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Lacunary Invariant Statistical Convergence of Fuzzy Numbers

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Abstract

In this paper, we introduce the concepts of invariant convergence, lacunary invariant statistical convergence of sequences of fuzzy numbers and lacunary strongly invariant convergence of sequences of fuzzy numbers. We give some relations related to these concepts.

Keywords: *Fuzzy numbers, lacunary sequence, almost convergence, statistical convergence, invariant mean.*

1 Introduction

Schaefer [17] defined the σ-convergence as follows:

Let σ be a one- to- one mapping of the set of positive integers into itself such that $\sigma^k(n) = \sigma\left(\sigma^{k-1}(n)\right)$, $k = 1, 2, 3, \cdots$. A continuous linear functional φ on l_∞, the set of all bounded sequences, is said to be an invariant mean or a σ-mean if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$ and
(iii) $\varphi(\{x_\sigma(n)\}) = \varphi(\{x_n\})$ all $x = (x_n) \in l_\infty$.

For certain kinds of mappings σ, every invariant mean φ extends the limit functional on the space c, the set of all convergent sequences, in the sense that $\varphi(x) = \lim x$ for all $x = (x_n) \in c$. Consequently, $c \subset v_\sigma$, where v_σ is the set of bounded sequences all of whose σ-means are equal. In the case σ is the translation mapping $n \to n+1$, a σ-mean is often called a Banach limit and v_σ is the set of almost convergent sequences. If $x = (x_k)$, write $Tx = (Tx_k) = \left(x_{\sigma(k)}\right)$. It can be shown that

$$v_\sigma = \left\{ x \in l_\infty : \lim_{k\to\infty} t_{kn}(x) = l,\ uniformly\ in\ n,\ l = \sigma - \lim x \right\},$$

where $t_{kn}(x) = (k+1)^{-1} \sum_{i=0}^{k} x_{\sigma^i(n)}$, here $\sigma^k(n)$ denotes the k^{th} iterate of the mapping σ at n. By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_o = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by $\theta = (k_r)$ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated as q_r. Lacunary sequences have been discussed by [3], [7], [9], [11] and many others.

The notion of statistical convergence was introduced by Fast [6] and Schoenberg [18], independently. Over the years and under different names statistical convergence has been discussed in the different theories such as the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [8], Salat [16], Connor [2] and many others. This concept extended the idea to apply to sequences of fuzzy numbers with Nuray [14], Kwon *et al.* [11], Altin *et al.* [1], Nuray and Savaş [15] and many others.

A sequence $x = (x_k)$ is said to be statistically convergent to l if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{-1} \left| \{ k \le n : |x_k - l| \ge \varepsilon \} \right| = 0$$

where the vertical bars denote the cardinality of the set which they enclose, in which case we write $S-\lim x = l$. The concept of fuzzy sets was first introduced by Zadeh [19]. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [12]. Matloka show that every convergent sequences of fuzzy numbers is bounded. Later on sequences of fuzzy numbers have been discussed by Nanda [13], Nuray [14], [15], [10], Esi [5] and many others. Briefly, we recall some of the basic notations in the theory of fuzzy numbers and we refer readers to Matloka [12] and Diamond and Kloeden [4] for more details.

Let $C(R^n) = \{A \subset R^n : A\ is\ compact\ and\ convex\ set\}$. The space $C(R^n)$ has a linear structure induced by the operations

$$A + B = \{a + b : a \in A,\ b \in B\}\ and\ \gamma A = \{\gamma a : a \in A\}$$

for A, $B \in C(R^n)$ and $\gamma \in R$. The Hausdorff distance between A and B in $C(R^n)$ is defined by

$$\delta_\infty(A, B) = \max\left\{ \sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\| \right\}.$$

