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Common Fixed Point Theorems for T-Contractive Mappings in D*-Generalized Cone Metric Space

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Abstract

In this note, we prove common fixed point for a Banach pair of mappings on D^*-Generalized Cone Metric Space.

Keywords: *Cone metric space, common fixed point, contractive mappings, sequentially convergent, Banach operator pair.*

1 Introduction and Preliminaries

Recently Huang and Zhang [6] generalized the concept of metric spaces replacing the set of real numbers by an ordered Banach space defining in this way a cone metric space.They have defined convergent, Cauchy sequence in terms of interior points of the underlying Cone. They later proved some fixed point theorems for different contractive mappings. Their results have been generalized and extended by several authors see for instance [11], [7], [3].

Dhage in [4] defined $D-$metric space as a generalization of metric space and claimed that $D-$metric defines a Hausdorff topology and $D-$metric is sequentially continuous with respect to all three variables . He proved some results on fixed points for a self-map satisfying a contraction for complete and bounded complete metric spaces. In 2003, Zead Mustafa and Brailey Sims [9] introduced a new structure of generalized metric spaces, which are called G-metric spaces. Recently Shaban Sedghi et al [12] introduced D^*-metric which is probable modification of the definition of $D-$metric space and prove some basic properties in D^*-metric space and some results on common fixed point theorems. In 2010,C.T Aage and J.N. Salunke [1] introducd generalized D^*-metric by replacing $\mathbb{R}$ by Banach space in D^*-metric Spaces.

In [**2**], A.Beiranvand, S. Moradi et al, introduced a new class of contractive mapping $T-$ contraction and $T-$ Contractive extending the Banach's contraction principle and the Edelstein's fixed point theorem.

In the following we always suppose that E is a Banach space, P is a cone in E with $\mathrm{int}P \neq \phi$ and $\leq$ is a partial ordering with respect to P

In this section we recall some definitions of cone D^*-metric space and some of their properties see [1] for more details.

Definition 1 *let E be a real Banach space and P a subset of E. The set P is called a cone if and only if*

1. P is closed, nonempty and $P \neq \{0\}$
2. $\forall a, b \in \mathbb{R}_+$, $x, y \in P$ implies $ax + by \in P$
3. $x \in P$ and $-x \in P$ then $x = 0$ that is $P \cap (-P) = \{0\}$

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y \iff y - x \in P$. We write $x < y$ and $x \neq y$. We shall write $x \ll y$ if and only if $y - x \in \mathrm{int}P$ where $\mathrm{int}P$ denotes the interior of P.

Definition 2 *let E be a real Banach space and P a subset of E.The cone P is called normal if there is a number $k > 0$ such that for all*

$$x, y \in E,\, 0 \leq x \leq y \text{ implies } \|x\| \leq k\,\|y\|$$

The least positive constant k satisfying the inequality above is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is , if $\{x_n\}_{n\geq1}$is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ for $y \in E$, then there is $x \in E$ such that $\lim \|x_n - x\| = 0$ as n $\longrightarrow \infty$.

Let X be a nonempty set. A generalized metric (or D^*-metric) on X is a function: $D^* : X^3 \to E$ that satisfies the following conditions for each for all $(x, y, z, a) \in X^4$.

1. $D^* (x, y, z) \geq 0$
2. $D^* (x, y, z) = 0$ if and only if $x = y = z$
3. $D^* (x, y, z) = D^* (P \{x, y, z\})$, (symmetry) where P is a permutation function
4. $D^* (x, y, z) \leq D^* (x, y, a) + D (a, z, z)$

Example 3 *Let* $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0 \}$, $X = \mathbb{R}$ *and* $D^* : X \times X \times X \longrightarrow E$ *defined by*

$$D^* (x, y, z) = (|x - y| + |y - z| + |z - x|\, , \ \alpha (|x - y| + |y - z| + |z - x|))$$

where $\alpha \in \mathbb{R}_+$. *Then* (X, D^*) *is a generalized cone* D^**- metric space.*

Proposition 4 *[1] If* (X, D^*) *be generalized cone* D^**- metric space, then for all* $x, y, z \in X$,*we have* $D^* (x, x, y) = D^* (x, \ y, \ y)$

Definition 5 *Let* (X, D^*) *be a generalized cone* D^**- metric space. Let* $\{x_n\}$ *be a sequence in* X *and* $x \in X$. *If for every* $c \gg 0$ *there is* $N \in \mathbb{N}$ *such that for all* $m, n \geq N$, $D^* (x_m, x_n, x) \ll c$, *then* $\{x_n\}$ *is said to be convergent,* $\{x_n\}$ *converges to* x *and* x *is the limite of* $\{x_n\}$. *We denote this convergence by* $x_n \rightarrow x$ $(as\ n \rightarrow \infty)$.

