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New Generalizations of Lucas Numbers

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Abstract

In this paper, we present new generalizations of the Lucas numbers by matrix representation, using Generalized Lucas Polynomials. These new generalizations include more powerful relationships with generalizations of Fibonacci numbers. We give some properties of these new generalizations and obtain some relations between the generalized order-k Lucas numbers and the generalized order-k Fibonacci numbers. In addition, we obtain Binet formula and combinatorial representation for generalized order-k Lucas numbers by using properties of generalized Fibonacci numbers.

Keywords: *Fibonacci numbers, Lucas numbers, k sequences of the generalized order-k Fibonacci numbers, k sequences of the generalized order-k Lucas numbers.*

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1 Background and Notation

The well-known Fibonacci sequence $\{f_n\}$ is defined recursively by the equation

$$f_n = f_{n-1} + f_{n-2}, \text{ for } n \geq 2$$

where $f_0 = 0$, $f_1 = 1$ and Lucas sequence $\{l_n\}$ is defined recursively by the equation

$$l_n = l_{n-1} + l_{n-2}, \text{ for } n \geq 2$$

where $l_0 = 2$, $l_1 = 1$.

There are various types of generalizations of Fibonacci and Lucas numbers. Falcon and Plaza [3] defined Fibonacci k-numbers $\{F_{k,n}\}$, for $k \geq 1$, $F_{k,0} = 0$, $F_{k,1} = 1$ and $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$. It is easy to see that for $k = 1$ Fibonacci k-sequence is reduced to the usual Fibonacci sequence and for $k = 2$, it is reduced to the usual Pell sequence. In [14], authors defined a generalized Fibonacci sequence as $F_{n+1} = pF_n + qF_{n-1}$, where p and q are natural numbers, which serve as the control parameters. Akbulak and Bozkurt [1] defined order-m generalized Fibonacci k-numbers and obtained sums, some identities and the generalized Binet formula of these numbers. In [15], authors studied on Fibonacci and Lucas p-numbers and their Binet formulas. In [17], authors defined bivariate Fibonacci and Lucas p-polynomials and studied on these polynomials.

Miles [13] defined generalized order-k Fibonacci numbers (GOkF) as,

$$f_{k,n} = \sum_{j=1}^{k} f_{k,n-j} \tag{1}$$

for $n > k \geq 2$, with boundary conditions: $f_{k,1} = f_{k,2} = f_{k,3} = \cdots = f_{k,k-2} = 0$, $f_{k,k-1} = f_{k,k} = 1$.
Er [2] defined k sequences of the generalized order-k Fibonacci numbers (kSOkF) as; for $n > 0$, $1 \leq i \leq k$

$$f_{k,n}^{\,i} = \sum_{j=1}^{k} c_j f_{k,n-j}^{\,i} \tag{2}$$

with boundary conditions for $1 - k \leq n \leq 0$,

$$f_{k,n}^{\,i} = \begin{cases} 1 & \text{if } \; i = 1 - n, \\ 0 & \text{otherwise,} \end{cases}$$

where c_j $(1 \leq j \leq k)$ are constant coefficients, $f_{k,n}^{\,i}$ is the n-th term of i-th sequence of order k generalization. k-th column of this generalization involves the Miles generalization for $i = k$, i.e. $f_{k,n}^{k} = f_{k,k+n-2}$. Er [2] showed

$$F_{n+1}^{\sim} = AF_n^{\sim}$$

where

$$A = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{k-1} & c_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

is a $k \times k$ companion matrix and

$$F_n^{\sim} = \begin{bmatrix} f_{k,n}^1 & f_{k,n}^2 & \cdots & f_{k,n}^k \\ f_{k,n-1}^1 & f_{k,n-1}^2 & \cdots & f_{k,n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ f_{k,n-k+1}^1 & f_{k,n-k+1}^2 & \cdots & f_{k,n-k+1}^k \end{bmatrix} \tag{3}$$

is a $k \times k$ matrix. Karaduman [5] showed $F_1^{\sim} = A$ and $F_n^{\sim} = A^n$ for $c_j = 1$, $(1 \leq j \leq k)$. Kalman [4] derived the Binet formula by using Vandermonde matrix as

$$f_{k,n}^k = \sum_{i=1}^{k} \frac{(\lambda_i)^n}{P'(\lambda_i)} \tag{4}$$

where λ_i $(1 \leq i \leq k)$ are roots of the polynomial

$$P(x; t_1, t_2, \ldots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k, \tag{5}$$

$t_{1,\ldots,}t_k$ are constants, $f_{k,n}^k$ is (for $c_j = 1$, $1 \leq j \leq k$ and $i = k$) k-th sequences of kSOkF and $P'(x)$ is the derivative of the polynomial (5).

