \textit{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \textit{if } \mu = \mu_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *defined by* $f(a) = p$, $f(b) = p$ *and* $f(c) = q$. *Then* f *is* $(1,2)$-GF-*continuous but not* $(1,2)$-θ-GF-*continuous.*

Now in order to discus the relation between (i,j)-S-θ-fuzzy continuous and j-fuzzy continuous mappings, we need to redefine the definition of j-fuzzy continuous mapping by using fuzzy points and the concept of Q-nbd in the following theorem.

Theorem 5.3. *A mapping* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *is* j-*fuzzy continuous iff for each* $x_t \in Pt(X)$ *and for each* $\mu \in \sigma_j(f(x_t), r)$, *there exists* $\eta \in Q_{\tau_j}(x_t, r)$ *such that* $f(\eta) \leq \mu$.

Proof. Suppose f is j-fuzzy continuous. Let $x_t \in Pt(X)$ and $\mu \in \sigma_j(f(x_t), r)$. Since f is j-fuzzy continuous then, $\tau_j(f^{-1}(\mu)) \geq \sigma_j(\mu)$. That implies, $f^{-1}(\mu) \in Q_{\tau_j}(x_t, r)$ i.e., $x_t\ q\ f^{-1}(\mu)$. Now let $\eta = f^{-1}(\mu)$. To obtain the required results, we must prove $f(\eta) \leq \mu$ i.e., $f(\eta)\ \bar{q}\ \bar{1} - \mu$.
Suppose $f(\eta)\ q\ \bar{1} - \mu$, implies $f(f^{-1}(\mu))\ q\ \bar{1} - \mu$. Consequently, $f(f^{-1}(\mu)) \leq \mu\ q\ \bar{1} - \mu$ which is a contradiction. Hence, $f(\eta) \leq \mu$.

Conversely, let $\mu \in I^Y$ such that μ is r-σ_j-open fuzzy set. Then, from our assumption we have, for each $x_t \in Pt(X)$ such that $\mu \in Q_{\sigma_j}(f(x_t), r)$, then there exists $\eta \in Q_{\tau_j}(x_t, r)$ such that $f(\eta) \leq \mu$. By take the inverse image of the last inequality, we get $\eta \leq f^{-1}(\mu)$ that implies $f^{-1}(\mu) \in Q_{\tau_j}(x_t, r)$. Thus, $\tau_j(f^{-1}(\mu)) \geq r$. Hence, f is j-fuzzy continuous. □

Theorem 5.4. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (j, i)-S-θ-fuzzy continuous, then f is j-fuzzy continuous.*

Proof. Let $x_t \in Pt(X)$ and $\mu \in \sigma_j(f(x_t), r)$. Since f is (j, i)-S-θ-fuzzy continuous then, there exists $\lambda \in Q_{\tau_j}(x_t, r)$ such that $f(C_{\tau_i}(\lambda, r)) \leq \mu$. Since $\lambda \leq C_{\tau_i}(\lambda, r)$ then, $f(\lambda) \leq f(C_{\tau_i}(\lambda, r)) \leq \mu$. In view of Theorem 5.3, f is j-fuzzy continuous. □

The converse of Theorem 5.4 is not true as the following example show.

Example 5.3. *Let $X = \{a, b\}$ and $Y = \{p, q\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:*

$$\lambda_1 = a_{\frac{3}{4}} \vee b_{\frac{1}{2}}, \qquad \lambda_2 = a_{\frac{1}{2}} \vee b_{\frac{1}{4}}, \qquad \mu_1 = p_{\frac{1}{4}} \vee q_{\frac{1}{2}}, \qquad \mu_2 = p_{\frac{1}{2}} \vee q_{\frac{1}{4}}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ and $\sigma_1, \sigma_2 : I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & if\ \lambda = \lambda_1, \\ 0 & otherwise; \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & if\ \lambda = \lambda_2, \\ 0 & otherwise; \end{cases}$$