It is well-known that $(C(R^n), \delta_\infty)$ is a complete metric space. A fuzzy number is a function X from R^n to $[0,1]$ which is normal, fuzzy convex, upper semicontinuous and the closure of $\{X \in R^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the α-level set $X^\alpha = \{X \in R^n : X(x) \geq \alpha\}$ is a non-empty compact, convex subset of R^n, with support X^0. Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces the addition $X + Y$ and the scalar multiplication λX in terms of α-level sets, by

$$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha \ \ and \ \ [\lambda X]^\alpha = \lambda [X]^\alpha$$

for each $0 \leq \alpha \leq 1$. Consider the Hausdorff metrics d_q and d_∞ defined by

$$d_q(X,Y) = \left(\int_0^1 \delta_\infty (X^\alpha, Y^\alpha)\, d\alpha \right)^{\frac{1}{q}} \quad (1 \leq q \leq \infty)$$

and

$$d_\infty(X,Y) = \sup_{0 \leq \alpha \leq 1} \delta_\infty (X^\alpha, y^\alpha).$$

Clearly, $d_\infty(X,Y) = \lim_{q\to\infty} d_q(X,Y)$ with $d_q(X,Y) \leq d_s(X,Y)$ if $q \leq s$, [4].

For simplicity in notation, we shall write throughout $\overline{d}$ instead of d_q with $1 \leq q \leq \infty$. The metric $\overline{d}$ has the following properties:

$$\overline{d}(cX, cY) = |c|\, \overline{d}(X,Y) \tag{1}$$

for $c \in R$ and

$$\overline{d}(X+Y, Z+W) \leq \overline{d}(X,Z) + \overline{d}(Y,W). \tag{2}$$

A metric on $L(R)$ is said to be translation invariant $\overline{d}(X+Z, Y+Z) = \overline{d}(X,Y)$ for all $X, Y, Z \in L(R)$. A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of natural numbers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ [12]. We denote by w^F the set of all sequences $X = (X_k)$ of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded [2]. We denote by l_∞^F the set of all bounded sequences $X = (X_k)$ of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_o if for every $\varepsilon > 0$ there is a positive integer k_o such that $\overline{d}(X_k, X_o) < \varepsilon$ for $k > k_o$ [12]. We denote by c^F the set of all convergent sequences $X = (X_k)$ of fuzzy numbers. It is straightforward to see that $c^F \subset l_\infty^F \subset w^F$. In [13], it was shown that c^F and l_∞^F are complete metric spaces.

In the present note, we introduce and examine the concepts of invariant convergence of sequences, lacunary invariant statistical convergence of sequences of fuzzy numbers and lacunary strongly invariant convergence of sequences of fuzzy numbers. We give some relations related to these concepts. Let $p = (p_k) \in l_\infty$, then the following well-known inequality will be used in the paper: For sequences (a_k) and (b_k) of complex numbers we have

$$|a_k + b_k|^{p_k} \leq K \left(|a_k|^{p_k} + |b_k|^{p_k}\right), \tag{3}$$

where $K = \max\left(1, 2^{H-1}\right)$ and $H = \sup_k p_k < \infty$.

Definition 1.1 *A sequence $X = (X_k)$ of fuzzy numbers is said to be invariant convergent to fuzzy number X_o if*

$$\lim_{k\to\infty} d\left(t_{kn}\left(X\right), X_o\right) = 0 \tag{4}$$

uniformly in n, where $t_{kn}\left(X\right) = (k+1)^{-1} \sum_{i=0}^{k} X_{\sigma^i(n)}$.

This means that for every $\varepsilon > 0$, there exists a $k_o \in N$ such that $d\left(t_{kn}\left(X\right), X_o\right) < \varepsilon$ whenever $k \geq k_o$ and for all n. If the limit in (4) exists, then we write $v_\sigma^F - \lim X = X_o$. Let v_σ^F be the space of all invariant convergent sequences of fuzzy numbers. It is evident that $v_{\sigma,0}^F \subset v_\sigma^F$, where $v_{\sigma,0}^F$ denotes the classes of all invariant convergent to zero of fuzzy numbers.

Definition 1.2 *Let $\theta = (k_r)$ be lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary invariant statistically convergent to fuzzy number X_o if, for every $\varepsilon > 0$*

$$\lim_{r\to\infty} h_r^{-1} \left|\left\{k \in I_r : d\left(t_{kn}\left(X\right), X_o\right) \geq \varepsilon\right\}\right| = 0$$

uniformly in n. In this case we write $X_k \to X_o\left(\widehat{S}_\theta^\sigma\right)$ or $\widehat{S}_\theta^\sigma - \lim X_k = X_o$.

