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On Some Two–Generator Finite Solvable Automorphism Groups of Compact Riemann Surfaces

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Abstract

A finite group G can be represented as a group of automorphisms of a compact Riemann surfaces. In this paper we prove the existence of some infinite families of two generator finite solvable groups with short derived series acting as Riemann surface automorphism groups.

Keywords: *Riemann surface, Fuchsian group, smooth homomorphism, smooth quotient.*

1 Introduction

For the last three decades , the study of automorphisms of Riemann surfaces has received considerable attention . The automorphisms of a compact Riemann surface of genus $g\geq 2$ form a finite group. On the other hand , every finite group is representable as the automorphism group of some Riemann surface of genus $g\geq 2$[2]. The theory of Fuchsian groups have an important role in the study of Riemann surface automorphism groups. Macbeath[8] in his Dundee summer school note proved that a finite group G is the automorphism group of a compact Riemann surface of genus g if and only if it is a quotient of a 'Fuchsian' group by a 'surface' group K. Macbeath also showed [8] that the maximal automorphism groups of compact Riemann surfaces occur as quotients of a 'special type' of 'Fuchsian triangle group'. Similarly Chetiya [3] in his Ph.D. thesis showed that the maximal solvable automorphism groups are also quotients of a Fuchsian triangle group having periods 2, 3, 8 .A quotient of a Fuchsian triangle group is a two – generator group. This shows the importance of finding quotients of different classes of Fuchsian triangle groups from a strictly theoretic point of view. This was the theme of the papers by Chetiya [4], [5], Chetiya and Kalita [6], [7] where the existence of several infinite classes of two generator finite solvable automorphism groups of compact Riemann surfaces was proved.

In this paper we consider another interesting class of Fuchsian triangle groups and give a technique to construct infinitely many finite solvable quotients of these groups. Some of the results given by Chetiya and Kalita [5] ,[6] come out as special cases of those of ours.

2 Prelimineries

An infinite group Γ generated by k elements $x_1, x_2, \ldots, x_k$ of finite orders $m_1, m_2, \ldots, m_k$ respectively and 2γ elements $a_1, b_1, \ldots, a_\gamma, b_\gamma$ of infinite orders satisfying:

$$x_1^{m_1} = \ldots\ldots = x_k^{m_k} = \prod_{i=1}^{K} x_i \prod_{j=1}^{\gamma} [a_j, b_j] = 1 \quad \ldots\ldots(2.1)$$

Where $[a_j, b_j]$ denotes the commutator of a_j, b_j ,is called a Fuchsian group if

$$\partial(\Gamma) = 2\gamma\_2 + \prod_{i=1}^{k} (1 - 1/_{m_i}) > 0 \quad \ldots\ldots(2.2)$$

Such a Fuchsian group is usually denoted by $\Delta(\gamma; m_1, \ldots, m_k)$ which is called the signature of the group. The non negative integer γ is called the genus of the group Γ. If γ=0 we simply use the symbol $\Delta(m_1, m_2, \ldots, m_k)$.The integers $m_i \geq 2$ are called the periods of the Fuchsian group. If γ=0, k=3, then $\Gamma = \Delta(m_1, m_2, m_3)$ is called a Fuchsian triangle group. A Fuchsian group having no elements of finite order except the identities is called a 'surface group'. A surface group K is generated by 2g elements of infinite order and is denoted by Δ(g;0). If K is a surface group of genus g, then

$$\partial(K) = 2\,(g - 1\,). \quad \ldots\ldots(2.3)$$

Moreover If Γ_1 is a subgroup of Γ of finite index then

$$[\Gamma:\ \Gamma_1] = \partial\,(\Gamma_1)/\partial(\Gamma) \quad \ldots\ldots(2.4)$$

If Φ is a homomorphism from the Fuchsian group Γ to any finite group G such that kerΦ is a surface group ,then Φ is called a smooth homomorphism and a factor Γ́/K where K is a normal surface subgroup of Γ is called a smooth quotient of Γ.

A Fuchsian group Γ is said to satisfy the l.c.m. condition if every period of it divides the l.c.m. of the remaining periods. The derived group of a Fuchsian group is a surface group if and only if it satisfies the l.c.m. condition [9]. It is to be noted that a necessary condition for a Fuchsian group to satisfy the l.c.m. condition is that it must have more than one period.

3 Existence of Solvable Smooth Quotients

We now proceed to find an infinite family of solvable finite smooth quotients of $\Delta(l,m,n)$ where

1. ℓ,m, n are positive integers ≥ 2
2. $(\ell, m) = d_1 > 1$ $(\ell, n) = d_2 > 1$ $(m, n) = 1$ and $(d_1, d_2) = 1$ (for any two positive integers k_1, k_2 the notation (k_1, k_2) is used to denote the h. c .f. of k_1 and k_2.)