Lemma 6 *[1] Let* (X, D^*) *be generalized cone* D^**- metric space,* P *be a normal cone with normal constant* k. *Let* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *converges to* x *if and only if* $D^* (x_m, x_n, x) \rightarrow 0$ $(as\ n, m \rightarrow \infty)$.

Lemma 7 *[1] Let* (X, D^*) *be a generalized cone* D^**-metric space,* P *be a normal cone with normal constant* k. *Let* $\{x_n\}$ *be a sequence in* X . *If* $\{x_n\}$ *converges to* x *and if* $\{x_n\}$ *converges to* y, *then* $x = y$, *that is the limit of* $\{x_n\}$ *if there exists, is unique.*

Definition 8 *Let* $(X,\ D^*)$ *be a generalized cone* D^**-metric space,* P *be a normal cone with normal constant* k. $\{x_n\}$ *be a sequence in* X. *If for every* $c \gg 0, c \in E$ *there is* $N \in \mathbb{N}$ *such that for all* $m, n, l \geq N$, $D^* (x_m, x_n,\ x_l) \ll c$, *then* $\{x_n\}$ *is called a Cauchy sequence in* X.

Definition 9 *Let* $(X,\ D^*)$ *be a generalized cone* D^**- metric space. If every Cauchy sequence in* X *is convergent in* X, *then* X *is called a complete generalized cone* D^**- metric space.*

Lemma 10 *[1] Let* (X, D^*) *be a generalized cone* D^**-metric space,* $\{x_n\}$ *be a sequence in* X. *If* $\{x_n\}$ *converges to* x, *then* $\{x_n\}$ *is a Cauchy sequence .*

Lemma 11 *[1] Let (X, D^*) be generalized cone D^*- metric space, P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $D^*(x_m, x_n, x_l) \to 0$ (as $n, m, l \to \infty$)*

Definition 12 *Let (X, D^*) be a generalized cone D^*- metric space. Then a function $f : X \to X'$ is said to be D^*continuous at a point $x \in X$ if and only if it is D^*- sequentially continuous at x, that is whenever $\{x_n\}$ is D^*-convergent to x, we have $\{fx_n\}$ is D^*-convergent to fx.*

Definition 13 *[1] Let (X, D^*) be a generalized cone D^*- metric space, P be a normal cone with normal constant k. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be three sequences in X and $x_n \to x$, $y_n \to y$, $x_n \to z$ (as $n \to \infty$). Then*

$$D^*(x_n, y_n, z_n) \to D^*(x, y, z) \quad (as\ n \to \infty)$$

Definition 14 *Let (X, D^*) be a generalized cone D^*- metric space, P be a normal cone with normal constant k and $T : X \to X$. Then*

1. T is said to be sequentially convergent if we have, for every sequence (y_n),if $T(y_n)$ is convergent, then (y_n) is also convergent.
2. T is said to be subsequentially convergent if we have, for every sequence (y_n),if $T(y_n)$ is convergent, then (y_n) has a convergent subsequence.

Definition 15 *Let T be a self mapping of a normed space X. Then T is called a Banach operator of type k if*

$$\left\|T^2x - Tx\right\| \leq k\left\|Tx - x\right\|$$

for some $k \geq 0$ and all $x \in X$.

This concept was introduced by Subrahmanyam [10], then Chen and Li [5] extended this as following:

Definition 16 *Let f and T be self mappings of a nonempty subset M of a normed linear space X. Then (f, T) is a Banach operator pair, if one of the following conditions is satisfied:*

1. $f[F(T)] \subseteq F(T)$
2. $Tfx = fx$ for each $x \in F(T)$
3. $Tfx = fTx$ for each $x \in F(T)$
4. $\|fTx - Tx\| \leq k\|Tx - x\|$ for some $k \geq 0$.

where $F(T)$ denotes the set of fixed points of T.

Lemma 17 *If (X, D^*) be a generalized compact cone metric space, then every function $T : X \to X$ is subsequentially convergent and every continuous function $T : X \to X$ is sequentially convergent.*

In this section, we introduce the notions of $T-$contraction and we extend the Banach Contraction principle and Edelstein point theorem given in [2]

Definition 18 *Let (X, D^*) be generalized cone D^*- metric space and $T, S : X \to X$ two functions. A mapping S is said to be a $T-$contraction if there is $\alpha \in [0, 1[$ constant such that*

$$D^*(TSx, TSy, TSz) \leq \alpha D^*(Tx, Ty, Tz)$$

for all $x, y, z \in X$.