The set of all lacunary invariant statistically convergent sequences of fuzzy numbers is denoted by S_θ^σ. In special case $\theta = (2^r)$, we shall write $\widehat{S}^\sigma$ instead of $\widehat{S}_\theta^\sigma$.

Definition 1.3 *Let $\theta = (k_r)$ be lacunary sequence and $p = (p_k)$ be any sequence of strictly positive real numbers. A sequence $X = (X_k)$ is said to be lacunary strongly invariant convergent if there is a fuzzy number X_o such that*

$$\lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} = 0$$

uniformly in n. In this case we write $X_k \to X_o\left(\widehat{M}_\theta^\sigma, p\right)$.

The set of all lacunary strongly invariant convergent sequences of fuzzy numbers is denoted by $\left(\widehat{M}_\theta^\sigma, p\right)$. In special case $\theta = (2^r)$ and $p_k = 1$ for all $k \in N$, we shall write $\left(\widehat{C}^\sigma, p\right)$ and $\widehat{C}_\theta^\sigma$ instead of $\left(\widehat{M}_\theta^\sigma, p\right)$, respectively.

2 Main Results

In this section, we prove the results of this paper.

Theorem 2.1 *Let $X = (X_k)$ and $Y = (Y_k)$ be sequences of fuzzy numbers.*

(i) If $\widehat{S}_\theta^\sigma - \lim X_k = X_o$ and $a \in R$, then $\widehat{S}_\theta^\sigma - \lim aX_k = aX_o$.
(ii) $\widehat{S}_\theta^\sigma - \lim X_k = X_o$ and $\widehat{S}_\theta^\sigma - \lim Y_k = Y_o$, then $\widehat{S}_\theta^\sigma - \lim (X_k + Y_k) = X_o + Y_o$.

Proof. (i) Let $X_k \to X_o \left(\widehat{S}_\theta^\sigma\right)$ and $a \in R$. Then by taking into account the properties (1) and (2) of the metric d, we have

$$h_r^{-1} \left|\left\{k \in I_r : d\left(t_{kn}\left(aX\right), aX_0\right) \geq \varepsilon\right\}\right| \leq h_r^{-1} \left|\left\{k \in I_r : d\left(t_{kn}\left(X\right), X_o\right) \geq \frac{\varepsilon}{a}\right\}\right|$$

which yields that $\widehat{S}_\theta^\sigma - \lim aX_k = aX_o$.
(ii) By combining the Minkowski's inequality with the property (2) of the metric d, we derive that

$$d\left(t_{kn}\left(X\right) + t_{kn}\left(Y\right), X_o + Y_o\right) \leq d\left(t_{kn}\left(X\right), X_o\right) + d\left(t_{kn}\left(Y\right), Y_o\right).$$

Therefore given $\varepsilon > 0$ we have,

$$\begin{aligned} & h_r^{-1} \left|\left\{k \in I_r : d\left(t_{kn}\left(X\right) + t_{kn}\left(Y\right), X_o + Y_o\right) \geq \varepsilon\right\}\right| \\ & \leq h_r^{-1} \left|\left\{k \in I_r : d\left(t_{kn}\left(X\right), X_o\right) + d\left(t_{kn}\left(Y\right), Y_o\right) \geq \varepsilon\right\}\right| \\ & \leq h_r^{-1} \left|\left\{k \in I_r : d\left(t_{kn}\left(X\right), X_o\right) \geq \frac{\varepsilon}{2}\right\}\right| \\ & + h_r^{-1} \left|\left\{k \in I_r : d\left(t_{kn}\left(X\right), Y_o\right) \geq \frac{\varepsilon}{2}\right\}\right|, \end{aligned}$$

which yields that that $\widehat{S}_\theta^\sigma - \lim (X_k + Y_k) = X_o + Y_o$.

The following result is a consequence of Theorem 2.1.