Kılıç and Taşcı [6] studied on $F_n^{\sim}$ and $f_{k,n}^k$ and gave some formulas and properties concerning kSOkF. One of these is Binet formula for kSOkF. That is,

$$f_{k,n}^k = \frac{\det(V_k^{(1)})}{\det(V)}. \tag{6}$$

In addition, Kılıç and Taşcı [7] defined the generalized order-k Pell numbers (GOkP) and Taşcı and Kılıç [16] defined the generalized order-k Lucas numbers. In [8], authors studied on generalized Pell $(p, i)-$numbers. In [9], authors studied on the m-extension of the Fibonacci and Lucas p-numbers.

MacHenry [10] defined generalized Fibonacci polynomials $(F_{k,n}(t))$, Lucas polynomials $(G_{k,n}(t))$ and obtained important relations between them. $F_{k,n}(t)$ is defined inductively by

$$\begin{aligned} F_{k,n}(t) &= 0, \ n < 0 \\ F_{k,0}(t) &= 1 \\ F_{k,1}(t) &= t_1 \\ F_{k,n+1}(t) &= t_1 F_{k,n}(t) + \cdots + t_k F_{k,n-k+1}(t) \end{aligned}$$

where $t = (t_1, t_2, \ldots, t_k)$.

$G_{k,n}(t)$ is defined inductively by

$$G_{k,n}(t) = 0, \ n < 0$$

$$\begin{aligned} G_{k,0}(t) &= k \\ G_{k,1}(t) &= t_1 \\ G_{k,n}(t) &= G_{k-1,n}(t),\ 1 \le n \le k \\ G_{k,n+1}(t) &= t_1 G_{k,n}(t) + \cdots + t_k G_{k,n-k+1}(t),\ n > k. \end{aligned}$$

In addition, in [12], authors obtained $F_{k,n}(t)$ and $G_{k,n}(t)$ as

$$F_{k,n}(t) = \sum_{a \vdash n} \begin{pmatrix} |a| \\ a_{1,\ldots,}a_k \end{pmatrix} t_1^{a_1} \ldots t_k^{a_k} \tag{7}$$

and

$$G_{k,n}(t) = \sum_{a \vdash n} \frac{n}{|a|} \begin{pmatrix} |a| \\ a_{1,\ldots,}a_k \end{pmatrix} t_1^{a_1} \ldots t_k^{a_k} \tag{8}$$

where a_i are nonnegative integers for all i $(1 \le i \le k)$, with initial conditions given by

$$F_{k,0}(t) = 1,\ F_{k,-1}(t) = 0,\ \ \ldots,\ F_{k,-k+1}(t) = 0$$

and

$$G_{k,0}(t) = k,\ G_{k,-1}(t) = 0,\ \ \cdots,\ G_{k,-k+1}(t) = 0.$$

Throughout this paper, the notations $a \vdash n$ and $|a|$ are used instead of $\sum\limits_{j=1}^{k} j a_j = n$ and $\sum\limits_{j=1}^{k} a_j$, respectively. A combinatorial representation for Fibonacci polynomials is given in [12] as

$$F_{2,n}(t) = \sum_{j=0}^{\left\lceil \frac{n}{2} \right\rceil} (-1)^j \begin{pmatrix} n-j \\ j \end{pmatrix} F_1^{n-2j} (-t_2)^j \tag{9}$$

for n is an integer, where $\left\lceil \frac{n}{2} \right\rceil = k$, either $n = 2k$ or $n = 2k-1$.

In [11], matrices $A_{(k)}^{\infty}$ and $D_{(k)}^{\infty}$ are defined by using the following matrix,

$$A_{(k)} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ t_k & t_{k-1} & t_{k-2} & \ldots & t_1 \end{bmatrix}.$$

They also record the orbit of the k-th row vector of $A_{(k)}$ under the action of $A_{(k)}$, below $A_{(k)}$, and the orbit of the first row of $A_{(k)}$ under the action of $A_{(k)}^{-1}$ on the first row of $A_{(k)}$ is recorded above $A_{(k)}$, and consider the $\infty \times k$

matrix whose row vectors are the elements of the doubly infinite orbit of $A_{(k)}$ acting on any one of them. For $k=3$, $A^{\infty}_{(k)}$ looks like this

$$A^{\infty}_{(3)} = \begin{bmatrix} \cdots & \cdots & \cdots \\ S_{(-n,1^2)} & -S_{(-n,1)} & S_{(-n)} \\ \cdots & \cdots & \cdots \\ S_{(-3,1^2)} & -S_{(-3,1)} & S_{(-3)} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ t_3 & t_2 & t_1 \\ \cdots & \cdots & \cdots \\ S_{(n-1,1^2)} & -S_{(n-1,1)} & S_{(n-1)} \\ S_{(n,1^2)} & -S_{(n,1)} & S_{(n)} \\ \cdots & \cdots & \cdots \end{bmatrix}$$

and

$$A^{n}_{(k)} = \begin{bmatrix} (-1)^{k-1}S_{(n-k+1,1^{k-1})} & \cdots & (-1)^{k-j}S_{(n-k+1,1^{k-j})} & \cdots & S_{(n-k+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{k-1}S_{(n,1^{k-1})} & \cdots & (-1)^{k-j}S_{(n,1^{k-j})} & \cdots & S_{(n)} \end{bmatrix}$$

where

$$S_{(n-r,1^r)} = (-1)^r \sum_{j=r+1}^{n} t_j S_{(n-j)},\ 0 \le r \le n. \tag{10}$$

