1

Gen. Math. Notes, Vol. 9, No. 1, March 2012, pp.44-51
ISSN 2219-7184; Copyright ©ICSRS Publication, 2012
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Lacunary Strongly Almost Generalized Convergence with Respect to Orlicz Function

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(Received: 24-1-12/Accepted: 28-2-12)

Abstract

Kizmaz [5] *defined the concept of difference sequence spaces. Later some authors introduced and studied some generalizations of this idea. In this paper, we study some properties of* $[\widehat{c}, M]^{\theta}(\Delta^m)$ *-convergence which was defined by Esi* [1] .

Keywords: *Lacunary sequence, difference sequence, Orlicz function, strongly almost convergence.*

1 Definitions and Notations

Let l_∞, c and c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_i)$, respectively. A sequence $x = (x_i) \in l_\infty$ is said to be almost convergent [8] if all Banach limits of $x = (x_i)$ coincide. In [8] ,it was shown that

$$\widehat{c} = \left\{ x = (x_i) : \lim_{n\to\infty} \sum_{i=1}^{n} x_{i+s} \text{ exists, uniformly in } s \right\}.$$

In [9, 10], Maddox defined a sequence $x = (x_i)$ strongly almost convergent to a number L, if

$$\lim_{n\to\infty} \sum_{i=1}^{n} |x_{i+s} - L| = 0, \text{ uniformly in } s.$$

By a lacunary sequence $\theta = (k_r)$, $r = 0, 1, 2, ...$,where $k_0 = 0$, we shall mean increasing sequence of non-negative integers $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al. [3] as follows:

$$N_\theta = \left\{ x = (x_i) : \ \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} |x_i - L| = 0 \ \text{ for some L} \right\}.$$

In [5] , Kizmaz defined the sequence spaces $Z(\Delta) = \{x = (x_i) : \ (\Delta x_i) \in Z\}$ for $Z = l_\infty, c$ and c_0,where $\Delta x = (\Delta x_i) = (x_i - x_{i-1})$. After, Et and Colak [2] defined generalized the difference sequence spaces as follows: $Z(\Delta^m) = \{x = (x_i) : \ (\Delta^m x_i) \in Z\}$ for $Z = l_\infty, c$ and c_0,where $m \in \mathbb{N}$, $\Delta^0 x = x_i, \Delta x = (x_i - x_{i-1})$, $\Delta^m x_i = (\Delta^m x_i) = (\Delta^{m-1} x_i - \Delta^{m-1} x_{i+1})$ and so that

$$\Delta^m x_i = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{i+v}.$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to be satisfy Δ_2-condition for all values of t, if there exists a constant $T > 0$ such that $M(2t) \le TM(t), (t \ge 0)$.

Remark 1. The Δ_2-condition is equivalent to the satisfaction of the inequality $M(Lt) \le TLM(t)$ for all values of t and for $L > 1$.This inequality was used in some published articles [12] ,[13] and many others. But this is not true, which is shown by the simplest example such as $M(t) = t^2$. Then the Orlicz function M satisfies Δ_2-condition with $T = 4$, but for $M(Lt) = L^2 t^2 > 4Lt^2$ when $L \ge 5$.

Remark 2. An Orlicz function M satisfies the inequality $M(\lambda t) \le \lambda M(t)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz functions to construct Orlicz sequence spaces

$$l_M = \left\{ x = (x_i) : \ \sum_i M\left(\frac{|x_i|}{\rho}\right) < \infty, \ \text{for some } \rho > 0 \right\}.$$

The sequence space l_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \ \sum_i M\left(\frac{|x_i|}{\rho}\right) \le 1 \right\}$$

becomes a Banach space with is called an Orlicz Sequence Space. The space l_M is closely related to the space l_p , which is an Orlicz Sequence Space with $M(x) = x^p$ for $1 \leq p < \infty$.

Let M be an Orlicz function. Göngör and Et [4] defined the following sequence spaces:

$$\left[\widehat{c}, M\right]\left(\Delta^m\right) = \left\{ \begin{array}{c} x = (x_i) : \ \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} M\left(\frac{|\Delta^m x_{i+s} - L|}{\rho}\right) = 0, \\ \text{uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \end{array} \right\},$$

$$\left[\widehat{c}, M\right]_0\left(\Delta^m\right) = \left\{ x = (x_i) : \ \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right) = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \right\},$$

$$\left[\widehat{c}, M\right]_\infty\left(\Delta^m\right) = \left\{ x = (x_i) : \ \sup_{n,s} \frac{1}{n}\sum_{i=1}^{n} M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