Example 19 *Let $E = \left(C_{[0,1]},\ \mathbb{R}\right)$, $P = \{\varphi \in E : \varphi \geq 0\ \} \subset E$, $X = \mathbb{R}$ and $D^* : X \times X \times X \longrightarrow E$ defined by*

$$D^*(x, y, z) = (|x - y| + |y - z| + |z - x|)\,\varphi$$

where $\varphi(t) = e^{-t} \in E$. Then (X, D^*) is a generalized cone D^*- metric space. We consider the functions $T, S : X \to X$ defined by $T(x) = \exp(-x)$ and $S(x) = 2x + 1$. Then

1. It is clear that S is not a contraction.
2. S is a $T-$contraction. Indeed

$$\begin{aligned}
D^*(TSx, TSy, TSz) &= \begin{pmatrix} |TSx - TSy| \\ +\,|TSy - TSz| \\ +\,|TSz - TSx| \end{pmatrix} e^{-t} \\
&= \frac{1}{e}\begin{pmatrix} |e^{-x} - e^{-y}|\,|e^{-x} + e^{-y}| \\ +\,|e^{-y} - e^{-z}|\,|e^{-y} + e^{-z}| \\ +\,|e^{-z} - e^{-x}|\,|e^{-z} + e^{-x}| \end{pmatrix} e^{-t} \\
&\leq \frac{2}{e}\begin{pmatrix} |e^{-x} - e^{-y}| \\ +\,|e^{-y} - e^{-z}| \\ +\,|e^{-z} - e^{-x}| \end{pmatrix} e^{-t} \\
&= \frac{2}{e} D^*(Tx, Ty, Tz)
\end{aligned}$$

2 Main Results

The following theorems are the main results of this paper

Theorem 20 *Let (X, D^*) be a generalized complete cone D^*- metric space , P be a normal cone with constant normal K and let $T, S : X \to X$ be two continuous mappings. Assume that T is injective and subsequentially convergent. If T and S satisfy*

$$\begin{aligned} D^*(TSx, TSy, TSz) &\leq a_1 D^*(Tx, Ty, Tz) + a_2 D^*(Tx, TSx, TSx) + \\ &\leq a_3 D^*(Tx, TSx, Tz) + a_4 D^*(Ty, TSy, TSy) + \\ &\leq a_5 D^*(Tx, TSy, TSy) + a_6 D^*(Tx, TSz, TSz) + \\ &\leq a_7 D^*(Ty, TSz, TSz) + a_8 D^*(Tx, TSy, TSz) + \\ &\leq a_9 D^*(Tx, Ty, TSz) + a_{10} D^*(TSx, TSx, TSz) \end{aligned}$$

for all $x, y, z \in X$, where $a_i, i \in \{1, 2, 3, \cdots, 10\}$ are all nonnegative constants such that

$$a_1 + a_2 + a_3 + a_4 + 3a_5 + 3a_6 + a_7 + 3a_8 + a_9 + a_{10} < 1$$

then S has a unique fixed point in X. Moreover, if (T, S) is a Banach pair, then T and S have a unique common fixed point in X.

Proof. : Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = Sx_n$; for each $n = 0, 1, 2, \cdots, \infty$. Consider

$$\begin{aligned} D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq\ & a_1 D^*(Tx_{n-1}, Tx_n, Tx_n) \\ &+ a_2 D^*(Tx_{n-1}, Tx_n, Tx_n) \\ &+ a_3 D^*(Tx_{n-1}, Tx_n, Tx_n) \\ &+ a_4 D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &+ a_5 D^*(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}) \\ &+ a_6 D^*(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}) \\ &+ a_7 D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &+ a_8 D^*(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}) \\ &+ a_9 D^*(Tx_{n-1}, Tx_n, Tx_{n+1}) \\ &+ a_{10} D^*(Tx_n, Tx_n, Tx_{n+1}) \end{aligned}$$

$$\begin{aligned} D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq\ & (a_1 + a_2 + a_3) D^*(Tx_{n-1}, Tx_n, Tx_n) \\ &+ (a_4 + a_7 + a_{10}) D^*(Tx_n, Tx_n, Tx_{n+1}) \\ &+ (a_5 + a_6 + a_8) D^*(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}) \\ &+ a_9 D^*(Tx_{n-1}, Tx_n, Tx_{n+1}) \end{aligned} \tag{1}$$

then by rectangle inequality

$$D^*(Tx_{n-1}, Tx_n, Tx_{n+1}) \leq D^*(Tx_{n-1}, Tx_{n+1}, Tx_n) + D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \quad (2)$$

and

$$D^*(Tx_{n-1}, Tx_{n+1}, Tx_n) \leq D^*(Tx_{n-1}, Tx_n, Tx_n) + D^*(Tx_n, Tx_{n+1}, Tx_{n+1})$$