Derivative of the core polynomial $P(x;t_1,t_2,\ldots,t_k) = x^k - t_1x^{k-1} - \cdots - t_k$ is $P'(x) = kx^{k-1} - t_1(k-1)x^{k-2} - \cdots - t_{k-1}$, which is represented by the vector $(-t_{k-1},\ldots,-t_1(k-1),k)$, and the orbit of this vector under the action of $A_{(k)}$ gives the standard matrix representation $D^{\infty}_{(k)}$. Right hand column of $A^{\infty}_{(k)}$ contains sequence of the generalized Fibonacci polynomials $F_{k,n}(t)$ and $tr(A^{n}_{(k)}) = G_{k,n}(t)$, where $G_{k,n}(t)$ is the generalized Lucas polynomials, which is also a t-linear recursion. In addition, the right hand column of $D^{\infty}_{(k)}$ contains the generalized Lucas polynomials $G_{k,n}(t)$.

We obtain $(-1)^r S_{(n,1^r)} = f^{\,r+1}_{k,n-1}$, from $A^{\infty}_{(k)}$, for $n \ge 0$, $c_i = t_i$ $(1 \le i \le k)$ and $0 \le r \le k-1$. Moreover, $A^{\infty}_{(k)}$ is reduced to GOkP when $t_1 = 2$ and $t_i = 1$ (for $2 \le i \le k$).

Generalized Fibonacci polynomials are reduced, by suitable substitutions, to Fibonacci k-numbers $\{F_{k,n}\}$, generalized Fibonacci sequence, Fibonacci p-numbers, generalized Pell $(p,i)-$numbers and bivariate Fibonacci p-polynomials etc.

Likewise, Generalized Lucas polynomials are reduced, by suitable substitutions, to Lucas p numbers, m-extension of the Lucas p-numbers, bivariate Lucas p-polynomials, Lucas p-polynomials, Lucas polynomials, Lucas p-numbers, Lucas numbers, bivariate Pell-Lucas p-polynomials, bivariate Pell-Lucas polynomials, Pell-Lucas p-polynomials, Pell-Lucas polynomials, First kind Chebyshev polynomials, Jacobsthal-Lucas polynomials and Pell-Lucas numbers etc. It is known that, numbers and polynomials listed above have applications, especially in theoretical physics for modeling processes. This analogy shows the importance of the matrix $A^{\infty}_{(k)}$ and Generalized Fibonacci and Lucas polynomials. However, Lucas generalization in [16] is not compatible with the matrix $A^{\infty}_{(k)}$ and Generalized Lucas polynomials. For that reason, we studied on generalized order-k Lucas numbers $l_{k,n}$ and k sequences of the generalized order-k Lucas numbers $l^i_{k,n}$ with the help of generalized Lucas polynomials $G_{k,n}(t)$ and the matrix $D^{\infty}_{(k)}$.

In this paper, after presenting a matrix representation of $l^i_{k,n}$, we derived a relation between GOkF and generalized order-k Lucas numbers, as well as a relation between k sequences of the generalized order-k Lucas numbers and kSOkF. Since many properties, applications of Fibonacci numbers and those of its generalizations are known, these relations are very important. Using these relations, properties and applications of Fibonacci numbers and its generalizations can be transferred to Lucas numbers and its generalizations. In addition to obtaining these relations, we give a generalized Binet formula and combinatorial representation for k sequences of the generalized order-k Lucas numbers with the help of properties of generalized Fibonacci numbers.

2 New Generalizations of Lucas Numbers

This section contains new generalizations of Lucas numbers with the help of Lucas polynomials.

Definition 2.1 *For $t_s = 1$, $1 \leq s \leq k$, the Lucas polynomials $G_{k,n}(t)$ and $D^{\infty}_{(k)}$ together are reduced to*

$$l_{k,n} = \sum_{j=1}^{k} l_{k,n-j} \tag{11}$$

with boundary conditions

$$l_{k,1-k} = l_{k,2-k} = \ldots = l_{k,-1} = -1 \text{ and } l_{k,0} = k,$$

which is called generalized order-k Lucas numbers (GOkL). When $k = 2$, it is reduced to ordinary Lucas numbers.