Let M be an Orlicz function. We defined in [1] new generalized difference sequence spaces as follows:

$$\left[\widehat{c}, M\right]^\theta\left(\Delta^m\right) = \left\{ \begin{array}{c} x = (x_i) : \ \lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} M\left(\frac{|\Delta^m x_{i+s} - L|}{\rho}\right) = 0, \\ \text{uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \end{array} \right\},$$

$$\left[\widehat{c}, M\right]_0^\theta\left(\Delta^m\right) = \left\{ x = (x_i) : \ \lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right) = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \right\},$$

$$\left[\widehat{c}, M\right]_\infty^\theta\left(\Delta^m\right) = \left\{ x = (x_i) : \ \sup_{r,s} \frac{1}{h_r}\sum_{i\in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

If $x = (x_i) \in \left[\widehat{c}, M\right]^\theta\left(\Delta^m\right)$, we say that $x = (x_i)$ is lacunary strongly almost generalized Δ^m-convergence to the number L with respect to Orlicz function M.In this case we write $\left[\widehat{c}, M\right]^\theta\left(\Delta^m\right) - \lim x = L$.When $M(x) = x$, then we write $\left[\widehat{c}\right]^\theta\left(\Delta^m\right), \left[\widehat{c}\right]_0^\theta\left(\Delta^m\right)$ and $\left[\widehat{c}\right]_\infty^\theta\left(\Delta^m\right)$ for the spaces $\left[\widehat{c}, M\right]^\theta\left(\Delta^m\right), \left[\widehat{c}, M\right]_0^\theta\left(\Delta^m\right)$ and $\left[\widehat{c}, M\right]_\infty^\theta\left(\Delta^m\right)$.

The purpose of this paper is to examine some properties of these new sequence spaces as a concept of lacunary almost generalized Δ^m-convergence using Orlicz function which also generalize the well known Orlicz sequence space l_M , strongly summable sequence spaces $[C,1], [C,1]_0$ and $[C,1]_\infty$.

2 Main Results

In this section we prove some results involving the sequence spaces $[\widehat{c}, M]^{\theta}(\Delta^m), [\widehat{c}, M]_0^{\theta}(\Delta^m)$ and $[\widehat{c}, M]_{\infty}^{\theta}(\Delta^m)$.

Theorem 2.1. The spaces $[\widehat{c}, M]^{\theta}(\Delta^m), [\widehat{c}, M]_0^{\theta}(\Delta^m)$ and $[\widehat{c}, M]_{\infty}^{\theta}(\Delta^m)$ are linear spaces over the complex field $\mathbf{C}$.

Proof. Let $x = (x_i), y = (y_i) \in [\widehat{c}, M]_0^{\theta}(\Delta^m)$ and $\alpha, \beta \in \mathbf{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho_1}\right) = 0, \text{ uniformly in } s$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} M\left(\frac{|\Delta^m y_{i+s}|}{\rho_2}\right) = 0, \text{ uniformly in } s.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$ .Since M is non-decreasing convex function, we have

$$\frac{1}{h_r} \sum_{i\in I_r} M\left(\frac{|\Delta^m(\alpha x_{i+s} + \beta y_{i+s})|}{\rho_3}\right) \le \frac{1}{h_r} \sum_{i\in I_r} M\left[\left(\frac{|\Delta^m(\alpha x_{i+s})|}{\rho_3}\right) + \left(\frac{|\Delta^m(\beta y_{i+s})|}{\rho_3}\right)\right]$$

$$\le \frac{1}{h_r} \sum_{i\in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho_1}\right) + \frac{1}{h_r} \sum_{i\in I_r} M\left(\frac{|\Delta^m y_{i+s}|}{\rho_2}\right).$$

Therefore $\alpha x + \beta y \in [\widehat{c}, M]_0^{\theta}(\Delta^m)$.

The proof for other two cases are routine work in view of above proof.

Theorem 2.2. For any Orlicz function M, $[\widehat{c}, M]_{\infty}^{\theta}(\Delta^m)$ is a semi-normed linear space, semi-normed by

$$h_{\Delta^m}(x) = \sum_{i=1}^{m} |x_i| + \inf\left\{\rho > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right) \le 1,\ r = 1, 2, ...s = 1, 2, ...\right\}.$$