$$\sigma_1(\mu) = \begin{cases} 1 & if\ \mu = \bar{0}, \bar{1}, \\ \frac{1}{3} & if\ \mu = \mu_1, \\ 0 & otherwise; \end{cases} \quad and \quad \sigma_2(\mu) = \begin{cases} 1 & if\ \mu = \bar{0}, \bar{1}, \\ \frac{1}{4} & if\ \mu = \mu_2, \\ 0 & otherwise. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ defined by $f(a) = p$, $f(b) = q$. Then f is 2-fuzzy continuous but not $(2, 1)$-S-θ-fuzzy continuous.

Theorem 5.5. *[22] If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is j-fuzzy continuous, then f is (i, j)-GF-continuous.*

Thus we have the following implication and none of them is reversible.

$$\begin{array}{ccc} (i,j)\text{-}\theta\text{-}GF\text{-continuous} & \Longrightarrow & (i,j)\text{-}GF\text{-continuous} \\ \Uparrow & & \Uparrow \\ (j,i)\text{-}S\text{-}\theta\text{-}fuzzy\ continuous & \Longrightarrow & j\text{-}fuzzy\ continuous \end{array}$$

Theorem 5.6. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \longrightarrow (Z, \delta_1, \delta_2)$. Then:*

(1) If g is j-fuzzy continuous and f is (i, j)-θ-GF-continuous then $g \circ f$ is (i, j)-θ-GF-continuous.

(2) If g is (i, j)-θ-GF-irresolute and f is (i, j)-θ-GF-irresolute then $g \circ f$ is (i, j)-θ-GF-irresolute.

(3) If g is (i, j)-θ-GF-continuous and f is (i, j)-θ-GF-irresolute then $g \circ f$ is (i, j)-θ-GF-continuous.

Proof. We prove (1) and the proof of (2) and (3) are similar to (1). Let ν be an r-δ_j-closed fuzzy set of Z. Since g is j-fuzzy continuous, then $g^{-1}(\nu)$ is r-σ_j-closed fuzzy set of Y. Moreover, f is (i, j)-θ-GF-continuous, then, $(g \circ f)^{-1}(\nu) = f^{-1}(g^{-1}(\nu))$ is an r-(τ_i, τ_j)-θ-gfc set of X. Hence, $g \circ f$ is (i, j)-θ-GF-continuous. □

Theorem 5.7. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is bijective, i-fuzzy open and (i, j)-θ-GF-continuous, then f is (i, j)-θ-GF-irresolute.*

Proof. Let ν be an r-(σ_i, σ_j)-θ-gfc set of Y and let $f^{-1}(\nu) \leq \mu$ where $\tau_i(\mu) \geq s$ for $0 < s \leq r$. Clearly $\nu \leq f(\mu)$. Since $\sigma_i(f(\mu)) \geq s$ and ν is an r-(σ_i, σ_j)-θ-gfc set in Y. Then, $T^{\sigma_j}_{\sigma_i}(\nu, s) \leq f(\mu)$ and thus, $f^{-1}(T^{\sigma_j}_{\sigma_i}(\nu, s)) \leq \mu$. Since $T^{\sigma_j}_{\sigma_i}(\nu, s)$ is an s-σ_j-closed fuzzy set in Y and f is (i, j)-θ-GF-continuous. Then, $f^{-1}(T^{\sigma_j}_{\sigma_i}(\nu, s))$ is s-(τ_i, τ_j)-θ-gfc set in X. Thus, from Definition 3.1(1), $T^{\tau_j}_{\tau_i}(f^{-1}(T^{\sigma_j}_{\sigma_i}(\nu, s)), s) \leq \mu$ this yields $T^{\tau_j}_{\tau_i}(f^{-1}(\nu), s) \leq \mu$. Therefore, $f^{-1}(\nu)$ is an r-(τ_i, τ_j)-θ-gfc set. Hence, f is (i, j)-θ-GF-irresolute. □

References

[1] R. Badard, Smooth axiomatics, *First IFSA Congress Palma de Mallorca*, July(1986).