Definition 2.2 *Positive direction of $D^{\infty}_{(k)}$ for $t_s = 1$, $1 \leq s \leq k$, is*

$$l^i_{k,n} = \sum_{j=1}^{k} l^i_{k,n-j} \tag{12}$$

for $n > 0$ *and* $1 \le i \le k$, *with boundary conditions*

$$l_{k,n}^{i} = \begin{cases} -i & \text{if } i-n < k, \\ -2n+i & \text{if } i-n = k, \\ k-i-1 & \text{if } i-n > k \end{cases}$$

for $1-k \le n \le 0$, *where* $l_{k,n}^{i}$ *is the* n*-th term of* i*-th sequence. This generalization is called* k *sequences of the generalized order-*k *Lucas numbers (kSOkL).*

Although definitions look similar, the initial conditions of this generalization are different from the generalization in [16]. These initial conditions arise from Lucas polynomials and $D_{(k)}^{\infty}$.

When $i = k = 2$, we obtain ordinary Lucas numbers and for $i = k$, we obtain $l_{k,n}^{k} = l_{k,n}$ from the definition.

Example 2.3 *Substituting* $k = 3$ *and* $i = 2$ *we obtain the generalized order-3 Lucas sequence as;*

$$l_{3,-2}^{2} = 0,\ l_{3,-1}^{2} = 4,\ l_{3,0}^{2} = -2,\ l_{3,1}^{2} = 2,\ l_{3,2}^{2} = 4,\ l_{3,3}^{2} = 4,\ \ldots$$

Lemma 2.4 *Matrix multiplication and (12) can be used to obtain*

$$L_{n+1}^{\sim} = A_1 L_n^{\sim}$$

where

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}_{k \times k} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ & & & 0 \\ & I & & \vdots \\ & & & 0 \end{bmatrix}_{k \times k} \tag{13}$$

where I *is a* $(k-1) \times (k-1)$ *identity matrix, and the matrix* $L_n^{\sim}$ *is;*

$$L_n^{\sim} = \begin{bmatrix} l_{k,n}^{1} & l_{k,n}^{2} & \ldots & l_{k,n}^{k} \\ l_{k,n-1}^{1} & l_{k,n-1}^{2} & \ldots & l_{k,n-1}^{k} \\ \vdots & \vdots & \ddots & \vdots \\ l_{k,n-k+1}^{1} & l_{k,n-k+1}^{2} & \ldots & l_{k,n-k+1}^{k} \end{bmatrix} \tag{14}$$

which is contained by $k \times k$ *block of* $D_{(k)}^{\infty}$ *for* $t_i = 1$, $1 \le i \le k$.

Lemma 2.5 *Let* A_1 *and* $L_n^{\sim}$ *be as in (13) and (14), respectively. Then,*

$$L_{n+1}^{\sim} = A_1^{n+1} L_0^{\sim}$$

where

$$L_0^{\sim} = \begin{bmatrix} -1 & -2 & -3 & \dots & -(k-2) & -(k-1) & k \\ -1 & -2 & -3 & \dots & -(k-2) & k+1 & -1 \\ \vdots & \vdots & \vdots & \dots & k+2 & 0 & -1 \\ -1 & -2 & 2k-3 & \dots & 1 & 0 & -1 \\ -1 & 2k-2 & k-4 & \dots & \vdots & \vdots & \vdots \\ 2k-1 & k-3 & k-4 & \dots & 1 & 0 & -1 \end{bmatrix}_{k\times k}.$$

Proof 2.6 *It is clear that $L_1^{\sim} = A_1 L_0^{\sim}$ and $L_{n+1}^{\sim} = A_1 L_n^{\sim}$. So by induction and properties of matrix multiplication, we have the result.*

Lemma 2.7 *Let $F_n^{\sim}$ and $L_n^{\sim}$ be as in (3) and (14), respectively. Then*

$$L_n^{\sim} = F_n^{\sim} L_0^{\sim}.$$

Proof 2.8 *Proof is trivial from $F_n^{\sim} = A_1^n$(see [5]) and Lemma 2.5.*

Example 2.9 *From Lemma 2.7 for $k = 2$, we have*

$$\begin{bmatrix} l_{2,n}^1 & l_{2,n}^2 \\ l_{2,n-1}^1 & l_{2,n-1}^2 \end{bmatrix} = \begin{bmatrix} f_{2,n}^1 & f_{2,n}^2 \\ f_{2,n-1}^1 & f_{2,n-1}^2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}.$$

