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The Inversion of an Integral Equation Pertaining to a Generalized Polynomial Set

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Abstract

The main object of this paper is to evaluate a certain class of convolution integral equation of Fredholm type with n-generalized polynomial sets. Here, the integral equation is solved by applying the Mellin transform.

Keywords: *Mellin Transform, convolution integral equation, generalized polynomial set.*

1 Introduction

This paper deals with the investigation of the inversion of the integral

$$\prod_{i=1,\ldots,n} g_i(x) = \prod_{i=1,\ldots,n} \int_0^\infty h_i\left(\frac{x}{y}\right) f_i(y) \left(\frac{dy}{y}\right), \ (x > 0), \qquad \ldots(1)$$

Where g_i is a prescribed function, f_i is an unknown function to be determined and the kernel h_i is given by

$$\prod_{i=1,\dots,n} h_i(x) = \prod_{i=1,\dots,n}\left[\frac{(a_i x^{\alpha_i}+b_i)^{p_i}(c_i x^{\beta_i}+d_i)^{q_i}}{e^{-t_i x^{\xi_i}}} K_{n_i}\right.$$

$$\left. . \, R_{n_i}^{p_i,q_i}[x, a_i, b_i, c_i, d_i; \alpha_i, \beta_i, \gamma_i, \delta_i; e^{-t_i x^{\xi_i}}\right]$$

$$= \prod_{i=1,\dots,n}\left[\{x^{\theta_i}(\phi + xD_x)\}^{n_i}\{(a_i x^{\alpha_i}+b_i)^{p_i+\gamma_i n_i}(c_i x^{\beta_i}+d_i)^{q_i+\delta_i n_i} e^{-t_i x^{\xi_i}}\}\right]$$
…(2)

Where the polynomial set $R_n^{p,q}[x]$ is introduced by Agrawal and Chaubey [1] by means of the following formula

$$R_n^{p,q}[x] = R_n^{p,q}[x, a, b, c, d; \alpha, \beta, \gamma, \delta; \omega(x)]$$

$$= \frac{(ax^{\alpha}+b)^{-p}(cx^{\beta}+d)^{-q}}{K_n\,\omega(x)} T_{\theta,\phi}^{n}[(ax^{\alpha}+b)^{p+\gamma n}(cx^{\beta}+d)^{q+\delta n}\,\omega(x)], n = 0,1,2,...$$
…(3)

Where

$$T_{\theta,\phi}^{n} = x^{\theta}(\phi + x\,D_x),\ D_x = \frac{d}{dx}$$
…(4)

$\{K_n\}_{n=0}^{\infty}$ is a sequence of constants, a,b,c,d, $\alpha,\beta,\gamma,\delta$,p,q are constants and $\omega(x)$ is any general function of x, differentiable an arbitrary number of times.

The polynomial $R_n^{p,q}[x]$ is general in nature and yields a number of known polynomials as its special cases. In particular $\alpha = \beta = 1$, $K_n = n!$, $\phi = 0$, $\theta = \&1$, the polynomial set $R_n^{p,q}[x]$ reduces to $S_n^{p,q}[x, a, b, c, d; \gamma, \delta; \omega(x)]$, given by Srivastava and Panda [6].

2 Lemma

We begin with Lemma involving the Mellin transform of $\prod_{i=1,\dots,n} h_i(x)$ which is as follows:

If $\prod_{i=1,\dots,n} H_i(s) = \prod_{i=1,\dots,n} M\{h_i(x); s\}$ where $\prod_{i=1,\dots,n} h_i(x)$ is defined by (2), then

$$\prod_{i=1,\dots,n} H_i(s) = \prod_{i=1,\dots,n}\left[\sum_{k_i=0}^{n_i}\sum_{\ell_i=0}^{p_i+\gamma_i n_i}\sum_{m_i=0}^{q_i+\delta_i n_i}(-1)^{k_i+\ell_i+m_i}\frac{(-n_i)_{k_i}}{k_i!}\frac{(-p_i-\gamma_i n_i)_{\ell_i}}{\ell_i!}\right.$$

$$\cdot\frac{(-q_i-\delta_i n_i)_{m_i}}{m_i!}(\phi)^{n_i-k_i}(b_i)^{p_i+\gamma_i n_i-\ell_i}(d_i)^{q_i+\delta_i n_i-m_i}(a_i)^{\ell_i}(c_i)^{m_i}(\theta_i)^{k_i}$$

$$\left.\cdot\left\{-\left(\frac{s+\theta_i n_i}{\theta_i}\right)\right\}_{k_i}\frac{1}{|\xi_i|}\Gamma\left(\frac{s+\theta_i n_i+\alpha_i\ell_i+\beta_i m_i}{\xi_i}\right)(t_i)^{-\left(\frac{s+\theta_i n_i+\alpha_i\ell_i+\beta_i m_i}{\xi_i}\right)}\right],$$