Corollary 2.2 *Let $X = (X_k)$ and $Y = (Y_k)$ be sequences of fuzzy numbers.*

(iii) If $\widehat{S}^\sigma - \lim X_k = X_o$ and $a \in R$, then $\widehat{S}^\sigma - \lim aX_k = aX_o$.
(iv) $\widehat{S}^\sigma - \lim X_k = X_o$ and $\widehat{S}^\sigma - \lim Y_k = Y_o$, then $\widehat{S}^\sigma - \lim (X_k + Y_k) = X_o + Y_o$.

Theorem 2.3 *Let $\theta = (k_r)$ be lacunary sequence and let $X = (X_k)$ be a sequence of fuzzy numbers. Then,*
(i) *For* $\liminf q_r > 1$*, we have* $\left(\widehat{C}^\sigma, p\right) \subset \left(\widehat{M}_\theta^\sigma, p\right)$.
(ii) *For* $\limsup q_r < \infty$*, we have* $\left(\widehat{C}^\sigma, p\right) \supset \left(\widehat{M}_\theta^\sigma, p\right)$.
(iii) $\left(\widehat{C}^\sigma, p\right) = \left(\widehat{M}_\theta^\sigma, p\right)$ *if* $1 < \liminf q_r \leq \limsup q_r < \infty$.

Proof. (i) Let $\liminf q_r > 1$, then there exists $\delta > 0$ such that $q_r \geq 1+\delta$ for all $r \geq 1$. Then for $X = (X_k) \in \left(\widehat{C}^{\sigma}, p\right)$, we write

$$\begin{aligned} h_r^{-1} \sum_{k \in I_r} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} &= h_r^{-1} \sum_{k=1}^{k_r} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \\ &\quad - h_r^{-1} \sum_{k=1}^{k_{r-1}} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \\ &= k_r h_r^{-1} \left(k_r^{-1} \sum_{k=1}^{k_r} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k}\right) \\ &\quad - k_{r-1} h_r^{-1} \left(k_{r-1}^{-1} \sum_{k=1}^{k_{r-1}} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k}\right). \end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have $k_r h_r^{-1} \leq (1+\delta)^{\delta-1}$ and $k_{r-1} h_r^{-1} \leq \delta^{-1}$. The terms $k_r^{-1} \sum_{k=1}^{k_r} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k}$ and $k_{r-1}^{-1} \sum_{k=1}^{k_r} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k}$ both converge to 0 as $r \to \infty$ uniformly in n. Therefore $X = (X_k) \in \left(\widehat{M}_{\theta}^{\sigma}, p\right)$.

(ii) Now suppose that $\limsup q_r < \infty$, then there exists $A > 0$ such that $q_r < A$ for all $r \geq 1$. Let $X = (X_k) \in \left(\widehat{M}_{\theta}^{\sigma}, p\right)$ and $\varepsilon > 0$ be given. Then there exists a number $R > 0$ such that

$$A_i = h_i^{-1} \sum_{k=I_r} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} < \varepsilon$$

for every $i \geq R$ and all n. We can also find $K > 0$ such that $A_i < K$ for all $i = 1, 2, 3, \cdots$. Now let m be any integer with $k_{r-1} < m \leq k_r$, where $r > R$. Then

$$\begin{aligned} & m^{-1} \sum_{k=1}^{m} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \leq k_{r-1}^{-1} \sum_{k=1}^{k_r} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \\ = \; & k_{r-1}^{-1} \left[\begin{array}{c} \sum_{k=I_1} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} + \sum_{k=I_2} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} + \cdots \\ + \sum_{k=I_r} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \end{array} \right] \\ = \; & (k_1 - k_o)\, k_{r-1}^{-1} k_1^{-1} \sum_{k=I_1} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \\ & + (k_2 - k_1)\, k_{r-1}^{-1} k_2^{-1} (k_2 - k_1)^{-1} \sum_{k=I_2} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} + \cdots \\ & + (k_R - k_{R-1})\, k_{r-1}^{-1} (k_R - k_{R-1})^{-1} \sum_{k=I_R} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} + \cdots \\ & + (k_r - k_{r-1})\, k_{r-1}^{-1} (k_r - k_{r-1})^{-1} \sum_{k=I_r} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \\ = \; & k_1 k_{r-1}^{-1} A_1 + (k_2 - k_1) k_{r-1}^{-1} A_2 + \cdots + (k_R - k_{R-1})\, k_{r-1}^{-1} A_R + \cdots \\ & + (k_r - k_{r-1})\, k_{r-1}^{-1} A_r \end{aligned}$$