Therefore, $l_{2,n}^2 = 2f_{2,n}^1 - f_{2,n}^2$. Since $f_{2,n}^1 = f_{2,n+1}^2$ for all positive integers n then we have

$$l_{2,n}^2 = 2f_{2,n+1}^2 - f_{2,n}^2$$

where $l_{2,n}^2$ and $f_{2,n}^2$ are ordinary Lucas and Fibonacci numbers, respectively. For $k = 3$, we have

$$\begin{bmatrix} l_{3,n}^1 & l_{3,n}^2 & l_{3,n}^3 \\ l_{3,n-1}^1 & l_{3,n-1}^2 & l_{3,n-1}^3 \\ l_{3,n-2}^1 & l_{3,n-2}^2 & l_{3,n-2}^3 \end{bmatrix} = \begin{bmatrix} f_{3,n}^1 & f_{3,n}^2 & f_{3,n}^3 \\ f_{3,n-1}^1 & f_{3,n-1}^2 & f_{3,n-1}^3 \\ f_{3,n-2}^1 & f_{3,n-2}^2 & f_{3,n-2}^3 \end{bmatrix} \begin{bmatrix} -1 & -2 & 3 \\ -1 & 4 & -1 \\ 5 & 0 & -1 \end{bmatrix}.$$

Therefore, $l_{3,n}^3 = 3f_{3,n}^1 - f_{3,n}^2 - f_{3,n}^3$. Since for $k = 3$, $f_{3,n}^1 = f_{3,n+1}^3$ and $f_{3,n}^2 = f_{3,n-1}^1 + f_{3,n-1}^3 = f_{3,n}^3 + f_{3,n-1}^3$ for all positive integers n we have

$$l_{3,n}^3 = 3f_{3,n+1}^3 - 2f_{3,n}^3 - f_{3,n-1}^3.$$

Theorem 2.10 *For $i = k$, $n \geq 0$ and $c_1 = \cdots = c_k = 1$,*

$$l_{k,n}^k = kf_{k,n+1}^k - \sum_{j=2}^{k}(k-j+1)f_{k,n+2-j}^k. \tag{15}$$

Proof 2.11 *We use mathematical induction to prove the equality*

$$l_{k,n}^k = kf_{k,n+1}^k - \sum_{j=2}^{k}(k-j+1)f_{k,n+2-j}^k.$$

First, we have $l_{k,0}^k = k$, $f_{k,0}^k = 0$ *and* $f_{k,1}^k = 1$ *for all positive integers* $k \geq 2$, *from the definition of kSOkL and kSOkF. So, the equation (15) is true for* $n = 0$, *i.e.,*

$$l_{k,0}^k = kf_{k,1}^k - (k-1)f_{k,0}^k - \cdots - f_{k,-k+2}^k = k.1 + 0 = k.$$

Now, suppose that the equation holds for all positive integers less than or equal to n *i.e., for all integer* n

$$l_{k,n}^k = kf_{k,n+1}^k - \sum_{j=2}^{k}(k-j+1)f_{k,n+2-j}^k$$

Then, from (2) and (12), for $c_1 = \cdots = c_k = 1$, *we get;*

$$\begin{aligned}
l_{k,n+1}^k &= l_{k,n}^k + l_{k,n-1}^k + l_{k,n-2}^k + \cdots + l_{k,n-k+1}^k \\
&= (kf_{k,n+1}^k - (k-1)f_{k,n}^k - \cdots - f_{k,n-k+2}^k) + \\
&\quad (kf_{k,n}^k - (k-1)f_{k,n-1}^k - \cdots - f_{k,n-k+1}^k) + \\
&\quad \ldots + (kf_{k,n-k+2}^k - (k-1)f_{k,n-k+1}^k - \cdots - f_{k,n-2k+3}^k) \\
&= kf_{k,n+2}^k - (k-1)f_{k,n+1}^k - \cdots - f_{k,n-k+3}^k \\
&= kf_{k,n+2}^k - \sum_{j=2}^{k}(k-j+1)f_{k,n+3-j}^k.
\end{aligned}$$

Hence, the equation holds for $(n+1)$ *and proof is complete.*

Since $f_{k,n}^k = f_{k,n+k-2}$ and $l_{k,n}^k = l_{k,n}$, the following relation is obvious

$$l_{k,n} = kf_{k,n+k-1} - \sum_{j=2}^{k}(k-j+1)f_{k,n+k-j}.$$

The following theorem shows that, the equality (15) is valid for Generalized Fibonacci and Lucas polynomials as well.

Theorem 2.12 *For* $k \geq 2$ *and* $n \geq 0$,

$$G_{k,n}(t) = kF_{k,n}(t) - \sum_{j=2}^{k}(k-j+1)t_{j-1}F_{k,n+1-j}(t).$$

Proof 2.13 *Proof is by induction as Theorem 2.10.*

Theorem 2.14 *For $i = k$ and $n \geq 0$,*

$$l_{k,n}^{k} = \sum_{j=1}^{k} j f_{k,n+1-j}^{k}. \tag{16}$$

Proof 2.15 *Proof is by induction as Theorem 2.10.*

Lemma 2.16 *For $k \geq 2$, i-th sequences of kSOkL, in terms of the k-th sequences of kSOkL, is*

$$l_{k,n}^{i} = \begin{cases} l_{k,n-1}^{k} & \text{if } i = 1 \\ \sum\limits_{m=1}^{i} l_{k,n-m}^{k} & \text{if } 1 < i < k \\ l_{k,n}^{k} & \text{if } i = k \end{cases}. \tag{17}$$