therefore from (2) we have

$$D^*(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}) \leq 2D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) + D^*(Tx_{n-1}, Tx_n, Tx_n)$$

so

$$\begin{aligned} D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq\ & (a_1 + a_2 + a_3) D^*(Tx_{n-1}, Tx_n, Tx_n) \\ & + (a_4 + a_7 + a_{10}) D^*(Tx_n, Tx_n, Tx_{n+1}) \\ & + (a_5 + a_6 + a_8) \begin{pmatrix} 2D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ +D^*(Tx_{n-1}, Tx_n, Tx_n) \end{pmatrix} \\ & + a_9 (D^*(Tx_{n-1}, Tx_n, Tx_n) + D^*(Tx_n, Tx_{n+1}, Tx_{n+1})) \end{aligned} \quad (3)$$

hence

$$\begin{aligned} D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq\ & (a_1 + a_2 + a_3 + a_9 + a_5 + a_6 + a_8) \\ & D^*(Tx_{n-1}, Tx_n, Tx_n) \\ & + (a_4 + a_7 + a_{10} + 2a_5 + 2a_6 + 2a_8) \\ & D^*(Tx_n, Tx_n, Tx_{n+1}) \end{aligned}$$

since

$$D^*(x, x, y) = D^*(x, y, y) \quad \text{for all} \quad x, y, z \in X \quad (4)$$

then

$$\begin{aligned} & (1 - (a_4 + a_7 + a_{10} + 2a_5 + 2a_6 + 2a_8)) D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ & \leq (a_1 + a_2 + a_3 + a_9 + a_5 + a_6 + a_8) D^*(Tx_{n-1}, Tx_n, Tx_n) \end{aligned}$$

hence

$$D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \frac{a_1 + a_2 + a_3 + a_9 + a_5 + a_6 + a_8}{1 - (a_4 + a_7 + a_{10} + 2a_5 + 2a_6 + 2a_8)} D^*(Tx_{n-1}, Tx_n, Tx_n)$$

This implies

$$D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq q D^*(Tx_{n-1}, Tx_n, Tx_n)$$

where $q=\frac{a_1+a_2+a_3+a_9+a_5+a_6+a_8}{1-(a_4+a_7+a_{10}+2a_5+2a_6+2a_8)}$, then $q\in[0,1[$.
by repeated the above inequality we obtain,

$$D^*\left(Tx_n,Tx_{n+1},Tx_{n+1}\right)\leq q^nD^*\left(Tx_0,Tx_1,Tx_1\right) \tag{5}$$

Then, for all $n,\ m\in\mathbb{N},\ n<m$ we have by repeated use the rectangle inequality and inequality (5) that:

$$\begin{aligned}D^*\left(Tx_n,Tx_m,Tx_m\right) &\leq D^*\left(Tx_n,Tx_n,Tx_{n+1}\right)\\ &+D^*\left(Tx_{n+1},Tx_{n+1},Tx_{n+2}\right)\\ &+D^*\left(Tx_{n+2},Tx_{n+2},Tx_{n+3}\right)\\ &+\cdots\\ &+D^*\left(Tx_{m-1},Tx_m,Tx_m\ \right)\\ &\leq\left(q^n+\cdots+q^m\right)D^*\left(Tx_0,Tx_1,Tx_1\right)\\ &\leq\frac{q^n}{1-q}D^*\left(Tx_0,Tx_1,Tx_1\right)\end{aligned}$$

hence

$$\left\|D^*\left(Tx_n,Tx_m,Tx_m\right)\right\|\leq K\frac{q^n}{1-q}\left\|D^*\left(Tx_0,Tx_1,Tx_1\right)\right\|$$

Since $K\frac{q^n}{1-q}\left\|D^*\left(Tx_0,Tx_1,Tx_1\right)\right\|\to 0$, as $n,m\to\infty$ then, it follows that

$$D^*\left(Tx_n,Tx_m,Tx_m\ \right)\to 0,\text{as } n,m\to\infty.\text{For } n,m,l\in\mathbb{N}$$

and from

$$D^*\left(Tx_n,Tx_m,Tx_l\right)\leq D^*\left(Tx_n,Tx_m,Tx_m\right)+D^*\left(Tx_m,Tx_l,Tx_l\right)$$

we have

$$\left\|D^*\left(Tx_n,Tx_m,Tx_l\right)\right\|\leq K\left(\left\|D^*\left(Tx_n,Tx_m,Tx_m\right)\right\|+\left\|D^*\left(Tx_m,Tx_l,Tx_l\right)\right\|\right)$$