$$\leq \left(\sup_{i\geq 1} A_i\right) k_R k_{r-1}^{-1} + \left(\sup_{i\geq R} A_i\right)(k_r - k_R)\, k_{r-1}^{-1} < K k_R k_{r-1}^{-1} + \varepsilon A.$$

Since $k_{r-1} \to \infty$ as $m \to \infty$, it follows that $m^{-1}\sum_{k=1}^{m} \left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \to 0$ uniformly in n. Hence $X = (X_k) \in \left(\widehat{C}^{\sigma}, p\right)$.

(iii) Follows from (i) and (ii).

Theorem 2.4 *Let $\theta = (k_r)$ be lacunary sequence and let $X = (X_k)$ be a sequence of fuzzy numbers. Then,*

(i) $X_k \to X_o\left(\widehat{M}_{\theta}^{\sigma}, p\right)$ implies $X_k \to X_o\left(\widehat{S}_{\theta}^{\sigma}\right)$.

(ii) $X = (X_k) \in l_{\infty}^{F}$ and $X_k \to X_o\left(\widehat{S}_{\theta}^{\sigma}\right)$ imply $X_k \to X_o\left(\widehat{M}_{\theta}^{\sigma}, p\right)$.

(iii) $\widehat{S}_{\theta}^{\sigma} = \left(\widehat{M}_{\theta}^{\sigma}, p\right)$ if $X = (X_k) \in l_{\infty}^{F}$,
where $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$.

Proof. (i) $\varepsilon > 0$ and $X_k \to X_0\left(\widehat{M}_{\theta}^{\sigma}, p\right)$. Then we can write

$$\begin{aligned}
h_r^{-1}\sum_{k\in I_r}\left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} &= h_r^{-1}\sum_{k\in I_r,\ d(t_{kn}(X),X_o)\geq\varepsilon}\left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \\
&\quad + h_r^{-1}\sum_{k\in I_r,\ d(t_{kn}(X),X_o)<\varepsilon}\left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \\
&\geq h_r^{-1}\sum_{k\in I_r,\ d(t_{kn}(X),X_o)\geq\varepsilon}\left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \\
&\geq h_r^{-1}\sum_{k\in I_r,\ d(t_{kn}(X),X_o)\geq\varepsilon}\left[\varepsilon\right]^{p_k} \\
&\geq h_r^{-1}\sum_{k\in I_r,\ d(t_{kn}(X),X_o)\geq\varepsilon}\min\left[\varepsilon^h, \varepsilon^H\right]^{p_k} \\
&\geq h_r^{-1}\left|\left\{k\in I_r : d\left(t_{kn}(X), X_o\right)\geq\varepsilon\right\}\right| \min\left[\varepsilon^h, \varepsilon^H\right].
\end{aligned}$$

Hence $X_k \to X_o\left(\widehat{S}_{\theta}^{\sigma}\right)$.

(ii) Suppose that $X = (X_k) \in l_{\infty}^{F}$ and $X_k \to X_o\left(\widehat{S}_{\theta}^{\sigma}\right)$. Since $X = (X_k) \in l_{\infty}^{F}$, there is a constant $B > 0$ such that $d\left(t_{kn}\left(X\right), X_o\right) \leq B$. Given $\varepsilon > 0$, we have

$$\begin{aligned}
h_r^{-1}\sum_{k\in I_r}\left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} &= h_r^{-1}\sum_{k\in I_r,\ d(t_{kn}(X),X_o)\geq\varepsilon}\left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \\
&\quad + h_r^{-1}\sum_{k\in I_r,\ d(t_{kn}(X),X_o)<\varepsilon}\left[d\left(t_{kn}\left(X\right), X_o\right)\right]^{p_k} \\
&\quad \leq h_r^{-1}\sum_{k\in I_r,\ d(t_{kn}(X),X_o)\geq\varepsilon}\max\left[B^h, B^H\right] \\
&\quad + h_r^{-1}\sum_{k\in I_r,\ d(t_{kn}(X),X_o)<\varepsilon}\left[\varepsilon\right]^{p_k}
\end{aligned}$$