Theorem 2.17 *i-th sequences of kSOkL can be written in terms of the k-th sequences of kSOkF (which is equal to GOkF with index iteration) in various ways;*
i)For $k \geq 3$ and $1 \leq i \leq k$

$$l_{k,n}^{i} = \sum_{j=1}^{k+i-1} d_j f_{k,n-j}^{k}$$

where $1 \leq i \leq k$, $n \geq 0$ and constant coefficients

$$d_j = \begin{cases} \frac{j(j+1)}{2} & \text{if } 1 \leq j \leq i \\ \frac{j(j+1)}{2} - \frac{(j-i)(j-i+1)}{2} & \text{if } i+1 \leq j \leq k-1 \\ \frac{k(k+1)}{2} - \frac{(j-i)(j-i+1)}{2} & \text{if } k \leq j \leq k+i-1 \end{cases}$$

ii)For $k \geq 2$ and $1 \leq i \leq k$

$$l_{k,n}^{i} = \begin{cases} k f_{k,n}^{k} - \sum\limits_{j=2}^{k} (k-j+1) f_{k,n+1-j}^{k} & \text{if } i = 1 \\ \sum\limits_{m=1}^{i} k f_{k,n-m+1}^{k} - \sum\limits_{m=1}^{i} \sum\limits_{j=2}^{k} (k-j+1) f_{k,n-m-j+2}^{k} & \text{if } 1 < i < k \\ k f_{k,n+1}^{k} - \sum\limits_{j=2}^{k} (k-j+1) f_{k,n+2-j}^{k} & \text{if } i = k \end{cases}$$

iii)For $k \geq 2$ and $1 \leq i \leq k$

$$l_{k,n}^{i} = \begin{cases} \sum\limits_{j=1}^{k} j f_{k,n-j}^{k} & \text{if } i = 1 \\ \sum\limits_{m=1}^{i} \sum\limits_{j=1}^{k} j f_{k,n-m-j+1}^{k} & \text{if } 1 < i < k \\ \sum\limits_{j=1}^{k} j f_{k,n+1-j}^{k} & \text{if } i = k \end{cases}.$$

Proof 2.18 *i)Proof is from Theorem 2.14 and Lemma 2.16.*
ii)Proof is from Theorem 2.10 and Lemma 2.16.
iii)Proof is from Theorem 2.14 and Lemma 2.16.

Example 2.19 *Let us obtain $l^i_{k,n}$ for $k=4$, $n=4$ and $i=3$ by using Theorem 2.17-iii.*

$$\begin{aligned} l^3_{4,4} &= \sum_{m=1}^{3}\sum_{j=1}^{4} j.f^4_{4,4-m-j+1} = \sum_{m=1}^{3}(f^4_{4,4-m}+2f^4_{4,3-m}+3f^4_{4,2-m}+4f^4_{4,1-m}) \\ &= f^4_{4,3}+2f^4_{4,2}+3f^4_{4,1}+4f^4_{4,0}+f^4_{4,2}+2f^4_{4,1}+f^4_{4,1}=11 \end{aligned}$$

since $f^4_{4,0}=0$, $f^4_{4,1}=f^4_{4,2}=1$ and $f^4_{4,3}=2$.

Theorem 2.20 *For integers m,n and $1\leq i\leq k-1$, we have*

$$l^i_{n+m}=\sum_{j=1}^{i}(l^k_{m-j}\sum_{s=1}^{j}f^s_n)+\sum_{j=i+1}^{k}(l^k_{m-j}\sum_{s=j-i+1}^{j}f^s_n)+\sum_{j=k+1}^{k+i-1}(l^k_{m-j}\sum_{s=j-i+1}^{k}f^s_n).$$

Where, we assume that, the sum is equal to zero, if the subscript is greater than the superscript in the sum.

Proof 2.21 *We know from Lemma 2.5 that $L^\sim_n=F^\sim_n L^\sim_0$. So,*

$$L^\sim_{n+m}=F^\sim_{n+m}L^\sim_0=A_1^{n+m}L^\sim_0=A_1^nA_1^mL^\sim_0=A_1^nL^\sim_m=F^\sim_nL^\sim_m.$$