Taking the limit as $n,m,l\to\infty$, we get $D^*\left(Tx_n,Tx_m,Tx_l\right)\to 0$. So $\{Tx_n\}$ is D^*- Cauchy sequence, since X is D^*- complete, there exists $a\in X$ such that

$$\lim_{n\to\infty}Tx_n=a \tag{6}$$

Since T is subsequentially convergent $\{x_n=S^{n-1}x_0\}$ has a convergent subsequence. So there exist $b\in X$ and $\{n_k\}_{k=1}^{\infty}$ such that $\lim\limits_{n\to\infty}S^{n_k-1}x_0=b$. Hence, since T is continuous, we have $\lim\limits_{n\to\infty}TS^{n_k-1}x_0=Tb$ and by (6) we conclude that $Tb=a$. Since S is continuous and $\lim\limits_{n\to\infty}S^{n_k}x_0=Sb$ therefore

$$\lim_{n\to\infty}TS^{n_k}x_0=TSb$$

Again by (6), $\lim\limits_{n\to\infty} TS^{n_k}x_0 = a$ and therefore by the unique limit we have $TSb = a.Since$ T is one to one and by (6)$Sb = b$, so S has a fixed point. Now we prove the uniqueness of the fixed point. If b is another fixed point of S,then $Sb = b$ and we have

$$\begin{aligned} D^*(TSa, TSa, TSb) &= D^*(Ta, Ta, Tb) \\ &\leq (a_1 + a_3 + a_7 + a_6 + a_8 + a_9 + a_{10}) D^*(TSa, TSa, TSb) \\ &< D^*(TSa, TSa, TSb) \end{aligned}$$

contradiction. Hence $D^*(TSa, TSa, TSb) = 0$ which implies that $Ta = TSa = TSb = Tb$. As T is injective, $a = b$ is the unique fixed point of S; As (T, S) is a Banach pair, T and S commutes at the fixed point of S which implies that $TSa = STa$ for $a \in F(S)$. that is $Ta = STa$ which implies that Ta is another fixed point of S. The uniqueness of fixed point of S implies that $a = Ta$. Hence $a = Sa = Ta$ is the unique common fixed point of S and T in X. ■

Corollary 21 *Let (X, D^*) be a generalized complete cone D^*- metric space , P be a normal cone with constant normal K and let $S : X \to X$ be a continuous mapping. If S satisfies*

$$\begin{aligned} D^*(Sx, Sy, Sz) \leq\ & a_1 D^*(x, y, z) + a_2 D^*(x, Sx, Sx) \\ &+ a_3 D^*(x, Sx, z) + a_4 D^*(y, Sy, Sy) \\ &+ a_5 D^*(x, Sy, Sy) + a_6 D^*(x, Sz, Sz) \\ &+ a_7 D^*(y, Sz, Sz) + a_8 D^*(x, Sy, Sz) \\ &+ a_9 D^*(x, y, Sz) + a_{10} D^*(Sx, Sx, Sz) \end{aligned}$$

for all $x, y, z \in X$,where a_i, $i \in \{1, 2, 3, \cdots, 12\}$ are all nonnegative constants such that

$$a_1 + a_2 + a_3 + a_4 + 3a_5 + 3a_6 + a_7 + 3a_8 + a_9 + a_{10} < 1$$

then S has a unique fixed point in X.

Proof. The proof of this corollary follows by taking $T = I$, the identity mapping in Theorem 20. ■

Corollary 22 *Let (X, D^*) be a generalized complete cone D^*-metric space, P be a normal cone with constant normal K and let $T, S^m : X \to X$ be two continuous mappings.Assume that T is injective and subsequentially convergent.*

If T and S^m satisfy

$$\begin{aligned}D^*(TS^m x, TS^m y, TS^m z) \leq\ & a_1 D^*(Tx, Ty, Tz) \\ & + a_2 D^*(Tx, TSx, TS^m x) \\ & + a_3 D^*(Tx, TS^m x, Tz) \\ & + a_4 D^*(Ty, TS^m y, TS^m y) \\ & + a_5 D^*(Tx, TS^m y, TSy) \\ & + a_6 D^*(Tx, TS^m z, TS^m z) \\ & + a_7 D^*(Ty, TS^m z, TS^m z) \\ & + a_8 D^*(Tx, TS^m y, TS^m z) \\ & + a_9 D^*(Tx, Ty, TS^m z) \\ & + a_{10} D^*(TS^m x, TS^m x, TS^m z)\end{aligned}$$

for all $x, y, z \in X$,where $a_i, i \in \{1, 2, 3, \cdots, 10\}$ are all nonnegative constants such that

$$a_1 + a_2 + a_3 + a_4 + 3a_5 + 3a_6 + a_7 + 3a_8 + a_9 + a_{10} < 1$$

then S has a unique fixed point in X. Moreover, if (T, S) is a Banach pair, then T and S have a unique common fixed point in X.