The following lemmas [lemma 3.1 , lemma 3.2 , lemma 3.3 and lemma 3.4] can be found in Bujalence et al [1] , Chetiya B. P. [4] and Das G .and Patra K [10].

Lemma 3.1 *Let $\Gamma = \Delta(\gamma; m_1, \ldots\ldots\ldots, m_k)$ be a Fuchsian group with generators $x_1, \ldots\ldots\ldots, x_k$ of finite order and $a_1, b_1, \ldots\ldots\ldots\ldots a_\gamma, b_\gamma$ of infinite order. Let N be a normal subgroup of Γ of finite index. Let p_i be the order of the image of x_i in the quotient $\acute{\Gamma}/N$ and let $I = \{1 \leq i \leq k;\ m_i \neq p_i\}$.Also let $n_i = m_i/p_i$ and $s_i = {}^{[\Gamma : N]}/p_i$ for every $i \in I$, then $N = \Delta(\gamma^{/}; n_1, \ldots\ldots\ldots, n_i)$, $i \in I$, where each n_i occurs s_i times and $\gamma^{/}$ is obtained from (2.4).*

Lemma 3.2 *Let K be a Fuchsian surface group of genus g and $K^{/}$ the derived group of K. If for each positive integer n , $K_n{}^*$ denotes the subgroup of K generated by the nth power of all the generators of K, then the product $K_n = K_n{}^* K^{/}$ is a normal surface subgroup of K such that $[K:K_n] = n^{2g}$.*

Lemma 3.3 *Let Γ be a Fuchsian group of non zero genus whose periods satisfy the l.c.m. condition and let l be the l.c.m. of the periods of Γ . If $\acute{\Gamma}$ is the derived group of Γ and $\Gamma_{k\ell}$ is the subgroup of Γ generated by the $k\ell^{-th}$ power ($k \geq 1$) of the generators of Γ, then $\Gamma_s = \Gamma_{k\ell}\acute{\Gamma}$ is a surface subgroup of Γ of finite index for any positive integer $k \geq 1$.*

Lemma 3.4 *Let $\Gamma = (\gamma; m, m, \ldots\ldots, m, n)$ be a Fuchsian group with s -1 occurrences of m, $s \geq 2$, $m \geq 2$, $n \geq 2$, $\gamma \geq 1$ and $(m, n) = 1$. Then for positive integer k, $k \geq 2$, Γ has a Fuchsian subgroup Γ_k whose periods satisfy the l. c. m. conditions and Γ_k is of index $d^{s-2} k^{2\gamma}$ where $d = (k,m)$ in Γ.*
We are now in a position to give the proof of the theorem on the existence of solvable smooth quotients of Fuchsian triangle group .

Theorem 3.1 *Let $\Gamma = (\ell, m, n)$ be a Fuchsian group where ℓ, m, n are positive integers ≥ 2 such that $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$, $(m, n) = 1$ and $(d_1, d_2) = 1$ (as $(m, n) = 1$). Then for each positive integer $k > 1$, Γ admits smooth quotients of order*
$d_1 d_2 k^{2\gamma^{/}} t^{2\gamma_k} a^{A+2\gamma_k-1}\, b^{A/h_1+2\gamma_k-1}\, c^{A/h_2+2\gamma_k-1}\, h_1{}^{d_2-A/h_1-2\gamma_k}\, h_2{}^{d_1-A/h_2-2\gamma_k}$, k>1
and of genus:
$(1/(2h_1^{A/h_1+2\gamma_k-1} h_2^{A/h_2+2\gamma_k-1}))$ [$a^{A+2\gamma_k-1} b^{A/h_1+2\gamma_k-1}$ $c^{A/h_2+2\gamma_k-1} t^{2\gamma_k}$][$2\gamma_k - 2 + A(1 + \frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c})$]+1, Where $A = h_1^{d_2-1} h_2^{d_1-1}$ $k^{2\gamma^{/}}$, $h_1 = (b, k)$ and $h_2 = (c, k)$.
Proof: Let us assume that $\ell = ad_1d_2$, $m = bd_1$, $n = cd_2$ where a, b , c are prime to each other. Let $\Gamma^{/}$ be generated by elements x_1, x_2 and x_3 satisfying : $x_1^{ad_1d_2} = x_2^{bd_1} = x_3^{cd_2} = x_1x_2x_3 = 1$

Or equivalently $x_1^{ad_1d_2} = x_2^{bd_1} = (x_1x_2)^{cd_2} = 1$.