Proof. Clearly; $h_{\Delta^m}(x) = h_{\Delta^m}(-x), x = \bar{0}$ implies $\Delta^m x_{i+s} = 0$ for all $i, s \in N$ and as such $M\left(\bar{0}\right) = 0$,where $\bar{0} = (0, 0, ...)$.Therefore $h_{\Delta^m}\left(\bar{0}\right)$=0. Next, let ρ_1 and ρ_2 be such that

$$\sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho_1}\right) \le 1$$

and

$$\sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M\left(\frac{|\Delta^m y_{i+s}|}{\rho_2}\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$.Then, we have

$$\sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M\left(\frac{|\Delta^m (x_{i+s} + y_{i+s})|}{\rho}\right)$$

$$\leq \frac{\rho_1}{\rho_1+\rho_2} \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2} \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M\left(\frac{|\Delta^m y_{i+s}|}{\rho_2}\right) \leq 1.$$

Since the $\rho' s$ non-negative, so we have

$$h_{\Delta^m}(x+y) = \sum_{i=1}^{m} |x_i + y_i| + \inf\left\{\rho > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M\left(\frac{|\Delta^m (x_{i+s} + y_{i+s})|}{\rho}\right) \leq 1,\ r = 1,2,...s = 1,2,\right.$$

$$\leq \sum_{i=1}^{m} |x_i| + \inf\left\{\rho_1 > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho_1}\right) \leq 1,\ r = 1,2,...s = 1,2,...\right\}$$

$$+ \sum_{i=1}^{m} |y_i| + \inf\left\{\rho_2 > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M\left(\frac{|\Delta^m y_{i+s}|}{\rho_2}\right) \leq 1,\ r = 1,2,...s = 1,2,...\right\}.$$

So, $h_{\Delta^m}(x+y) \leq h_{\Delta^m}(x) + h_{\Delta^m}(y)$. Finally for $\lambda \in \mathbf{C}$, without loss of generality $\lambda \neq 0$, then

$$h_{\Delta^m}(\lambda x) = \sum_{i=1}^{m} |\lambda x_i| + \inf\left\{\rho > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M\left(\frac{|\Delta^m \lambda x_{i+s}|}{\rho}\right) \leq 1,\ r = 1,2,...s = 1,2,...\right\}$$

$$= |\lambda| \sum_{i=1}^{m} |x_i| + \inf\left\{|\lambda| r > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{r}\right) \leq 1,\ r = 1,2,...s = 1,2,...\right\}, \text{where } r = \frac{\rho}{\lambda}$$

$$= |\lambda| h_{\Delta^m}(\lambda x).$$

This completes the proof.

Theorem 2.3. If $\theta = (k_r)$ be a lacunary sequence with $\liminf q_r > 1$,then

$$[\widehat{c}, M](\Delta^m) \subset [\widehat{c}, M]^{\theta}(\Delta^m)$$

where

$$[\widehat{c}, M](\Delta^m) = \left\{ \begin{array}{c} x = (x_i) : \ \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} M\left(\frac{|\Delta^m x_{i+s} - L|}{\rho}\right) = 0, \\ \text{uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \end{array} \right\}.$$

Proof. Let $\liminf q_r > 1$. Then there exists $\delta > 0$ such that $q_r > 1 + \delta$ and hence

$$\frac{h_r}{k_r} = 1 - \frac{k_{r-1}}{k_r} > 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}.$$

Therefore,

$$\frac{1}{k_r}\sum_{i=1}^{k_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right) \geq \frac{1}{k_r}\sum_{i\in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right) \geq \frac{\delta}{1+\delta}\frac{1}{h_r}\sum_{i\in I_r} M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right)$$

and if $x = (x_i) \in [\widehat{c}, M](\Delta^m)$,then it follows that $x = (x_i) \in [\widehat{c}, M]^{\theta}(\Delta^m)$.

Theorem 2.4. If $\theta = (k_r)$ be a lacunary sequence with $\limsup q_r < \infty$,then

$$[\widehat{c}, M]^{\theta}(\Delta^m) \subset [\widehat{c}, M](\Delta^m).$$

Proof. Let $x = (x_i) \in [\widehat{c}, M]^{\theta}(\Delta^m)$ .Then for $\varepsilon > 0$, there exists j_0 such that for every $j \geq j_0$ and for all $s \in N$

$$a_{js} = \frac{1}{h_j}\sum_{i\in I_j} M\left(\frac{|\Delta^m x_{i+s}|}{\rho}\right) < \varepsilon$$

that is, we can find some positive constant T, such that

$$a_{js} < T \tag{1}$$

for all j and s. Given $\limsup q_r < \infty$ implies that there exists some positive number K such that

$$q_r < K \tag{2}$$

for all $r \geq 1$. Therefore, for $k_{r-1} < n \leq k_r$, we have by (1) and (2)