$$\leq \max\left[B^h, B^H\right] h_r^{-1} \left|\left\{k \in I_r : d\left(t_{kn}\left(X\right), X_o\right) \geq \varepsilon\right\}\right| \\ + \max\left[\varepsilon^h, \varepsilon^H\right].$$

Hence $X_k \to X_o\left(\widehat{M}_\theta^\sigma, p\right)$.

(iii) Follows from (i) and (ii).

Theorem 2.5 *Let $\theta = (k_r)$ be lacunary sequence and let $X = (X_k)$ be a sequence of fuzzy numbers. Then,*

(i) For $\limsup q_r < \infty$, we have $\widehat{S}_\theta^\sigma \subset \widehat{S}^\sigma$.

(ii) For $\liminf q_r > 1$, we have $\widehat{S}^\sigma \subset \widehat{S}_\theta^\sigma$.

(iii) if $1 < \liminf q_r \leq \limsup q_r < \infty$, then $\widehat{S}_\theta^\sigma = \widehat{S}^\sigma$.

Proof. (i) If $\limsup q_r < \infty$, then there is a $B > 0$ such that $q_r < B$ for all $r \geq 1$. Suppose that $X_k \to X_o\left(\widehat{S}_\theta^\sigma\right)$ and for each $n \geq 1$ set $N_{rn} = \left|\left\{k \in I_r : d\left(t_{kn}\left(X\right), X_o\right) \geq \varepsilon\right\}\right|$. Then there exists an $r_o \in N$ such that

$$N_{rn} h_r^{-1} < \varepsilon \ for\ all\ r > r_o\ and\ n \geq 1. \tag{5}$$

Now let $M = \max\left\{N_{rn} : 1 \leq r \leq r_o\right\}$ and choose m such that $k_{r-1} < m \leq k_r$, then for each $n \geq 1$ we have

$$\begin{aligned}
& m^{-1}\left|\left\{k \leq m : d\left(t_{kn}(X), X_o\right) \geq \varepsilon\right\}\right| \\
\leq\ & k_{r-1}^{-1}\left|\left\{k \leq k_r : d\left(t_{kn}(X), X_o\right) \geq \varepsilon\right\}\right| \\
=\ & k_{r-1}^{-1}\left\{N_{1n} + N_{2n} + \cdots + N_{r_o n} + N_{(r_o+1)n} + \cdots + N_{rn}\right\} \\
\leq\ & k_{r-1}^{-1} M r_o + k_{r-1}^{-1}\left\{h_{r_o+1}\left(h_{r_o+1}\right)^{-1} N_{(r_o+1)n} + \cdots + h_r h_r^{-1} N_{rn}\right\} \\
\leq\ & k_{r-1}^{-1} M r_o + k_{r-1}^{-1}\left(\sup_{r>r_o} h_r^{-1} N_{rn}\right)\left\{h_{r_o+1} + \cdots + h_r\right\} \\
\leq\ & k_{r-1}^{-1} M r_o + k_{r-1}^{-1}\varepsilon\left(k_r - k_{r_o}\right) \quad by (5) \\
\leq\ & k_{r-1}^{-1} M r_o + \varepsilon q_r \\
\leq\ & k_{r-1}^{-1} M r_o + B\varepsilon.
\end{aligned}$$

This completes the proof.