…(5)

Provided that

$0 < \mathrm{Re}(s+\theta_i n_i+\alpha_i\ell_i+\beta_i m_i) < \xi_i$, when $\xi_i > 0$;

$\xi_i < \mathrm{Re}(s+\theta_i n_i+\alpha_i\ell_i+\beta_i m_i) < 0$, when $\xi_i < 0$;

$\theta_i \neq 0$ and $n_i, (p_i+\gamma_i n_i), (q_i+\delta_i n_i) \in N_0$, where $i = 1,\dots,n$

Proof:

Using binomial expansions for $[x^{\theta_i}(\phi + x D_x)]^{n_i}, (a_i x^{\alpha_i} + b_i)^{p_i+\gamma_i n_i}$ and $(c_i x^{\beta_i} + d_i)^{q_i+\delta_i n_i}$ in (2), we get

$$\prod_{i=1,\dots,n} h_i(x) = \prod_{i=1,\dots,n}\left[\sum_{k_i=0}^{n_i}\sum_{\ell_i=0}^{p_i+\gamma_i n_i}\sum_{m_i=0}^{q_i+\delta_i n_i}(-1)^{k_i+\ell_i+m_i}\frac{(-n_i)_{k_i}}{k_i!}\frac{(-p_i-\gamma_i n_i)_{\ell_i}}{\ell_i!}\right.\cdot$$

$$\cdot\frac{(-q_i-\delta_i n_i)_{m_i}}{m_i!}(\phi)^{n_i-k_i}(b_i)^{p_i+\gamma_i n_i-\ell_i}(d_i)^{q_i+\delta_i n_i-m_i}(a_i)^{\ell_i}(c_i)^{m_i}x^{\theta_i(n_i-k_i)}$$

$$. \, (x^{\theta_i+1} D_x)^{k_i} \{ x^{\alpha_i \ell_i + \beta_i m_i} e^{-t_i x^{\xi_i}} \} \Bigg]. \qquad \ldots(6)$$

Now applying Mellin transform of both sides of the equation (6) and using known results [9, p.14, eq. (2.2)]; [5, p.307,eq.(7)].

We find that

$$\prod_{i=1,\ldots,n} H_i(s) = \prod_{i=1,\ldots,n} \left[\sum_{k_i=0}^{n_i} \sum_{\ell_i=0}^{p_i+\gamma_i n_i} \sum_{m_i=0}^{q_i+\delta_i n_i} (-1)^{k_i+\ell_i+m_i} \frac{(-n_i)_{k_i}}{k_i!} \frac{(-p_i-\gamma_i n_i)_{\ell_i}}{\ell_i!} \right.$$

$$. \frac{(-q_i-\delta_i n_i)_{m_i}}{m_i!} (\phi)^{n_i-k_i} (b_i)^{p_i+\gamma_i n_i-\ell_i} (d_i)^{q_i+\delta_i n_i-m_i} (a_i)^{\ell_i} (c_i)^{m_i} (\theta_i)^{k_i}$$

$$\left. . \left\{ -\left(\frac{s+\theta_i n_i}{\theta_i} \right) \right\}_{k_i} M\left\{ x^{\alpha_i \ell_i+\beta_i m_i} e^{-t_i x^{\xi_i}}, s+\theta_i n_i \right\} \right] \qquad \ldots(7)$$

Again, using [5, p.307, eq. (7)] and the known result [5, p.313, eq. (15)]

$$M\{(e^{-ax^h});s\} = \frac{1}{|h|} a^{-(s/h)} \Gamma\left(\frac{s}{h}\right), \qquad \ldots(8)$$

We arrive at the required result (3).

3 Solution of the Integral Equation (1)

Theorem: *Let the Mellin transforms $F_i(s)$, $G_i(s)$ and $H_i(s) \neq 0$ of the functions $f_i(x)$, $g_i(x)$ and $h_i(x)$ defined by (2) exist and be analytic in some infinite strip $\eta_i < Re(s) < \lambda_i$ of the complex s-plane. Also suppose that for a fixed $\sigma_i \in (\eta_i, \lambda_i)$,* $h_i^*(x)$ *is defined by*

$$\prod_{i=1,\ldots,n} h_i^*(x) = \prod_{i=1,\ldots,n} [M^{-1}\{H_i^*(s);x\}]$$

$$= \prod_{i=1,\ldots,n} \frac{1}{2\pi\omega} \int_{\sigma_i-\omega\infty}^{\sigma_i+\omega\infty} x^{-s} H_i^*(s)\, ds, \qquad \ldots(9)$$