Let u_1, u_2, u_3 be the images of x_1, x_2 and x_3 respectively under the abelianizing homomorphism From Γ onto Γ/Γ'. Then Γ/Γ' is generated by u_1, u_2 and u_3 satisfying:

$u_1^{ad_1d_2} = u_2^{bd_1} = u_3^{cd_2} = u_1u_2u_3 = 1$.

Or $u_1^{ad_1d_2} = u_2^{bd_1} = (u_1u_2)^{cd_2} = 1$.

The above relation gives ,

$u_1^{d_1d_2} = 1$ as

$(u_1u_2)^{cd_2} = 1$

$\Rightarrow u_1^{cd_2} u_2^{cd_2} = 1$

$\Rightarrow u_1^{d_2} u_2 = 1$

$\Rightarrow u_2 = u_1^{-d_2}$.

Now, $u_1^{ad_1d_2} = u_2^{bd_1} = 1$

$\Rightarrow u_1^{ad_1d_2} = (u_1^{-d_2})^{bd_1} = 1$

$\Rightarrow u_1^{ad_1d_2} = u_1^{-bd_1d_2} = 1$.

Therefore $u_1^{d_1d_2} = 1$, as $(a, b) = 1$.

So, $\Gamma/\Gamma' \cong Z_{d_1d_2}$

By lemma 3.1

$$\Gamma' = (\gamma'; a, b, \underbrace{\ldots\ldots\ldots}_{d_1\text{-times}}, b\ c, \underbrace{\ldots\ldots\ldots}_{d_2\text{-times}} c) \quad . \qquad (3.1.1)$$

And we get $\gamma' = 1/2\,[\,(d_1-1)(d_2-1)\,]$.

By lemma 3.4 , Γ' has a subgroup Γ_k, $k>1$ 0f finite index whose periods satisfy the l.c.m. condition.

Let $\Gamma^*_k = \{x^k, x\in \Gamma'\}$ be a subgroup of Γ', for $k\geq 2$. Let $\Gamma_k = \Gamma^*_k \Gamma''$, then by lemma (3.2) Γ_k is normal in Γ' of finite index .

As $\Gamma'' \subseteq \Gamma_k \subseteq \Gamma'$, so Γ'/Γ_k is abelian .

Consider an abelianizing homomorphism :

$\Phi : \Gamma' \rightarrow \Gamma'/\Gamma_k$

Let $u, u'_1, \ldots\ldots, u'_{d2}, v'_1, \ldots\ldots, v'_{d1}$, a_1, b_1, ……….. , a'_γ, b'_γ be the images of x, $x'_1, \ldots\ldots, x'_{d2}, y'_1, \ldots, y'_{d_1}$, α_1, β_1,……. α'_γ, β'_γ respectively under the above mentioned homomorphism satisfying the conditions:

$u^a = u_1'^{\,b} = \ldots\ldots = u'^{\,b}_{d_2} = v_1'^{\,c} = \ldots\ldots = v'^{\,c}_{d_1} = uu'_1 \ldots\ldots u'_{d_2}\, v'_1 \ldots\ldots v'_{d_1} = 1$

$u^k = u_1'^{\,k} = \ldots\ldots = u'^{\,k}_{d_2} = v_1'^{\,k} = \ldots\ldots = v'^{\,k}_{d_1} = a_1^k = b_1^k = \ldots\ldots = a^k_{\gamma'} = b^k_{\gamma'} = 1$,

where the elements commute with each other.

If $(b,k)=h_1\geq 1$, $(c,k)=h_2\geq 1$, then the above relation give:

$u_1'^{\,h1} = \ldots\ldots = u'^{\,h1}_{d2} = v_1'^{\,h2} = \ldots\ldots = v'^{\,h2}_{d1} = 1$

and the elements commutes with each other.

We conclude that ,

$$\Gamma'/\Gamma_k \cong Z_{h_1} \oplus \ldots\ldots \oplus Z_{h_1} \oplus Z_{h_2} \ldots\ldots \oplus Z_{h_2} \oplus z_k \oplus \ldots\ldots \oplus z_k$$

$\underbrace{\qquad\qquad}_{(d_2\text{-1})\text{summands}}$ $\underbrace{\qquad\qquad}_{(d_1\text{-1})\text{summands}}$ $\underbrace{\qquad\qquad}_{2\gamma' - \text{summands}}$

Therefore

$$|\acute{\Gamma}/\Gamma_k| = h_1^{d_2-1} h_2^{d_1-1} k^{2\gamma'} = A, \text{ say} \quad . \qquad (3.1.2)$$

By lemma 3.1 we get,

$$\Gamma_k = \left(\gamma_k;\ \underbrace{a,\ldots\ldots,a}_{\text{A-times}},\underbrace{b/h_1,\ldots\ldots,b/h_1}_{A/h_1-times},\underbrace{c/h_2,\ldots\ldots,c/h_2}_{A/h_2-times} \right)$$

Since the periods of Γ_k satisfy the l.c.m. condition, therefore $\acute{\Gamma}_k-$ is a surface group. The genus of γ_k calculated from lemma 3.1 is,

$$\gamma_k = \frac{A}{2}\left[2\gamma' - 2 + d_2\left(1-\frac{1}{b}\right) + d_1\left(1-\frac{1}{c}\right) - \frac{1}{h_1} - \frac{1}{h_2} + \frac{1}{b} + \frac{1}{c}\right] + 1.$$

By lemma 3.3 let us construct another subgroup Γ_k^* generated by t^{-th} (t>1) power of the infinite order generators of Γ_k.