[2] G. Balasubramanian and P. Sundaram, On some generalizations of fuzzy continuous functions, *Fuzzy Sets and Systems*, 86(1997), 93-100.

[3] C.L. Chang, Fuzzy topological spaces, *Journal of Mathematical Analysis and Applications*, 24(1968), 182-190.

[4] K.C. Chattopadhyay, R.N. Hazar and S.K. Samanta, Gradation of openness: Fuzzy topology, *Fuzzy Sets and Systems*, 94(1992), 237-242.

[5] K.C. Chattopadhyay and S.K. Samanta, Fuzzy topology: Fuzzy closure operator, fuzzy compactness and fuzzy connectedness, *Fuzzy Sets and Systems*, 54(1993), 207-212.

[6] J. Dontchev and H. Maki, On θ-generalized closed sets, *International Journal of Mathematics and Mathematical Sciences*, 22(1999), 239-249.

[7] J. Dontchev and H. Maki, Groups of θ-generalized homeomorphisms and the digital line, *Topology and its Applications*, 95(1999), 113-128.

[8] M.E. El-Shafei and A. Zakari, θ-Generalized closed sets in fuzzy topological spaces, *The Arabian Journal for Science and Engineering*, 31(2A) (2006), 197-206.

[9] T. Fukutake, On generalized closed sets in bitopological spaces, *Bulletin of Fukuoka University of Education Part III*, 35(1986), 19-28.

[10] R.N. Hazra, S.K. Samanta and K. Chattopadhyay, Fuzzy topology redefined, *Fuzzy Sets and Systems*, 45(1992), 79-82.

[11] A. Kandil, Biproximities and fuzzy bitopological spaces, *Simon Stevin*, 63(1989), 45-66.

[12] F.H. Khedr and H.S. Al-Saadi, On pairwise θ-generalized closed sets, *Journal of International Mathematical Virtual Institute*, 1(2011), 37-51.

[13] Y.C. Kim, A.A. Ramadan and S.E. Abbas, Separation axioms in terms of θ-closure and δ-closure operators, *Indian Journal of Pure and Applied Mathematics*, 34(7) (2003), 1067-1083.

[14] Y.C. Kim and J.M. Ko, Fuzzy G-closure operators, *Communications Korean Mathematical Society*, 18(2) (2003), 325-340.

[15] E.P. Lee, Y.B. Im and H. Han, Semiopen sets on smooth bitopological spaces, *Far East Journal of Mathematical Sciences*, 3(2001), 493-511.

[16] N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo*, 19(1970), 89-96.

[17] T. Noiri, Generalized θ-closed sets of almost paracompact spaces, *Tour. of Math. and Comp. Sci. Math. Ser.*, 9(1996), 157-161.

[18] A.A. Ramadan, Smooth topological spaces, *Fuzzy Sets and Systems*, 48(1992), 371-375.

[19] A.P. Šostak, On a fuzzy topological structure, *Suppl. Rend. Circ. Matem. Palermo, Ser.II*, 11(1985), 89-103.

[20] A.P. Šostak, Basic structures of fuzzy topology, *Journal of Mathematical Sciences*, 78(1996), 662-701.

[21] O.A. Tantawy, F. Abelhalim, S.A. El-Sheikh and R.N. Majeed, Two new approaches to generalized supra fuzzy closure operators on smooth bitopological spaces, *Wulfenia Journal*, 21(5) (2014), 221-241.

[22] O.A. Tantawy, F. Abelhalim, S.A. El-Sheikh and R.N. Majeed, r-(τ_i, τ_j)-generalized fuzzy closed sets in smooth bitopological spaces, *Annals of Fuzzy Mathematics and Informatics*, (in press).