From this matrix product and Lemma 2.16 we obtain

$$\begin{aligned} l^i_{k,n+m} &= f^1_{k,n}l^i_{k,m}+\cdots+f^k_{k,n}l^i_{k,m-k+1} \\ &= f^1_{k,n}(l^k_{k,m-1}+\cdots+l^k_{k,m-i})+\cdots+f^k_{k,n}(l^k_{k,m-k}+\cdots+l^k_{k,m-k-i-1}) \\ &= l^k_{k,m-1}f^1_n+l^k_{k,m-2}(f^1_{k,n}+f^2_{k,n})+\cdots l^k_{k,m-i}(f^1_{k,n}+f^2_{k,n}+\cdots+f_{k,n}^{\,i})+ \\ &\quad l^k_{k,m-i-1}(f^2_{k,n}+f^3_{k,n}+\cdots+f_{k,n}^{\,i+1})+\cdots+l^k_{k,m-k}(f^{k-i+1}_{k,n}+\cdots+f^k_{k,n})+ \\ &\quad l^k_{k,m-k-1}(f^{k-i+2}_{k,n}+\cdots+f^k_{k,n})+\cdots+l^k_{k,m-k-i-1}f^k_{k,n} \\ &= \sum_{j=1}^{i}(l^k_{k,m-j}\sum_{t=1}^{j}f^t_{k,n})+\sum_{j=i+1}^{k}(l^k_{k,m-j}\sum_{t=j-i+1}^{j}f^t_{k,n})+\sum_{j=k+1}^{k+i-1}(l^k_{k,m-j}\sum_{t=j-i+1}^{k}f^t_{k,n}). \end{aligned}$$

Example 2.22 *Let us obtain $l^i_{k,n+m}$ for $k=5$, $i=3$, $n=3$ and $m=4$, by using Theorem 2.20;*

$$\begin{aligned} l^3_{5,3+4} &= l^3_7=\sum_{j=1}^{3}(l^5_{5,4-j}\sum_{t=1}^{j}f^t_{5,3})+\sum_{j=4}^{5}(l^5_{5,4-j}\sum_{t=j-2}^{j}f^t_{5,3})+\sum_{j=6}^{7}(l^5_{5,4-j}\sum_{t=j-2}^{5}f^t_{5,3}) \\ &= l^5_{5,3}f^1_{5,3}+l^5_{5,2}(f^1_{5,3}+f^2_{5,3})+l^5_{5,1}(f^1_{5,3}+f^2_{5,3}+f^3_{5,3})+l^5_{5,0}(f^2_{5,3}+f^3_{5,3}+f^4_{5,3}) \\ &\quad +l^5_{5,-1}(f^3_{5,3}+f^4_{5,3}+f^5_{5,3})+l^5_{5,-2}(f^4_{5,3}+f^5_{5,3})+l^5_{5,-3}f^5_{5,3} \\ &= 28+24+12+55-9-5-2=103. \end{aligned}$$

2.1 Binet Formula

In this subsection, we give two different Binet formula to find any term of kSOkL. We have the following corollary by (4) and Theorem 2.17-*iii*.

Corollary 2.23 *For all positive integers m, n and $1 \leq i \leq k$ we obtain,*

$$l^i_{k,n} = \begin{cases} \sum\limits_{j=1}^{k} j \sum\limits_{i=1}^{k} \frac{(\lambda_i)^{n-j}}{P'(\lambda_i)} & \text{for } i = 1 \\ \sum\limits_{m=1}^{i} \sum\limits_{j=1}^{k} j \sum\limits_{i=1}^{k} \frac{(\lambda_i)^{n-m-j+1}}{P'(\lambda_i)} & \text{for } 1 < i < k \\ \sum\limits_{j=1}^{k} j \sum\limits_{i=1}^{k} \frac{(\lambda_i)^{n-j+1}}{P'(\lambda_i)} & \text{for } i = k \end{cases}.$$

We have the following Corollary by (6) and Theorem 2.17-*iii*.

Corollary 2.24 *For all positive integers m, n and $1 \leq i \leq k$ we obtain,*

$$l^i_{k,n} = \begin{cases} \sum\limits_{j=1}^{k} j \frac{det(V^{(1)}_{k,n-j})}{det(V)} & \text{for } i = 1 \\ \sum\limits_{m=1}^{i} \sum\limits_{j=1}^{k} j \frac{det(V^{(1)}_{k,n-m-j+1})}{det(V)} & \text{for } 1 < i < k \\ \sum\limits_{j=1}^{k} j \frac{det(V^{(1)}_{k,n-j+1})}{det(V)} & \text{for } i = k \end{cases}.$$