Let $N = \Gamma_k^* \Gamma_k'$, then N is normal in Γ_k of finite index and Γ_k/N is abelian as $\Gamma_k' \subseteq \Gamma_k^* \Gamma_k' = N$.

We now have,

$$\Gamma_k/N \cong \underbrace{z_a \oplus \ldots\ldots \oplus z_a}_{(A-1)\text{summands}} \oplus \underbrace{z_{b/h_1} \ldots\ldots \oplus z_{b/h_1}}_{(A/h_1-1)\text{summands}} \oplus \underbrace{z_{c/h_2} \oplus \ldots\ldots \oplus z_{c/h_2}}_{(A/h_2-1)\text{summands}} \oplus \underbrace{z_{t\ell'} \oplus \ldots\ldots \oplus z_{t\ell'}}_{2\gamma_k \text{ summands}}$$

Where $\ell' = \frac{abc}{h_1 h_2}$ is the l.c.m. of the periods of Γ_k.

Therefore

$$|\Gamma_k/N| = a^{A-1} (b/h_1)^{A/h_1 - 1} (c/h_2)^{A/h_2 - 1} (t\,\ell')^{2\gamma k}$$

$$= \frac{t^{2\gamma_k}}{h_1^{A/h_1 + 2\gamma_k - 1} h_2^{A/h_2 + 2\gamma_k - 1}} [a^{A+2\gamma k-1} b^{A/h_1 + 2\gamma k-1} c^{A/h2+2\gamma k-1}]$$

$$= B, \quad \text{say} \qquad (3.1.3)$$

Where A is given by (3.1.2).

By lemma 3.1 we have $N = (\gamma_{n;}\ \ldots\ldots)$.

Since N is a surface group and therefore $G = \Gamma/N$ is a smooth quotient of Γ and the genus obtained from (2.4) is

$$\gamma_n = \frac{B}{2}\left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right)\right] + 1.$$

,

Where A and B are given by (3.1.2) and (3.1.3) respectively. Therefore we have

$$\gamma_n = \frac{t^{2\gamma_k}}{2h_1^{A/h_1 + 2\gamma_k - 1} h_2^{A/h_2 + 2\gamma_k - 1}} [a^{A+2\gamma k-1} b^{A/h_1 + 2\gamma k-1} c^{A/h_2 + 2\gamma k-1}] \left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right)\right] + 1.$$

Now the order of G is

$$|G| = |\Gamma/N| = |\Gamma/\Gamma'| \, |\Gamma'/\Gamma_k| \, |\Gamma_k/N|$$

$$= d_1 d_2 \; k^{2\gamma'} t^{2\gamma_k} \; a^{A+2\gamma_k-1} \; b^{A/h_1+2\gamma_k-1} \; c^{A/h_2+2\gamma_k-1} \; h_1^{d_2-A/h_1-2\gamma_k} \; h_2^{d_2-A/h_2-2\gamma_k}, \quad k>1$$

Thus Γ admits smooth quotients ,but we cannot say whether it is solvable or not. This completes the proof of the theorem.

Remark 1: *If a=b=c=1, then Γ admits abelian smooth quotient of order d_1d_2 and of genus*
$\frac{1}{2}[(d_1-1)(d_2-1)]$ as well as metabelian smooth quotient of order $d_1d_2k^{2\gamma'}$ and of genus,
$k^{(d_1-1)}[\frac{1}{2}(d_1-1)(d_2-1)-1]+1$, for $k\geq1$.

Remark 2: *If a=1 and b, c >1 then Γ admits a metabelian smooth quotient of order*
$d_1d_2b^{d_2+2\gamma'-1} \; c^{d_1+2\gamma'-1} \, k^{2\gamma'}$, $k\geq1$ and of genus
$\frac{1}{2}b^{d_2+2\gamma'-1}c^{d_1+2\gamma'-1} \, k^{2\gamma'} [d_1d_2-{d_2}/{b}-{d_1}/{c}-1]+1$; $k\geq1$,

Where $\gamma'=\frac{1}{2}(d_1-1)(d_2-1)$.

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