Proof. By theorem 20 applied by $S = S^m$, S^m has a unique fixed point u_0 that is $S^m u_0 = u_0$.Therefore

$$S(S^m u_0) = Su_0 = S^m(Su_0)$$

hence Su_0 is a fixed point for S^m, so by the uniqueness fixed point of S^m, we have $Su_0 = u_0$.Hence S has a fixed point.The remainder of the proof is obvious. ■

Corollary 23 *Let (X, D^*) be a generalized complete cone D^*- metric space , P be a normal cone with constant normal K and let $T, S : X \to X$ be two continuous mappings.Assume that T is injective and subsequentially convergent. If T and S satisfy*

$$D^*(TSx, TSy, TSz) \leq a_1 D^*(Tx, Ty, Tz)$$

for all $x, y, z \in X$,where $a_1 \in [0, 1[$. Then S has a unique fixed point in X. Moreover, if (T, S) is a Banach pair, then T and S have a unique common fixed point in X.

Proof. It follows from the proof of theorem 1 by taking.

$$a_i = 0 \text{ for } i \in \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

■

Definition 24 *Let (X, D^*) be a generalized cone metric space, P be a normal cone with constant normal K and let $T, S : X \to X$ be two functions. A mapping S is said to be a $T-$contractive if for $x, y, z \in X$ such that*

$$D^*(TSx, TSy, TSz) < D^*(Tx, Ty, Tz)$$
$$\forall\, x, y, z \in X : Tx \neq Ty \text{ or } Tx \neq Tz \text{ or } Tz \neq Ty$$

Obviously, every $T-$contraction function is $T-$ contractive but the converse is not true.

Example 25 *$X = [1, \infty[$, $D^*(x, y, z) = |x - y| + |x - z| + |y - z|$, $Sx = \sqrt{x}$ and $Tx = x$, then S is $T-$ contractive but S is not $T-$ contraction.*

Theorem 26 *Let (X, D^*) be a generalized compact cone D^*- metric space, P be a normal cone with constant normal K and let $T, S : X \to X$ be two continuous mappings.Assume that T is injective and subsequentially convergent. If T and S satisfy*

$$\begin{aligned} D^*(TSx, TSy, TSz) \leq\ & a_1 D^*(Tx, Ty, Tz) + a_2 D^*(Tx, TSx, TSx) \\ & + a_3 D^*(Tx, TSx, Tz) + a_4 D^*(Ty, TSy, TSy) \\ & + a_5 D^*(Tx, TSy, TSy) + a_6 D^*(Tx, TSz, TSz) \\ & + a_7 D^*(Ty, TSz, TSz) + a_8 D^*(Tx, TSy, TSz) \\ & + a_9 D^*(Tx, Ty, TSz) + a_{10} D^*(TSx, TSx, TSz) \end{aligned}$$

for all $x, y, z \in X$,where $a_i, i \in \{1, 2, 3, \cdots, 10\}$ are all nonnegative constants such that

$$a_1 + a_2 + a_3 + a_4 + 3a_5 + 3a_6 + a_7 + 3a_8 + a_9 + a_{10} < 1$$

then S has a unique fixed point in X. Moreover, if (T, S) is a Banach pair, then T and S have a unique common fixed point in X.

Proof. First, we prove that S is continuous. Let $\lim_{n\to\infty} x_n = x$,we prove that $\lim_{n\to\infty} Sx_n = Sx$,

$$\begin{aligned} D^*(TSx_n, TSx_n, TSx) =\ & D^*(Tx_{n+1}, Tx_{n+1}, TSx) \qquad (7) \\ & + a_1 D^*(Tx_n, Tx_n, Tx) + a_2 D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ & + a_3 D^*(Tx_n, Tx_{n+1}, Tx) + a_4 D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ & + a_5 D^*(Tx_n, Tx_{n+1}, Tx_{n+1}) + a_6 D^*(Tx_n, TSx, TSx) \\ & + a_7 D^*(Tx_n, TSx, TSx) + a_8 D^*(Tx_n, Tx_{n+1}, TSx) \\ & + a_9 D^*(Tx_n, Tx_n, TSx) + a_{10} D^*(Tx_{n+1}, Tx_{n+1}, TSx) \end{aligned}$$

from (7) and from (4) we have

$$\begin{aligned}(1-a_{10})\,D^*(TSx_n,TSx_n,TSx) &\leq a_1D^*(Tx_n,Tx_n,Tx)\\ &+(a_2+a_4+a_5)\,D^*(Tx_n,Tx_{n+1},Tx_{n+1})\\ &+(a_6+a_7+a_9)\,D^*(Tx_{n+1},TSx,TSx)\\ &+a_3D^*(Tx_n,Tx_{n+1},Tx)\\ &+a_8D^*(Tx_n,Tx_{n+1},TSx)\end{aligned} \tag{8}$$