Where $\omega = \sqrt{-1}$,

$$\prod_{i=1,\ldots,n} H_i^*(s)$$

$$= \prod_{i=1,\dots,n} \left[B_i^{E_i} \frac{\Gamma\left(-\frac{s}{B_i}\right)}{\Gamma\left(-E_i-\frac{s}{B_i}\right)} \sum_{k_i=0}^{n_i} \sum_{\ell_i=0}^{p_i+\gamma_i n_i} \sum_{m_i=0}^{q_i+\delta_i n_i} (-1)^{k_i+\ell_i+m_i} \frac{(-n_i)_{k_i}}{k_i!} \frac{(-p_i-\gamma_i n_i)_{\ell_i}}{\ell_i!} \right.$$

$$\cdot \frac{(-q_i-\delta_i n_i)_{m_i}}{m_i!} (\phi)^{n_i-k_i} (b_i)^{p_i+\gamma_i n_i-\ell_i} (d_i)^{q_i+\delta_i n_i-m_i} (a_i)^{\ell_i} (c_i)^{m_i} \frac{(\theta_i)^{k_i}}{|\xi_i|}$$

$$\left. \cdot \frac{\Gamma\left(1+\frac{s+\theta_i n_i+B_i E_i+C_i}{\theta_i}\right)\Gamma\left(\frac{s+B_i E_i+C_i+\theta_i n_i+\alpha_i \ell_i+\beta_i m_i}{\xi_i}\right)}{\Gamma\left(\frac{s+\theta_i n_i+B_i E_i+C_i}{\theta_i}-k_i+1\right)(t_i)^{\left(\frac{s+B_i E_i+C_i+\theta_i n_i+\alpha_i \ell_i+\beta_i m_i}{\xi_i}\right)}} \right]^{-1}, \quad \dots(10)$$

provided that

$0 < \mathrm{Re}\,(s+B_i E_i+C_i+\theta_i n_i+\alpha_i \ell_i+\beta_i m_i) < \xi_i$, when $\xi_i > 0$;

$\xi_i < \mathrm{Re}(s+B_i E_i+C_i+\theta_i n_i+\alpha_i \ell_i+\beta_i m_i) < 0$, when $\xi_i < 0; \xi_i, B_i; \theta_i \neq 0$

$n_i, (p_i+\gamma_i n_i), (q_i+\delta_i n_i) \in N_0$, where $i = 1,\dots,n$.

Then the integral equation (1) has its solution given by

$$\prod_{i=1,\dots,n} f_i(x) = \prod_{i=1,\dots,n} \left[x^{-B_i E_i - C_i} \int_0^\infty y^{-1} h_i^*\left(\frac{x}{y}\right) (y^{B_i+1} D_y)^{E_i} \{y^{C_i} g_i(y)\} dy \right], \quad \dots(11)$$

provided that the integral exists.

Proof: By convolution Theorem for Mellin transforms [5, p.308, eq.(14)], equation (1) changes into

$$\prod_{i=1,\dots,n} H_i(s) F_i(s) = \prod_{i=1,\dots,n} G_i(s), \quad \dots(12)$$

Where $H_i(s)$, $F_i(s)$ and $G_i(s)$ are Mellin transforms of $h_i(x)$, $f_i(x)$ and $g_i(x)$ respectively.

Replacing s in (12) by $(s+B_iE_i+C_i)$, we have

$$\prod_{i=1,\ldots,n} F_i(s+B_iE_i+C_i)$$

$$= \prod_{i=1,\ldots,n}\left[H_i^*(s)\left\{B_i^{E_i}\left(-\left(\frac{s+B_iE_i}{B_i}\right)\right)_{E_i}\right\}G_i(s+B_iE_i+C_i)\right], \qquad \ldots(13)$$

Where $H_i^*(s)$ is given by (10).

Now by using known results [6, p.14, eq.(2.2)] and [5, p.307, eq. (7)], we obtain

$$\prod_{i=1,\ldots,n} F_i(s+B_iE_i+C_i) = \prod_{i=1,\ldots,n} H_i^*(s)[M\{(y^{B_i+1}D_y)^{E_i}\, y^{C_i} g_i(y)\}dy\,;s] \qquad \ldots(14)$$

and by using the known results [5, p.307, eq. (7) and [5, p.308, eq. (14)] in (14), we have

$$\prod_{i=1,\ldots,n} M[x^{B_iE_i+C_i}\, f_i(x);s]$$

$$= \prod_{i=1,\ldots,n}\left[M\left\{\int_0^\infty y^{-1}\, h_i^*\left(\frac{x}{y}\right)(y^{B_i+1}\, D_y)^{E_i}\,(y^{C_i} g_i(y))\,dy\,;s\right\}\right], \qquad \ldots(15)$$

Inverting both sides of (15) by using the Mellin inversion Theorem [5, p.307, eq. (1)], we arrive at the required solution (11).