2.2 Combinatorial Representation of the Generalized Order-k Fibonacci and Lucas Numbers

In this subsection, we obtain some combinatorial representations of i-th sequences of kSOkF and kSOkL with the help of combinatorial representations of Generalized Fibonacci and Lucas polynomials.
i-th sequences of kSOkF can be stated in terms of k-th sequences of kSOkF as follows. For $c_i = 1$ $(1 < i < k)$,

$$f^{\,i}_{k,n} = \sum_{m=1}^{k-i+1} f^k_{k,n-m+1}.$$

Through this equality, studies on k-th sequences are portable to i-th$(1 < i < k)$ sequence. For $t_i = c_i$ $(1 < i < k)$, $F_{k,n-1}(t)$ is reduced to sequence $f^k_{k,n}$. So for $t_i = c_i$ $(1 < i < k)$, $f^{\,i}_{k,n} = \sum\limits_{m=1}^{k-i+1} t_{(i+m-1)} F_{k,n-m+1}(t)$, and using (7) we have

$$f^{\,i}_{k,n} = \sum_{m=1}^{k-i+1} t_{(i+m-1)} \sum_{a \vdash (n-m)} \begin{pmatrix} |a| \\ a_{1,\ldots,}a_k \end{pmatrix}. \tag{18}$$

We remind once more that, Generalized Fibonacci polynomials, by suitable substitutions, are reduced to Fibonacci k-numbers $\{F_{k,n}\}$, generalized Fibonacci sequence, order-m generalized Fibonacci k-numbers, m-extension of the Fibonacci p-numbers, Fibonacci p-numbers, generalized Pell $(p,i)-$numbers and bivariate Fibonacci p polynomials, generalized Order-k Pell Numbers etc. Hence, (18) is applicable for any sequences and polynomials mentioned above, and other Fibonacci like sequences and polynomials.

Then, we have the following corollary using Theorem2.17. *iii*.

Corollary 2.25 *For all positive integers m,n and $1 \leq i \leq k$ we obtain,*

$$l_{k,n}^{i} = \begin{cases} \sum\limits_{a \vdash (n-1)} \frac{n-1}{|a|} \begin{pmatrix} |a| \\ a_{1,\ldots,}a_k \end{pmatrix} & \text{if } i = 1 \\ \sum\limits_{m=1}^{i} \sum\limits_{a \vdash (n-m)} \frac{n-m}{|a|} \begin{pmatrix} |a| \\ a_{1,\ldots,}a_k \end{pmatrix} & \text{if } 1 < i < k \\ \sum\limits_{a \vdash n} \frac{n}{|a|} \begin{pmatrix} |a| \\ a_{1,\ldots,}a_k \end{pmatrix} & \text{if } i = k \end{cases} .$$

Proof 2.26 *For $t_i = 1 (1 \leq i \leq k)$, $G_{k,n}$ is reduced to $l_{k,n}^{k}$. So, by using (8)and (17) the proof is completed.*

Corollary 2.27 *For all positive integers m,n and $1 \leq i \leq k$ we obtain,*

$$l_{k,n}^{i} = \begin{cases} \sum\limits_{j=1}^{k} j \sum\limits_{a \vdash (n-1-j)} \begin{pmatrix} |a| \\ a_{1,\ldots,}a_k \end{pmatrix} & \text{if } i = 1 \\ \sum\limits_{m=1}^{i} \sum\limits_{j=1}^{k} j \sum\limits_{a \vdash (n-m-j)} \begin{pmatrix} |a| \\ a_{1,\ldots,}a_k \end{pmatrix} & \text{if } 1 < i < k \\ \sum\limits_{j=1}^{k} j \sum\limits_{a \vdash (n-j)} \begin{pmatrix} |a| \\ a_{1,\ldots,}a_k \end{pmatrix} & \text{if } i = k \end{cases} .$$

Proof 2.28 *Proof is trivial from (7) and (17).*

Corollary 2.29 *Let $l_{2,n}^{2}$ be the second sequence of the $2SO2L$. Then,*

$$l_{2,n}^{2} = \sum_{j=1}^{2} j \sum_{s=0}^{\left\lceil \frac{n-j}{2} \right\rceil} \begin{pmatrix} n-j-s \\ s \end{pmatrix}$$

where $\begin{pmatrix} n \\ s \end{pmatrix}$ *is combinations s of n objects, such that* $\begin{pmatrix} n \\ s \end{pmatrix} = 0$ *if $n < s$.*

Proof 2.30 *If we write $t_i = c_i (1 \leq i \leq k)$ in (9), $F_{2,n-1}(t)$ is reduced to the sequence $f_{k,n}^{2}$. Proof is completed by using $f_{k,n}^{2}(c_i = 1$ for $1 \leq i \leq k)$ and Theorem 2.17-iii.*

3 Conclusion

There are a number of studies on Fibonacci numbers, golden ratio and generalized Fibonacci numbers. Lately, researchers realized that generalized Fibonacci and Lucas polynomials are important for Fibonacci and Lucas generalizations. In this article, we generalized the Lucas numbers by the help of generalized Lucas polynomials and matrix $D_{(k)}^{\infty}$. We obtained some relations between generalized Fibonacci and Lucas numbers. Using these relations, properties and applications of Fibonacci numbers can be transferred to Lucas numbers and its generalizations. Since our definitions are polynomial based, it has a great number of application areas and it is more suitable to extend studies on number sequences.

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