Since

$$\begin{aligned}D^*(Tx_n,Tx_{n+1},Tx_{n+1}) &\leq D^*(Tx_n,Tx_n,Tx)+D^*(Tx_{n+1},Tx_{n+1},Tx) \quad (9)\\ D^*(Tx_n,Tx_{n+1},TSx) &\leq D^*(Tx_n,Tx_{n+1},Tx_{n+1})+D^*(Tx_{n+1},Tx,Tx)\\ &\leq D^*(Tx_n,Tx_n,Tx)+2D^*(Tx_{n+1},Tx_{n+1},Tx).\\ D^*(Tx_n,Tx_{n+1},TSx) &\leq D^*(Tx_n,Tx_n,Tx)+D^*(Tx_{n+1},Tx_{n+1},Tx)\\ &+D^*(Tx_{n+1},Tx_{n+1},TSx)\end{aligned}$$

by (7), (8) and (9) we obtain

$$\begin{aligned}(1-a_6-a_7-a_8-a_9-a_{10})\,D^*(TSx_n,TSx_n,TSx) \leq\; &(a_1+a_2+a_3+a_4+a_5+a_8)\\ &D^*(Tx_n,Tx_n,Tx)\\ &+(a_2+2a_3+a_4+a_5+a_8)\\ &D^*(Tx_{n+1},Tx_{n+1},Tx)\end{aligned}$$

hence

$$\begin{aligned}D^*(TSx_n,TSx_n,TSx) &\leq \frac{(a_1+a_2+a_3+a_4+a_5+a_8)}{(1-a_6-a_7-a_8-a_9-a_{10})}D^*(Tx_n,Tx_n,Tx)\\ &+\frac{(a_2+2a_3+a_4+a_5+a_8)}{(1-a_6-a_7-a_8-a_9-a_{10})}D^*(Tx_{n+1},Tx_{n+1},Tx)\end{aligned}$$

setting

$$\alpha=\frac{a_1+a_2+a_3+a_4+a_5+a_8}{1-a_6-a_7-a_8-a_9-a_{10}} \text{ and } \beta=\frac{a_2+2a_3+a_4+a_5+a_8}{1-a_6-a_7-a_8-a_9-a_{10}}$$

we get

$$\begin{aligned}\|D^*(TSx_n,TSx_n,TSx)\| &\leq K\,\|\alpha D^*(Tx_n,Tx_n,Tx)+\beta D^*(Tx_{n+1},Tx_{n+1},Tx)\| \\ &K\alpha\,\|D^*(Tx_n,Tx_n,Tx)\|+K\beta\,\|D^*(Tx_{n+1},Tx_{n+1},Tx)\|\end{aligned} \tag{10}$$

Since T is continuous, it follows from (10), that

$$D^*(Tx_n,Tx_n,Tx)\to 0$$

and

$$D^*(Tx_{n+1}, Tx_{n+1}, Tx) \to 0$$

thus

$$D^*(TSx_n, TSx_n, TSx) \to 0$$

this shows that

$$TSx_n \to TSx$$

Let $\{Sx_{n_k}\}$ be arbitrary convergence subsequence of $\{Sx_n\}$. there exists $y \in X$ such $\lim\limits_{n\to\infty} Sx_{n_k} = y$.Since T is continuous

$$\lim_{n\to\infty} TSx_{n_k} = Ty$$

By the unique limit $TSx = Ty$, T is injective, therefore $Sx = y$. Hence, every convergence subsequence of $\{Sx_n\}$ converges to $Sx.Since$ X is a compact generalized cone metric space, S is continuous. Now we prove the uniqueness: we assume that there exist a and b such that $a \neq b$ and $Sa = a$, $Sb = b$. Then

$$\begin{aligned} D^*(TSa, TSb, TSa) &\leq (a_1 + a_3 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}) D^*(Ta, Tb, Ta) \\ &< D^*(Ta, Tb, Ta) \end{aligned}$$

then necessarily $Ta = Tb$, Hence, since T is one to one , we have $a = b$ contradiction. For the existence, we consider the function $\varphi : X \to E$ defined by $\varphi(y) = D^*(TSy, TSy, Ty)$, φ is continuous and hence by compacteness attains its minimum say at x. If $Sx \neq x$, therefore since T is injective $TSx \neq Tx$. Since