4 Applications

4.1. By setting i = 1 to 3 in (2), we have the following corollary

Corollary: *The convolution integral equation*

$$\prod_{i=1,2,3} g_i(x) = \prod_{i=1,2,3}\int_0^\infty y^{-1} h_i\left(\frac{x}{y}\right) f_i(y)\,dy, \quad (x > 0) \qquad \ldots(16)$$

Where, the kernel

$$\prod_{i=1,2,3} h_i(x) = \prod_{i=1,2,3}\left[\frac{(a_i x^{\alpha_i}+b_i)^{p_i}(c_i x^{\beta_i}+d_i)^{q_i}}{e^{-t_i x^{\xi_i}}} K_{n_i}\right.$$

$$\left. .R_{n_i}^{p_i,q_i}[x,a_i,b_i,c_i,d_i;\alpha_i,\beta_i,\gamma_i,\delta_i;e^{-t_i x^{\xi_i}}]\right] \qquad \ldots(17)$$

$$=\prod_{i=1,2,3}\left[\{x^{\theta_i}(\phi+x D_x)\}^{n_i}\{(a_i x^{\alpha_i}+b_i)^{p_i+\gamma_i n_i}(c_i x^{\beta i}+d_i)^{q_i+\delta_i n_i}\; e^{-t_i x^{\xi_i}}\}\right]$$

has the solution

$$\prod_{i=1,2,3} f_i(x)$$

$$=\prod_{i=1,2,3}\left[x^{-B_iE_i-C_i}\int_0^\infty y^{-1}\, h_i^*\left(\frac{x}{y}\right)(y^{B_i+1}D_y)^{E_i}\{y^{c_i}\, g_i(y)\}dy\right], \qquad \ldots(18)$$

provided that the integral exists and $\prod_{i=1,2,3} h_i^*(x)$ is the Mellin inverse transform

$$\prod_{i=1,2,3} H_i^*(s)$$

$$=\prod_{i=1,2,3}\left[B_i^{E_i}\frac{\Gamma\left(-\frac{s}{B_i}\right)}{\Gamma\left(-E_i-\frac{s}{B_i}\right)}\sum_{k_i=0}^{n_i}\sum_{\ell_i=0}^{p_i+\gamma_i n_i}\sum_{m_i=0}^{q_i+\delta_i n_i}(-1)^{k_i+\ell_i+m_i}\frac{(-n_i)_{k_i}}{k_i!}\frac{(-p_i-\gamma_i n_i)_{\ell_i}}{\ell_i!}\right.$$

$$.\frac{(-q_i-\delta_i n_i)_{m_i}}{m_i!}(\phi)^{n_i-k_i}(b_i)^{p_i+\gamma_i n_i-\ell_i}(d_i)^{q_i+\delta_i n_i-m_i}(a_i)^{\ell_i}(c_i)^{m_i}\frac{(\theta_i)^{k_i}}{|\xi_i|}$$

$$\left..\frac{\Gamma\left(1+\frac{s+\theta_i n_i+B_iE_i+C_i}{\theta_i}\right)\Gamma\left(\frac{s+B_iE_i+C_i+\theta_i n_i+\alpha_i\ell_i+\beta_i m_i}{\xi_i}\right)}{\Gamma\left(\frac{s+\theta_i n_i+B_iE_i+C_i}{\theta_i}-k_i+1\right)t_i^{\frac{s+B_iE_i+C_i+\theta_i n_i+\alpha_i\ell_i+\beta_i m_i}{\xi_i}}}\right]^{-1},$$

$$\ldots(19)$$

provided that

$$0 < \mathrm{Re}\,(s + B_i E_i + C_i + \theta_i n_i + \alpha_i \ell_i + \beta_i m_i) < \xi_i, \text{ when } \xi_i > 0;$$

$$\xi_i < \mathrm{Re}(s + B_i E_i + C_i + \theta_i n_i + \alpha_i \ell_i + \beta_i m_i) < 0, \text{ when } \xi_i < 0; \xi_i, B_i; \theta_i \neq 0$$

$$n_i, (p_i + \gamma_i n_i), (q_i + \delta_i n_i) \in N_0, \text{ where } i = 1,2,3.$$

4.2. By taking i = 1,2 in (2), the main theorem reduces to a known result recently obtained by Chaurasia and Agnihotri in [4].

4.3. Since the polynomial set $R_n^{p,q}(x)$ incorporate in itself several classical as well as other polynomials, solutions of a large number of convolution integral equations for the above mentioned polynomials may be obtained by assigning different values to the parameters in $R_n^{p,q}(x)$.

By making suitable substitution, we get the known results obtained by Srivastava [9] and Chaurasia and Patni [3].

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