$$\frac{1}{n}\sum_{i=1}^{n} M\left(\frac{|\Delta^m x_{i+s} - L|}{\rho}\right) \leq \frac{1}{k_{r-1}}\sum_{i=1}^{k_r} M\left(\frac{|\Delta^m x_{i+s} - L|}{\rho}\right)$$

$$\leq \frac{1}{k_{r-1}}\sum_{j=1}^{r}\sum_{i\in I_j} M\left(\frac{|\Delta^m x_{i+s} - L|}{\rho}\right) = \frac{1}{k_{r-1}}\left[\sum_{j=1}^{j_0} + \sum_{j=j_0+1}^{r}\right]\sum_{i\in I_j} M\left(\frac{|\Delta^m x_{i+s} - L|}{\rho}\right)$$

$$\leq \frac{1}{k_{r-1}}\left(\sup_{1\leq p\leq j_0} a_{ps}\right) k_{j_0} + \varepsilon\frac{k_r - k_{j_0}}{k_{r-1}}$$

$$\leq T\frac{k_{j_0}}{k_{r-1}}+\varepsilon K.$$

Since $k_{r-1}\to\infty$ as $r\to\infty$,we get $x=(x_i)\in[\widehat{c},M](\Delta^m)$.This completes the proof.

Theorem 2.5. Let $\theta=(k_r)$ be a lacunary sequence with 1¡$\liminf q_r\leq\limsup q_r<\infty$, then

$$[\widehat{c},M]^{\theta}(\Delta^m)=[\widehat{c},M](\Delta^m).$$

Proof. It follows from Theorem 2.3. and Theorem 2.4..

Theorem 2.6. Let $x=(x_i)\in[\widehat{c},M](\Delta^m)\cap[\widehat{c},M]^{\theta}(\Delta^m)$. Then

$$[\widehat{c},M]^{\theta}(\Delta^m)-\lim x=[\widehat{c},M](\Delta^m)-\lim x$$

and $[\widehat{c},M]^{\theta}(\Delta^m)-\lim x$ is unique for any lacunary sequence $\theta=(k_r)$.

Proof. Let $x=(x_i)\in[\widehat{c},M](\Delta^m)\cap[\widehat{c},M]^{\theta}(\Delta^m)$ and $[\widehat{c},M]^{\theta}(\Delta^m)-\lim x=L_o$, $[\widehat{c},M](\Delta^m)-\lim x=L$.Suppose that $L\neq L_0$.We can see that

$$M\left(\frac{|L-L_o|}{\rho}\right)\leq\frac{1}{h_r}\sum_{i\in I_r}M\left(\frac{|\Delta^m x_{i+s}-L|}{\rho}\right)+\frac{1}{h_r}\sum_{i\in I_r}M\left(\frac{|\Delta^m x_{i+s}-L_o|}{\rho}\right)$$

$$\leq\lim_r\sup_s\frac{1}{h_r}\sum_{i\in I_r}M\left(\frac{|\Delta^m x_{i+s}-L|}{\rho}\right)+0.$$

Hence, there exists r_o, such that for $r>r_o$

$$\frac{1}{h_r}\sum_{i\in I_r}M\left(\frac{|\Delta^m x_{i+s}-L|}{\rho}\right)>\frac{1}{2}M\left(\frac{|L-L_o|}{\rho}\right).$$

Since $[\widehat{c},M](\Delta^m)-\lim x=L$, it follows that

$$0\geq\limsup_r\left(\frac{h_r}{k_r}\right)M\left(\frac{|L-L_o|}{\rho}\right)\geq\liminf_r M\left(\frac{|L-L_o|}{\rho}\right)\geq 0$$

and so $\lim_r q_r=1$. Hence by Theorem 2.2., $[\widehat{c},M]^{\theta}(\Delta^m)\subset[\widehat{c},M](\Delta^m)$ and $[\widehat{c},M]^{\theta}(\Delta^m)-\lim x=L_o=[\widehat{c},M](\Delta^m)-\lim x=L$. Further

$$\frac{1}{n}\sum_{i=1}^{n}M\left(\frac{|\Delta^m x_{i+s}-L|}{\rho}\right)+\frac{1}{n}\sum_{i=1}^{n}M\left(\frac{|\Delta^m x_{i+s}-L_o|}{\rho}\right)\geq M\left(\frac{|L-L_o|}{\rho}\right)\geq 0$$

and taking the limit on both sides as $n\to\infty$, we have $M\left(\frac{|L-L_o|}{\rho}\right)=0, i.e., L=L_o$ for any Orlicz function M. This completes the proof.

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