$$\frac{a_1 + a_3}{1 - a_2 - a_3 - a_4 - a_5 - a_8 - a_{10}} < 1$$

it is easy to see that

$$\begin{aligned} \varphi(Sx) = D^*\left(TS^2x, TS^2x, TSx\right) &\leq \\ &\frac{a_1 + a_3}{1 - a_2 - a_3 - a_4 - a_5 - a_8 - a_{10}} D^*(TSx, TSx, Tx) \\ &< D^*(TSx, TSx, Tx) = \varphi(x) \end{aligned}$$

which is a contradiction to the definition of x, hence $Sx = x$. Now, let $x_0 \in X$, and set $\alpha_n = D^*(TS^n x_0, TS^n x_0, Tx)$.

$$\begin{aligned} \alpha_{n+1} &= D^*\left(TS^{n+1}x_0, TS^{n+1}x_0, Tx\right) = D^*\left(TS^{n+1}x_0, TS^{n+1}x_0, Tx\right) \qquad (11) \\ &\leq a_1 D^*(TS^n x_0, TS^n x_0, Tx) + a_2 D^*\left(TS^n x_0, TS^{n+1}x_0, TS^{n+1}x_0\right) \\ &+ a_3 D^*\left(TS^n x_0, TS^{n+1}x_0, Tx\right) + a_4 D^*\left(TS^n x_0, TS^{n+1}x_0, TS^{n+1}x_0\right) \\ &+ a_5 D^*\left(TS^n x_0, TS^{n+1}x_0, TS^{n+1}x_0\right) + a_6 D^*(TS^n x_0, Tx, Tx) \\ &+ a_7 D^*(TS^n x_0, Tx, Tx) + a_8 D^*\left(TS^n x_0, TS^{n+1}x_0, Tx\right) \\ &+ a_9 D^*(TS^n x_0, TS^n x_0, Tx) + a_{10} D^*\left(TS^{n+1}x_0, TS^{n+1}x_0, Tx\right) \end{aligned}$$

by using rectangular inequality we get

$$\begin{aligned} a_2 D^*\left(TS^n x_0, TS^{n+1}x_0, TS^{n+1}x_0\right) &\leq a_2\left(\alpha_n + \alpha_{n+1}\right) \\ a_3 D^*\left(TS^n x_0, TS^{n+1}x_0, Tx\right) &\leq a_3\left(\alpha_n + \alpha_{n+1}\right) \\ a_4 D^*\left(TS^n x_0, TS^{n+1}x_0, TS^{n+1}x_0\right) &\leq a_4\left(\alpha_n + \alpha_{n+1}\right) \\ a_5 D^*\left(TS^n x_0, TS^{n+1}x_0, TS^{n+1}x_0\right) &\leq a_5\left(\alpha_n + \alpha_{n+1}\right) \\ a_8 D^*\left(TS^n x_0, TS^{n+1}x_0, Tx\right) &\leq a_8\left(\alpha_n + \alpha_{n+1}\right) \end{aligned} \tag{12}$$

from 11 and 12 , it follows that

$$\alpha_{n+1} \leq \frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9}{1 - a_2 - a_3 - a_4 - a_5 - a_8 - a_{10}} \alpha_n$$

Since $\frac{a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9}{1-a_2-a_3-a_4-a_5-a_8-a_{10}} < 1$, then

$$\alpha_{n+1} \leq \alpha_n$$

this implies that $\{\alpha_n\}$ is a nonincreasing sequence of nonnegative real numbers and so has a limit denoted by α. By compactness $\{TS^n x_0\}$ has a convergent subsequence $\{TS^{n_k} x_0\}$ say

$$\lim_{n\to\infty} TS^{n_k} x_0 = z \tag{13}$$

Since T is sequentially convergent for a $\beta \in X$, we have

$$\lim_{n\to\infty} S^{n_k} x_0 = \beta \tag{14}$$

By (13) and (14) we have $T\beta = z$. So $D^*\left(T\beta, T\beta, Tx\ \right) = a$, Next we show that $S\beta = x$, if $S\beta \neq x$ then ■

$$\begin{aligned} a &= \lim_{n\to\infty} D^*\left(TS^n x_0, TS^n x_0, Tx\right) \\ &= \lim_{n\to\infty} D^*\left(TS^{n_k} x_0, TS^{n_k} x_0, Tx\right) \\ &= D^*\left(TS\beta, TS\beta, Tx\right) \\ &= D^*\left(TS\beta, TS\beta, TSx\right) < D^*\left(T\beta, T\beta, Tx\right) = a \end{aligned}$$

contradiction. Hence $S\beta = x$.

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