(ii) Suppose that $\liminf_{q_r} > 1$. Then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large $r \geq 1$. Since $h_r = k_r - k_{r-1}$, we have $h_r k_r^{-1} \leq (1 + \delta)^{-1}\delta$. Let $X_k \to X_o\left(\widehat{S}^\sigma\right)$. Then for every $\varepsilon > 0$ and for all n, we have

$$k_r^{-1}\left|\left\{k \leq k_r : d\left(t_{kn}\left(X\right), X_o\right) \geq \varepsilon\right\}\right| \geq k_r^{-1}\left|\left\{k \in I_r : d\left(t_{kn}\left(X\right), X_o\right) \geq \varepsilon\right\}\right| \\ \geq (1 + \delta)^{-1}\delta h_r^{-1}\left|\left\{k \in I_r : d\left(t_{kn}(X), X_o\right) \geq \varepsilon\right\}\right|$$

Hence $\widehat{S}^\sigma \subset \widehat{S}_\theta^\sigma$.

(iii) Follows from (i) and (iii).

Theorem 2.6 *Let* $0 < p_k \leq q_k$ *and* $\left(p_k q_k^{-1}\right)$ *be bounded. Then* $\left(\widehat{M}_\theta^\sigma, q\right) \subset \left(\widehat{M}_\theta^\sigma, p\right)$.

Proof. Let $X = (x_k) \in \left(\widehat{M}_\theta^\sigma, q\right)$. Let $w_{k,n} = \left(d\left(t_{kn}(X), X_o\right)\right)^{q_k}$ and $\lambda_k = p_k q_k^{-1}$ for all $k \in N$. We define the sequences $(u_{k,n})$ as follows: For $w_{k,n} \geq 1$, let $u_{k,n} = w_{k,n}$ and $v_{k,n} = 0$ and for $w_{k,n} < 1$, let $u_{k,n} = 0$ and $v_{k,n} = w_{k,n}$. Then it is clear that for all $k \in N$, we have $w_{k,n} = u_{k,n} + v_{k,n}$ and $w_{k,n}^{\lambda_k} = u_{k,n}^{\lambda_k} + v_{k,n}^{\lambda_k}$. Now it follows that $u_{k,n}^{\lambda_k} \leq u_{k,n} \leq w_{k,n}$ and $v_{k,n}^{\lambda_k} \leq v_{k,n}^{\lambda}$. Therefore

$$h_r^{-1} \sum_{k \in I_r} w_{n,m}^{\lambda_n} = h_r^{-1} \sum_{k \in I_r} \left(u_{n,m} + v_{n,m}\right)^{\lambda_n} \leq h_r^{-1} \sum_{k \in I_r} w_{n,m} + h_r^{-1} v_{n,m}^{\lambda}.$$

Now for each r,

$$\begin{aligned} h_r^{-1} \sum_{k \in I_r} v_{n,m}^{\lambda} &= \sum_{k \in I_r} \left(h_r^{-1} v_{n,m}^{\lambda}\right)^{\lambda} \left(h_r^{-1}\right)^{1-\lambda} \\ &\leq \left(\sum_{k \in I_r} \left(\left(h_r^{-1} v_{n,m}^{\lambda}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\lambda} \left(\sum_{k \in I_r} \left(\left(h_r^{-1} v_{n,m}^{\lambda}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\lambda} \\ &= \left(h_r^{-1} \sum_{k \in I_r} v_{n,m}\right)^{\lambda} \end{aligned}$$

and so

$$h_r^{-1} \sum_{k \in I_r} w_{n,m}^{\lambda_n} \leq h_r^{-1} \sum_{k \in I_r} w_{n,m} + \left(h_r^{-1} \sum_{k \in I_r} v_{n,m}\right)^{\lambda}.$$

Hence $X = (X_k) \in \left(\widehat{M}_\theta^\sigma, p\right)$, i.e. $\left(\widehat{M}_\theta^\sigma, q\right) \subset \left(\widehat{M}_\theta^\sigma, p\right)$. The proof of the following result is easy and thus is omitted.

Theorem 2.7 (i) *Let* $0 < \inf_k p_k \leq 1$, *then* $\left(\widehat{M}_\theta^\sigma, p\right) \subset \left(\widehat{M}_\theta^\sigma\right)$.

(ii) Let $0 < p_k \leq \sup_k p_k \leq \infty$, then $\left(\widehat{M}_\theta^\sigma\right) \subset \left(\widehat{M}_\theta^\sigma, p\right)$.

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